NOTES ON PSEUDO-SEQUENCE-COVERING MAPS

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Abstract. In this paper, we prove that each sequentially-quotient and boundary-compact map on $g$-metrizable spaces is pseudo-sequence-covering, and each finite subsequence-covering (or 1-sequentially-quotient) map on $snf$-countable spaces is pseudo-sequence-covering.

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1. Introduction

Pseudo-sequence-covering maps and sequentially-quotient maps play an important role in the study of images of metric spaces. It is well known that every pseudo-sequence-covering on metric spaces is sequentially-quotient. But none of these implications can be reversed. In 2005, S. Lin proved that each sequentially-quotient and compact map on metric spaces is pseudo-sequence-covering, and there exists a sequentially-quotient $\pi$-map on metric spaces is not pseudo-sequence-covering (\cite{16}). After that, F. C. Lin and S. Lin proved each sequentially-quotient and boundary-compact map on metric spaces is pseudo-sequence-covering (\cite{12}). Recently, the authors proved that if $X$ is an open image of metric spaces, then each sequentially-quotient and boundary-compact map on $X$ is pseudo-sequence-covering (\cite{13}).

In this paper, we prove that each sequentially-quotient and boundary-compact map on $g$-metrizable spaces is pseudo-sequence-covering, and each finite subsequence-covering (or 1-sequentially-quotient) map on $snf$-countable spaces is pseudo-sequence-covering.

Throughout this paper, all spaces are assumed to be Hausdorff, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers. Let $\mathcal{P}$ be a collection of subsets of $X$, we denote $\bigcup \mathcal{P} = \bigcup \{P : P \in \mathcal{P}\}$.

Definition 1.1. Let $X$ be a space, $\{x_n\} \subset X$ and $P \subset X$.

1. $\{x_n\}$ is called eventually in $P$, if $\{x_n\}$ converges to $x$, and there exists $m \in \mathbb{N}$ such that $\{x\} \bigcup \{x_n : n \geq m\} \subset P$. 

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2. \( \{x_n\} \) is called frequently in \( P \), if some subsequence of \( \{x_n\} \) is eventually in \( P \).

3. \( P \) is called a sequential neighborhood of \( x \) in \( X \), if whenever \( \{x_n\} \) is a sequence converging to \( x \) in \( X \), then \( \{x_n\} \) is eventually in \( P \).

**Definition 1.2.** Let \( \mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\} \) be a cover of a space \( X \). Assume that \( \mathcal{P} \) satisfies the following (a) and (b) for every \( x \in X \).

(a) \( \mathcal{P}_x \) is a network at \( x \).
(b) If \( P_1, P_2 \in \mathcal{P}_x \), then there exists \( P \in \mathcal{P}_x \) such that \( P \subset P_1 \cap P_2 \).

1. \( \mathcal{P} \) is a weak base of \( X \), if for \( G \subseteq X \), \( G \) is open in \( X \) if and only if for every \( x \in G \), there exists \( P \in \mathcal{P}_x \) such that \( P \subseteq G \); \( \mathcal{P}_x \) is said to be a weak neighborhood base at \( x \) in \( X \).

2. \( \mathcal{P} \) is an sn-network for \( X \), if each element of \( \mathcal{P}_x \) is a sequential neighborhood of \( x \) for all \( x \in X \); \( \mathcal{P}_x \) is said to be an sn-network at \( x \) in \( X \).

**Definition 1.3.** Let \( X \) be a space. Then,

1. \( X \) is \( gf \)-countable (resp., snf-countable \( \mathbb{N} \)), if \( X \) has a weak base (resp., sn-network) \( \mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\} \) such that each \( \mathcal{P}_x \) is countable.

2. \( X \) is \( g \)-metrizable \( \mathbb{R} \), if \( X \) is regular and has a \( \sigma \)-locally finite weak base.

3. \( X \) is sequential \( \mathbb{I} \), if whenever \( A \) is a non closed subset of \( X \), then there is a sequence in \( A \) converging to a point not in \( A \).

4. \( X \) is strongly \( g \)-developable \( \mathbb{M} \), if \( X \) is sequential has a \( \sigma \)-locally finite strong network consisting of cs-covers.

**Remark 1.4.**

1. Each strongly \( g \)-developable space is \( g \)-metrizable.

2. A space \( X \) is \( gf \)-countable if and only if it is sequential and snf-countable.

**Definition 1.5.** Let \( f : X \to Y \) be a map.

1. \( f \) is a compact map \( \mathbb{E} \), if each \( f^{-1}(y) \) is compact in \( X \).

2. \( f \) is a boundary-compact map \( \mathbb{E} \), if each \( \partial f^{-1}(y) \) is compact in \( X \).

3. \( f \) is a pseudo-sequence-covering map \( \mathbb{U} \), if for each convergent sequence \( L \) in \( Y \), there is a compact subset \( K \) in \( X \) such that \( f(K) = \overline{L} \).

4. \( f \) is a sequentially-quotient map \( \mathbb{E} \), if whenever \( \{y_n\} \) is a convergent sequence in \( Y \), there is a convergent sequence \( \{x_k\} \) in \( X \) with each \( x_k \in f^{-1}(y_{n_k}) \).

5. \( f \) is a finite subsequence-covering map \( \mathbb{U} \), if for each \( y \in Y \), there is a finite subset \( F \) of \( f^{-1}(y) \) such that for any sequence \( S \) converging to \( y \) in \( Y \), there is a sequence \( L \) converging to some \( x \in F \) in \( X \) and \( f(L) \) is a subsequence of \( S \).
6. \( f \) is a \( 1\)-sequentially-quotient map \([1]\), if for each \( y \in Y \), there exists \( x_y \in f^{-1}(y) \) such that whenever \( \{y_n\} \) is a sequence converging to \( y \) in \( Y \), there is a sequence \( \{x_{n_k}\} \) converging to \( x_y \) in \( X \) with each \( x_{n_k} \in f^{-1}(y_{n_k}) \).

Remark 1.6. 1. Each compact map is a compact-boundary map.

2. Each 1-sequentially-quotient map is a finite subsequence-covering map.

Definition 1.7 ([11]). A function \( g : \mathbb{N} \times X \to \mathcal{P}(X) \) is a CWC-map, if it satisfies the following conditions.

1. \( x \in g(n,x) \) for all \( x \in X \) and \( n \in \mathbb{N} \).
2. \( g(n+1,x) \subset g(n,x) \) for all \( n \in \mathbb{N} \).
3. \( \{g(n,x) : n \in \mathbb{N}\} \) is a weak neighborhood base at \( x \) for all \( x \in X \).

2. Main results

Theorem 2.1. Let \( f : X \to Y \) be a boundary-compact map. If \( X \) is a g-metrizable space, then \( f \) is a sequentially-quotient map if and only if it is a pseudo-sequence-covering map.

Proof. Necessity. Let \( f \) be a sequentially-quotient map and \( \{y_n\} \) be a non-trivial sequence converging to \( y \) in \( Y \). Since \( X \) is g-metrizable, it follows from Theorem 2.5 in [21] that there exists a CWC-map \( g \) on \( X \) satisfying for the sequences \( \{x_n\} \) and \( \{y_n\} \) of \( X \), if \( x_n \to x \) and \( y_n \in g(n,x_n) \) for all \( n \in \mathbb{N} \), then \( y_n \to x \). For \( n \in \mathbb{N} \), let

\[
U_{y,n} = \bigcup \{g(n,x) : x \in \partial f^{-1}(y)\}, \quad \text{and} \quad P_{y,n} = f(U_{y,n}).
\]

It is obvious that \( \{P_{y,n} : n \in \mathbb{N}\} \) is a decreasing sequence in \( X \). Furthermore, \( P_{n,y} \) is a sequential neighborhood of \( y \) in \( Y \) for all \( n \in \mathbb{N} \). If not, there exists \( n \in \mathbb{N} \) such that \( P_{y,n} \) is not a sequential neighborhood of \( y \) in \( Y \). Thus, there exists a sequence \( L \) converging to \( y \) in \( Y \) such that \( L \cap P_{y,n} = \emptyset \). Since \( f \) is sequentially-quotient, there exists a sequence \( S \) converging to \( x \in \partial f^{-1}(y) \) such that \( f(S) \) is a subsequence of \( L \). On the other hand, because \( g(n,x) \) is a sequential neighborhood of \( x \) in \( X \), \( S \) is eventually in \( g(n,x) \). Thus, \( S \) is eventually in \( U_{y,n} \). Therefore, \( L \) is frequently in \( P_{y,n} \). This contradicts to \( L \cap P_{y,n} = \emptyset \).

Then for each \( n \in \mathbb{N} \), there exists \( i_n \in \mathbb{N} \) such that \( y_i \in P_{y,n} \) for all \( i \geq i_n \). So \( f^{-1}(y_i) \cap U_{y,n} \neq \emptyset \). We can suppose that \( 1 < i_n < i_{n+1} \). For each \( j \in \mathbb{N} \), we take

\[
x_j \in \begin{cases} f^{-1}(y_j), & \text{if } j < i_1, \\ f^{-1}(y_j) \cap U_{y,n}, & \text{if } i_n \leq j < i_{n+1}. \end{cases}
\]

Let \( K = \partial f^{-1}(y) \cup \{x_j : j \in \mathbb{N}\} \). Clearly, \( f(K) = \{y\} \cup \{y_n : n \in \mathbb{N}\} \). Furthermore, \( K \) is a compact subset in \( X \). In fact, let \( \mathcal{U} \) be an open cover for \( K \) in \( X \). Since \( \partial f^{-1}(y) \) is a compact subset in \( X \), there exists a finite subfamily \( \mathcal{H} \subset \mathcal{U} \) such that \( \partial f^{-1}(y) \subset \bigcup \mathcal{H} \). Then there exists \( m \in \mathbb{N} \) such that \( U_{n,y} \subset \bigcup \mathcal{H} \).
∪ ∀ \mathcal{H} \text{ for all } n \geq m. \text{ If not, for each } n \in \mathbb{N}, \text{ there exists } v_n \in U_{y,n} - \bigcup \mathcal{H}. \text{ It implies that } v_n \in g(n, u_n) - \bigcup \mathcal{H} \text{ for some } u_n \in \partial f^{-1}(y). \text{ Since } \{u_n\} \subset \partial f^{-1}(y) \text{ and each compact subset of } X \text{ is metrizable, there exists a subsequence } \{u_{n_k}\} \text{ of } \{u_n\} \text{ such that } u_{n_k} \rightarrow x \in \partial f^{-1}(y). \text{ Now, for each } i \in \mathbb{N}, \text{ we put }

a_i = \begin{cases} u_{n_1} & \text{if } i \leq n_1 \\ u_{n_k+1} & \text{if } n_k < i \leq n_{k+1}; \end{cases}

b_i = \begin{cases} v_{n_1} & \text{if } i \leq n_1 \\ v_{n_k+1} & \text{if } n_k < i \leq n_{k+1}. \end{cases}

Then } a_i \rightarrow x. \text{ Because } g(n+1, x) \subset g(n, x) \text{ for all } x \in X \text{ and } n \in \mathbb{N}, \text{ it implies that } b_i \in g(i, a_i) \text{ for all } i \in \mathbb{N}. \text{ By property of } g, \text{ it implies that } b_i \rightarrow x. \text{ Thus, } v_{n_k} \rightarrow x. \text{ This contradicts to } \bigcup \mathcal{H} \text{ is a neighborhood of } x \text{ and } v_{n_k} \notin \bigcup \mathcal{H} \text{ for all } k \in \mathbb{N}.

Because } P_{y,i+1} \subset P_{y,i} \text{ for all } i \in \mathbb{N}, \text{ it implies that } \partial f^{-1}(y) \cup \{x_i : i \geq m\} \subset \bigcup \mathcal{H}. \text{ For each } i < m, \text{ take } V_i \in \mathcal{U} \text{ such that } x_i \in V_i. \text{ Put } V = \mathcal{U} \cup \{V_i : i < m\}. \text{ Then } V \subset \mathcal{U} \text{ and } K \subset \bigcup V. \text{ Therefore, } K \text{ is compact in } X, \text{ and } f \text{ is pseudo-sequence-covering.}

Sufficiency. Suppose that } f \text{ is a pseudo-sequence-covering map. If } \{y_n\} \text{ is a convergent sequence in } Y, \text{ then there is a compact subset } K \text{ in } X \text{ such that } f(K) = \overline{\{y_1\}}. \text{ For each } n \in \mathbb{N}, \text{ take a point } x_n \in f^{-1}(y_n) \cap K. \text{ Since } K \text{ is compact and metrizable, } \{x_n\} \text{ has a convergent subsequence } \{x_{n_k}\}, \text{ and } \{f(x_{n_k})\} \text{ is a subsequence of } \{y_n\}. \text{ Therefore, } f \text{ is sequentially-quotient.} \quad \square

By Remark 1.3, Remark 1.6 and Theorem 2.1, the following corollaries.

**Corollary 2.2.** Let } f : X \rightarrow Y \text{ be a boundary-compact map. If } X \text{ is strongly } g\text{-developable, then } f \text{ is sequentially-quotient if and only if it is pseudo-sequence-covering.}

**Corollary 2.3.** Let } f : X \rightarrow Y \text{ be a compact map. If } X \text{ is } g\text{-metrizable, then } f \text{ is sequentially-quotient if and only if it is pseudo-sequence-covering.}

**Corollary 2.4.** Let } f : X \rightarrow Y \text{ be a compact map. If } X \text{ is strongly } g\text{-developable, then } f \text{ is sequentially-quotient if and only if it is pseudo-sequence-covering.}

**Theorem 2.5.** Let } f : X \rightarrow Y \text{ be a finite subsequence-covering map. If } X \text{ is an snf-countable space, then } f \text{ is a pseudo-sequence-covering map.}

**Proof.** Let } \{y_n\} \text{ be a non-trivial sequence converging to } y \text{ in } Y \text{ and } \mathcal{B} = \bigcup \{\mathcal{B}_x : x \in X\} \text{ be an sn-network for } X \text{ such that each } \mathcal{B}_x \text{ is countable. Since } f \text{ is finite subsequence-covering, there exists a finite subset } F_y \subset f^{-1}(y) \text{ such that for each sequence } S \text{ converging to } y \text{ in } Y, \text{ there is a sequence } L \text{ in } X \text{ such that } L \text{ converging to some } x \in F_y \text{ and } f(L) \text{ is a subsequence of } S. \text{ Since each } \mathcal{B}_x \text{ is a countable sn-network at } x, \text{ for each } x \in X, \text{ we can choose a decreasing countable network } \{\mathcal{B}_{x,n} : n \in \mathbb{N}\} \subset \mathcal{B}_x. \text{ Put }
\[ U_y, n = \bigcup \{ B_{x, n} : x \in F_y \} \text{ and } P_{y, n} = f(U_{y, n}). \]

Then \( P_{y, n+1} \subset P_{y, n} \) for all \( n \in \mathbb{N} \). Furthermore, each \( P_{y, n} \) is a sequential neighborhood of \( y \) in \( Y \). If not, there exists \( n \in \mathbb{N} \) such that \( P_{y, n} \) is not a sequential neighborhood of \( y \) in \( Y \). Thus, there exists a sequence \( L \) converging to \( y \) in \( Y \) such that \( L \cap P_{y, n} = \emptyset \). Since \( f \) is finite subsequence-covering, there exists a sequence \( S \) in \( X \) such that \( S \) converges to some \( x \in F_y \) and \( f(S) \) is a subsequence of \( L \). On the other hand, because \( B_{x, n} \) is a sequential neighborhood of \( x \) in \( X \), \( S \) is eventually in \( B_{x, n} \). It implies that \( S \) is eventually in \( U_{y, n} \). Therefore, \( L \) is frequently in \( P_{y, n} \). This contradicts to \( L \cap P_{y, n} = \emptyset \).

Thus, for each \( n \in \mathbb{N} \), there exists \( i_n \in \mathbb{N} \) such that \( y_{i_n} \in f(U_{y, n}) \) for all \( i \geq i_n \). So \( g^{-1}(y_i) \cap U_{y, n} \neq \emptyset \). We can suppose that \( 1 < i_n < i_{n+1} \). For each \( j \in \mathbb{N} \), we take

\[ x_j \in \begin{cases} f^{-1}(y_j), & \text{if } j < i_1, \\ f^{-1}(y_j) \cap U_{y, n}, & \text{if } i_n \leq j < i_{n+1}. \end{cases} \]

Let \( K = F_y \cup \{ x_j : j \in \mathbb{N} \} \). Clearly, \( f(K) = \{ y \} \cup \{ y_n : n \in \mathbb{N} \} \). Furthermore, \( K \) is a compact subset in \( X \). In fact, let \( U \) be an open cover for \( K \) in \( X \). Since \( F_y \) is finite, there exists a finite subfamily \( \mathcal{H} \subset \mathcal{U} \) such that \( F_y \subset \bigcup \mathcal{H} \).

For each \( x \in F_y \), since \( \{ B_{x, n} : n \in \mathbb{N} \} \) is a decreasing network at \( x \) and \( \bigcup \mathcal{H} \) is a neighborhood of \( x \) in \( X \), \( B_{x, n_x} \subset \bigcup \mathcal{H} \) for some \( n_x \in \mathbb{N} \). If put \( k = \max \{ n_x : x \in F_y \} \), then \( U_{y, k} \subset \bigcup \mathcal{H} \). Furthermore, because \( U_{y, i+1} \subset U_{y, i} \) for all \( i \in \mathbb{N} \), \( F_y \cup \{ x_i : i \geq k \} \subset \bigcup \mathcal{H} \). For each \( i < k \), take \( V_i \in \mathcal{U} \) such that \( x_i \in V_i \), and put \( \mathcal{V} = \mathcal{H} \cup \{ V_i : i < k \} \). Then \( \mathcal{V} \subset \mathcal{U} \), \( K \subset \bigcup \mathcal{V} \) and \( K \) is compact in \( X \). Therefore, \( f \) is a pseudo-sequence-covering map.

By Remark 1.3, Remark 1.4 and Theorem 2.3, we have

**Corollary 2.6.** Let \( f : X \to Y \) be a 1-sequentially-quotient map. If \( X \) is an snf-countable space, then \( f \) is a pseudo-sequence-covering map.

**Corollary 2.7.** Let \( f : X \to Y \) be a finite subsequence-covering map. If \( X \) is a gf-countable space, then \( f \) is a pseudo-sequence-covering map.

**References**


