LOCAL CONVERGENCE OF MODIFIED HALLEY-LIKE METHODS WITH LESS COMPUTATION OF INVERSION

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Abstract. We present a local convergence analysis of a Modified Halley-Like Method of high convergence order in order to approximate a solution of a nonlinear equation in a Banach space. Our sufficient convergence conditions involve only hypotheses on the first Fréchet-derivative of the operator involved. Earlier studies use hypotheses up to the third Fréchet-derivative [26]. Numerical examples are also provided in this study.


Key words and phrases: Jarratt-type methods, Banach space, Local Convergence, Fréchet-derivative.

1. Introduction

In this study we are concerned with the problem of approximating a solution \( x^* \) of the nonlinear equation

\[
F(x) = 0,
\]

where \( F \) is a Fréchet-differentiable operator defined on a subset \( D \) of a Banach space \( X \) with values in a Banach space \( Y \).

Many problems in computational sciences and other disciplines can be brought in a form like (1.1) using mathematical modeling [3]. The solutions of equation (1.1) can rarely be found in closed form. That is why most solution methods for these equations are usually iterative. In particular, the practice of Numerical Functional Analysis for finding such solution is essentially connected to Newton-like methods [1]-[27]. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analyses. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and semi-local convergence analyses of Newton-like methods such as [1]-[27].

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We present a local convergence analysis for the modified Halley-Like Method \cite{15} defined for each \( n = 0, 1, 2, \cdots \) by

\[
\begin{align*}
  y_n &= x_n - F'(x_n)^{-1}F(x_n), \\
  u_n &= x_n - \theta F'(x_n)^{-1}F(x_n), \\
  z_n &= x_n - \gamma A_{\theta,n} F'(x_n)^{-1}F(x_n), \\
  x_{n+1} &= z_n - \alpha B_{\theta,n} F'(x_n)^{-1}F(z_n),
\end{align*}
\]

(1.2)

where \( x_0 \) is an initial point, \( \alpha, \gamma, \theta \in (-\infty, \infty) - \{0\} \) are given parameters, \( H_{\theta,n} = \frac{1}{2}F'(x_n)^{-1}(F'(u_n) - F'(x_n)) \), \( A_{\theta,n} = I - \frac{1}{2} H_{\theta,n} (I - \frac{1}{2} H_{\theta,n}) \) and \( B_{\theta,n} = I - H_{1,n} + H_{\theta,n}^2 \). The semi-local convergence of method (1.2) was studied in \cite{15} in the special case when \( \alpha = \gamma = 1 \) and \( \theta \in [0,1] \). Moreover, if \( \gamma = 1, \alpha = 0 \) and \( \theta \in (0,1) \), the semi-local convergence of the resulting method (1.2) was given in \cite{15}.

The semi-local convergence results in \cite{15} were given in a non-affine invariant form. However, the results obtained in our paper are given in affine invariant form. The sufficient semi-local convergence conditions (given in affine invariant form) used in \cite{15} are \((C)\):

\begin{enumerate}
  \item There exists \( F'(x_0)^{-1} \in L(Y, X) \) and \( \| F'(x_0)^{-1} \| \leq \beta \);
  \item \( \| F'(x_0)^{-1} F(x_0) \| \leq \beta_1 \);
  \item \( \| F'(x_0)^{-1} F''(x) \| \leq \beta_2 \) for each \( x \in D \);
  \item \( \| F'(x_0)^{-1} (F''(x) - F''(y)) \| \leq \beta_3 \| x - y \|^{q} \)
\end{enumerate}

for each \( x, y \in D \), and some \( q \in [0,1] \).

Under the \((C)\) conditions for \( \alpha = \gamma = 1 \) and \( \theta \in (0,1] \) the convergence order was shown to be \( 3 + 2q \) in \cite{15}. Moreover, for \( \gamma = 1, \alpha = 0 \) and \( \theta \in (0,1) \) the convergence order was shown to be \( 2 + q \) in \cite{3}.

Similar conditions have been used by several authors on other high convergence order methods \cite{1-15,26}. The corresponding conditions for the local convergence analysis are given by simply replacing \( x_0 \) by \( x^* \) in the preceding \((C)\) conditions. These conditions however are very restrictive. As a motivational example, let us define function \( f \) on \( D = [-\frac{1}{2}, \frac{5}{2}] \) by

\[
f(x) = \begin{cases}
  x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\
  0, & x = 0
\end{cases}
\]

Choose \( x^* = 1 \). We have that

\[
\begin{align*}
f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \quad F'(1) = 3, \\
f''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\
f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22.
\end{align*}
\]
Then, e.g., function $f$ cannot satisfy condition $(C_4)$, say for $q = 1$, since function $f'''$ is unbounded on $D$. In the present paper we only use hypotheses on the first Fréchet derivative (see conditions $(2.12)$-$(2.15)$). Notice that they used $\theta \in (0, 1]$, whereas in this paper $\theta$ can belong in a wider than $(0, 1]$ interval and $\gamma = \alpha = 1$ in [20]. This way we expand the applicability of method (1.2).

The paper is organized as follows. The local convergence of method (1.2) is given in Section 2, whereas the numerical examples are given in Section 3. Finally, some remarks are given in the concluding Section 4.

2. Local convergence analysis

We present the local convergence analysis of method (1.2) in this section. Denote by $U(v, \rho), \bar{U}(v, \rho)$ the open and closed balls, respectively, in $X$ of center $v \in X$ and of radius $\rho > 0$.

Let $L_0 > 0$, $L > 0$, $\theta \in (-\infty, \infty) - \{0\}$, $\alpha, \gamma \in (-\infty, \infty)$ and $M > 0$ be given parameters. Define functions on the interval $[0, 1/L_0)$ by

$$
g_1(r) = \frac{Lr}{2(1 - L_0r)},
$$

$$
g_2(r) = g_1(r) + \frac{M|1 - \theta|}{1 - L_0r},
$$

$$
g_3(r) = \frac{L_0(1 + g_2(r))}{2|\theta|(1 - L_0r)},
$$

$$
g_4(r) = 1 + g_3(r)r + g_3^2(r)r^2,
$$

$$
g_5(r) = g_1(r) + \frac{|\gamma|Mg_4(r)}{1 - L_0r},
$$

$$
g_6(r) = 1 + 2g_{1,3}(r)r + 4g_3^2(r)r^2,
$$

$$
g_{1,3}(r) = \frac{L_0(1 + g_1(r))}{2(1 - L_0r)}
$$

and

$$
g_7(r) = [1 + \frac{\alpha|Mg_6(r)}{1 - L_0r}]g_5(r).
$$

Moreover, define parameter

$$
r_2 = \frac{2(1 - M|1 - \theta|)}{2L_0 + L}.
$$

Suppose

$$M|1 - \theta| < 1.
$$

Then, it follows from the definition of functions $g_1$ and $g_2$ that

$$
0 < g_1(r) < 1, \text{ and } 0 < g_2(r) < 1, \text{ for each } r \in (0, r_2).
$$
Evidently, \( g_5(r) \in (0, 1) \), if for each \( r \in (0, r_5) \) and \( r_5 < \frac{1}{L_0} \) to be determined, we have that
\[
0 < g_1(r) + \frac{|\gamma|g_4(r)M}{1 - L_0r} < 1 \text{ for each } r \in (0, r_5).
\]

Define function \( p_5 \) on the interval \([0, \frac{1}{L_0}]\) by
\[
p_5(r) = |\gamma|Mg_4(r) - (1 - L_0r)(1 - g_1(r)).
\]
We have that
\[
p_5((\frac{1}{L_0})^-) = |\gamma|Mg_4((\frac{1}{L_0})^-) > 0.
\]
Suppose that
\[
|\gamma|M < 1.
\]
Then, we have that
\[
p_5(0) = M|\gamma| - 1 < 0.
\]
It follows from the intermediate value theorem that function \( p_5 \) has zeros in the interval \((0, \frac{1}{L_0})\). Denote by \( r_5 \) the smallest such zero. Then, we have that
\[
p_5(r) < 0 \Rightarrow 0 < g_5(r) < 1 \text{ for each } r \in (0, r_5).
\]
Similarly, function \( g_7 \in (0, 1) \) for each \( r \in (0, r_7) \) and \( r_7 < \frac{1}{L_0} \) to be determined, if function \( p_7(r) \in (0, 1) \) for each \( r \in [0, r_7] \), where
\[
p_7(r) = (1 - L_0r + |\alpha|Mg_6(r)g_5(r) - (1 - L_0r).
\]
We get that
\[
p_7((\frac{1}{L_0})^-) = |\gamma|Mg_6((\frac{1}{L_0})^-)g_5((\frac{1}{L_0})^-) > 0.
\]
and
\[
p_7(0) = (1 + |\alpha|Mg_6(0))|\gamma|g_5(0) - 1 = (1 + |\alpha|M)|\gamma|M - 1.
\]
Suppose that
\[
(1 + |\alpha|M)|\gamma|M < 1.
\]
Then, we have \( p_7(0) < 0 \). It follows that function \( p_7 \) has zeros in the interval \((0, \frac{1}{L_0})\). Denote by \( r_7 \) the smallest such zero. Then, we obtain that
\[
p_7(0) < 0 \Rightarrow 0 < g_7(r) < 1, \text{ for each } r \in (0, r_7).
\]
Set
\[
(2.1) \quad r^* = \min\{r_2, r_5, r_7\}.
\]
Then, we have that
\[
(2.2) \quad 0 < g_1(r) < 1,
\]
\[
(2.3) \quad 0 < g_2(r) < 1
\]
\[
(2.4) \quad 0 < g_3(r)
\]
\[
(2.5) \quad 0 < g_4(r)
\]
\[
(2.6) \quad 0 < g_5(r) < 1
\]
\[
(2.7) \quad 0 < g_6(r)
\]
and
\begin{align}
0 < g_7(r) < 1, \text{ for each } r \in (0, r^*).
\end{align}

Next, we present the local convergence analysis of method (1.2).

**THEOREM 2.1.** Let $F : D \subseteq X \rightarrow Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, parameters $L_0 > 0, L > 0, M > 0, \theta \in (-\infty, \infty) - \{0\}$ and $\alpha, \gamma \in (-\infty, \infty)$ such that for each $x \in D$
\begin{align}
M|1 - \theta| < 1,
\end{align}
\begin{align}
M|\gamma| < 1,
\end{align}
\begin{align}
(1 + |\alpha|M)|\gamma|M < 1,
\end{align}
\begin{align}
F(x^*) = 0, \quad F'(x^*)^{-1} \in L(Y, X),
\end{align}
\begin{align}
\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0\|x - x^*\|,
\end{align}
\begin{align}
\|F'(x^*)^{-1}(F(x) - F(x^*) - F'(x)(x - x^*))\| \leq \frac{L}{2}\|x - x^*\|^2,
\end{align}
\begin{align}
\|F'(x^*)^{-1}F'(x)\| \leq M
\end{align}
and
\begin{align}
\bar{U}(x^*, r^*) \subseteq D,
\end{align}

where $r^*$ is given in (2.1). Then, the sequence $\{x_n\}$ generated by method (1.2) for $x_0 \in \bar{U}(x^*, r^*)$ is well defined, remains in $U(x^*, r^*)$ for each $n = 0, 1, 2, \cdots$ and converges to $x^*$. Moreover, the following estimates hold for each $n = 0, 1, 2, \cdots$,
\begin{align}
\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r^*,
\end{align}
\begin{align}
\|u_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|,
\end{align}
\begin{align}
\|H_{\theta,n}\| \leq 2g_3(\|x_n - x^*\|)\|x_n - x^*\|,
\end{align}
\begin{align}
\|A_{\theta,n}\| \leq g_4(\|x_n - x^*\|)
\end{align}
\begin{align}
\|z_n - x^*\| \leq g_5(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|,
\end{align}
\begin{align}
\|B_{\theta,n}\| \leq g_6(\|x_n - x^*\|)
\end{align}
and
\begin{align}
\|x_{n+1} - x^*\| \leq g_7(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|.
\end{align}

where the "g" functions are defined above Theorem 2.1.
Proof. Using (2.13), the definition of \( r^* \) and the hypothesis \( x_0 \in U(x^*, r^*) \) we get that

\[
\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0\|x_0 - x^*\| < L_0r^* < 1.
\]

It follows from (2.24) and the Banach Lemma on invertible operators [3, 4] that \( F'(x_0)^{-1} \in L(Y, X) \) and

\[
\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|} < \frac{1}{1 - L_0r^*}.
\]

Hence, \( y_0 \) and \( u_0 \) are well defined. Using the first substep in method (1.2) for \( n = 0 \), (2.2), (2.14), (2.25) and the definition of function \( g_1 \) we obtain in turn that

\[
y_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0)
\]

\[= -F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)]\]

so,

\[
\|y_0 - x^*\| \leq \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)]\| \\
\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \\
= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r^*,
\]

which shows (2.17) for \( n = 0 \). We also have from the second substep of method (1.2) for \( n = 0 \), (2.3), (2.4), (2.5) and the definition of functions \( g_1 \) and \( g_2 \) that

\[
u_0 - x^* \leq \|y_0 - x^*\| + |1 - \theta|\|F'(x_0)^{-1}F'(x^*)\| \\
\times\|\int_0^1 F'(x^* + t(x_0 - x^*))dt\|\|x_0 - x^*\| \\
\leq [g_1(\|x_0 - x^*\|) + \frac{M|1 - \theta|}{1 - L_0\|x_0 - x^*\|}]\|x_0 - x^*\| \\
= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r^*,
\]

which shows (2.18) for \( n = 0 \).

Next, we need an estimate on \( \frac{1}{2}\|H_{\theta, 0}\| \). We have from (2.3), (2.4), (2.5),
(2.27) and the definition of functions $g_2$ and $g_3$ that

$$
\frac{1}{2} \|H_{\theta,0}\| \leq \frac{1}{2|\theta|} \|F'(x_0)^{-1}F'(x^*)\| (\|F'(x^*)^{-1}(F'(u_0) - F'(x^*))\| + \|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\|)
$$

$$
\leq \frac{L_0(\|u_0 - x^*\| + \|x_0 - x^*\|)}{2|\theta|(1 - L_0\|x_0 - x^*\|)}
$$

$$
\leq \frac{L_0(\|x_0 - x^*\| + g_2(\|x_0 - x^*\|)\|x_0 - x^*\|)}{2|\theta|(1 - L_0\|x_0 - x^*\|)}
$$

$$
\leq \frac{L_0(1 + g_2(\|x_0 - x^*\|))\|x_0 - x^*\|}{2|\theta|(1 - L_0\|x_0 - x^*\|)}
$$

(2.27)

$$
g_3(\|x_0 - x^*\|)\|x_0 - x^*\|,
$$

which shows (2.21) for $n = 0$. We also need an estimate on $\|A_{\theta,0}\|$. It follows from (2.27) and the definition of $A_{\theta,0}$, $g_3$, $g_4$ that

$$
\|A_{\theta,0}\| \leq 1 + \frac{1}{2} \|H_{\theta,0}\| + \frac{1}{4} \|H_{\theta,0}\|^2
$$

$$
\leq 1 + g_3(\|x_0 - x^*\|)\|x_0 - x^*\| + g_3(\|x_0 - x^*\|)\|x_0 - x^*\|^2
$$

(2.28)

$$
g_4(\|x_0 - x^*\|),
$$

which shows (2.21) for $n = 0$. Then, from the third substep of method (1.2) for $n = 0$, (2.19), (2.20), (2.28) the definition of functions $g_1$, $g_5$ and radius $r^*$, we have that

$$
\|z_0 - x^*\| \leq \|y_0 - x^*\| + |\gamma|\|A_{\theta,0}\||F'(x_0)^{-1}F'(x^*)|
$$

$$
\leq \|F'(x^*)^{-1}F'(x^* + t(x_0 - x^*)dt\|\|x_0 - x^*\|
$$

$$
\leq [g_1(\|x_0 - x^*\|) + M|g_4(\|x_0 - x^*\|)\|x_0 - x^*\|]
$$

(2.29)

$$
g_5(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r^*,
$$

which shows (2.21) for $n = 0$. Next, we need an estimate on $\|B_{\theta,0}\|$. We have by the definition of operator $B_{\theta,0}$ and functions $g_{1,3}$, $g_3$, $g_6$ that

(2.30)

$$
\|B_{\theta,0}\| \leq 1 + 2g_{1,3}(\|x_0 - x^*\|)\|x_0 - x^*\| + 4g_3^2(\|x_0 - x^*\|)\|x_0 - x^*\|^2
$$

$$
= g_6(\|x_0 - x^*\|),
$$

which shows (2.22) for $n = 0$. Using the fourth substep in method (1.2) for $n = 0$, (2.19), (2.20), (2.21), (2.22), (2.23) the definition of functions $g_5$, $g_6$, $g_7$
and radius \( r^* \), we obtain that

\[
\| x_1 - x^* \| \leq \| z_0 - x^* \| + |\alpha| ||B_{0,0}|| \| F'(x_0)^{-1} F'(x^*) \|
\]

\[
\| \int_0^1 F'(x^*)^{-1} F'(x^* + t(z_0 - x^*)) dt \| \| z_0 - x^* \|
\]

\[
\leq (1 + M|\alpha| g_6(\| x_0 - x^* \|)) \| z_0 - x^* \|
\]

\[
= (1 + M|\alpha| g_6(\| x_0 - x^* \|)) g_5(\| x_0 - x^* \|) \| x_0 - x^* \|,
\]

(2.31)

which shows (2.22) for \( n = 0 \). By simply replacing \( y_0, u_0, z_0, x_1 \) by \( y_k, u_k, z_k, x_{k+1} \) in the preceding estimates we arrive at estimates (2.17)-(2.23). Finally, from the estimate \( \| x_{k+1} - x^* \| < \| x_k - x^* \| \), we deduce that \( \lim_{k \to \infty} x_k = x^* \). 

\[ \square \]

**Remark 2.2.** 1. In view of (2.13) and the estimate

\[
\| F'(x^*)^{-1} F'(x) \| = \| F'(x^*)^{-1} (F'(x) - F'(x^*)) + I \|
\]

\[
\leq 1 + \| F'(x^*)^{-1} (F'(x) - F'(x^*)) \| \leq 1 + L_0 \| x - x^* \|
\]

condition (2.13) can be dropped and \( M \) can be replaced by

\[
M(r) = 1 + L_0 r.
\]

Moreover, condition (2.14) can be replaced by the popular but stronger conditions

\[
(2.32) \quad \| F'(x^*)^{-1} (F'(x) - F'(y)) \| \leq L \| x - y \| \text{ for each } x, y \in D
\]

or

\[
\| F'(x^*)^{-1} (F'(x^* + t(x - x^*)) - F'(x)) \| \leq L(1 - t) \| x - x^* \| \text{ for each } x, y \in D \text{ and } t \in [0,1].
\]

2. The results obtained here can be used for operators \( F \) satisfying autonomous differential equations [3] of the form

\[
F'(x) = P(F(x))
\]

where \( P \) is a continuous operator. Then, since \( F'(x^*) = P(F(x^*)) = P(0) \), we can apply the results without actually knowing \( x^* \). For example, let \( F(x) = e^x - 1 \). Then, we can choose: \( P(x) = x + 1 \).

3. The local results obtained here can be used for projection methods such as the Arnoldi’s method, the generalized minimum residual method (GM-RES), the generalized conjugate method (GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [3, 2].
4. The radius \( r_A \) given by

\[
(2.33) \quad r \leq r_A = \frac{1}{L_0 + L}.
\]

was shown by us to be the convergence radius of Newton’s method \[3, 4\]

\[
(2.34) \quad x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad \text{for each } n = 0, 1, 2, \ldots
\]

under the conditions (2.13) and (2.32). It follows from (2.1) and (2.33) that the convergence radius \( r^* \) of the method (1.2) cannot be larger than the convergence radius \( r_A \) of the second order Newton’s method (2.33). As already noted in \[3, 4\], \( r_A \) is at least as large as the convergence ball given by Rheinboldt \[3, 4\]

\[
(2.35) \quad r_R = \frac{2}{3L}.
\]

In particular, for \( L_0 < L \) we have that

\[
r_R < r_A
\]

and

\[
\frac{r_R}{r_A} \to \frac{1}{3} \text{ as } \frac{L_0}{L} \to 0.
\]

That is our convergence ball \( r_A \) is at most three times larger than Rheinboldt’s. The same value for \( r_R \) was given by Traub \[3, 4\].

5. It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger (C) conditions used in \[26\]. Moreover, we can compute the computational order of convergence (COC) defined by

\[
\xi = \ln \left( \frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|} \right) / \ln \left( \frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|} \right)
\]

or the approximate computational order of convergence

\[
\xi_1 = \ln \left( \frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|} \right) / \ln \left( \frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|} \right).
\]

This way we obtain in practice the order of convergence in a way that avoids the bounds given in \[26\] involving estimates up to the second Fréchet derivative of operator \( F \).

3. Numerical Examples

We present numerical examples in this section.
EXAMPLE 3.1. Let $X = Y = \mathbb{R}^2$, $D = \bar{U}(0, 1)$, $x^* = 0$ and define function $F$ on $D$ by

\begin{equation}
F(x) = (\sin x, \frac{1}{3}(e^x + 2x - 1)).
\end{equation}

Then, using (2.9)-(2.15), we get $L_0 = L = 1$, $M = \frac{1}{3}(e + 2)$, $\theta = \frac{3}{4}$, $\gamma = \frac{3}{4}$, $\alpha = \frac{3}{100}$. Then, by (2.11) we obtain

$$ r^* = 0.3161 < r_R = r_A = 0.6667 $$

EXAMPLE 3.2. Let $X = Y = \mathbb{R}^3$, $D = \bar{U}(0, 1)$. Define $F$ on $D$ for $v = x, y, z$ by

\begin{equation}
F(v) = (e^x - 1, \frac{e - 1}{2}y^2 + y, z).
\end{equation}

Then, the Fréchet-derivative is given by

$$ F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e - 1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. $$

Notice that $x^* = (0, 0, 0)$, $F'(x^*) = F'(x^*)^{-1} = \text{diag}\{1, 1, 1\}$, $L_0 = e - 1 < L = e$, $M = e$, $\theta = \frac{3}{4}$, $\gamma = \frac{3}{10}$, $\alpha = \frac{3}{100}$. Then, by (2.11) we obtain

$$ r^* = 0.2136 < r_R = 0.2453 < r_A = 0.3249. $$

EXAMPLE 3.3. Returning back to the motivational example at the introduction of this study, we see that conditions (2.12)-(2.15) are satisfied for $x^* = 1$, $f'(x^*) = 3$, $f(1) = 0$, $L_0 = L = 146.6629073$ and $M = 101.5578008$. Hence, the results of Theorem 2.1 can apply but not the ones in [26]. In particular, for $\theta = 0.9902$, $\alpha = 0.008$ and $\gamma = 0.005$ hypotheses (2.9)-(2.15) are satisfied. Moreover, we obtain

$$ r^* = 0.0032 < r_R = 0.0045 \leq r_A = 0.0045. $$

4. Conclusion

We present a local convergence analysis of Modified Halley-Like Methods with less computation of inversion in order to approximate a solution of an equation in a Banach space setting. Earlier convergence analysis is based on Lipschitz and Holder-type hypotheses up to the second Fréchet-derivative [11–27]. In this paper the local convergence analysis is based only on Lipschitz hypotheses of the first Fréchet-derivative. Hence, the applicability of these methods is expanded under less computational cost of the constants involved in the convergence analysis.
References


[21] H. Ren, Q. Wu, W. Bi, New variants of Jarratt method with sixth-order conver-

[22] X. Wang, J. Kou, Y. Li, Modified Jarratt method with sixth order convergence,

[23] X. Ye, C. Li, Convergence of the family of the deformed Euler-Halley iterations

iteration family under the Hölder condition of the first derivative, J. Comput.

[25] Y. Zhao, Q. Wu, Newton-Kantorovich theorem for a family of modified Hal-
ley’s method under Hölder continuity condition in Banach spaces, Appl. Math.

[26] X. Wang, J. Kou, Convergence for modified Halley-like methods with less com-

method in Banach spaces under general continuity conditions, Numer. Algo-