D-HOMOTHETIC DEFORMATION OF LP-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study a transformation called D-homothetic deformation of LP-Sasakian manifolds. Among others it is shown that in an LP-Sasakian manifold, the Ricci operator $Q$ commutes with the structure tensor $\phi$. We also discuss about the invariance of $\eta$-Einstein manifolds, $\phi$-sectional curvature, the locally $\phi$-Ricci symmetry and $\eta$-parallelity of the Ricci tensor under the D-homothetic deformation. Finally, we give an example of such a manifold.

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1. Introduction

The notion of Lorentzian para-Sasakian manifold was introduced by Matsumoto [5] in 1989. Then Mihai and Rosca [7] defined the same notion independently and they obtained several results in this manifold. LP-Sasakian manifolds have also been studied by Matsumoto and Mihai [6], De, and Shaikh [3], Ozgur [8] and others.

An LP-Sasakian manifold is said to be $\eta$-Einstein if its Ricci tensor $S$ is of the form

$$\begin{align*}
S &= \lambda g + \mu \eta \otimes \eta
\end{align*}$$

where $\lambda$ and $\mu$ are smooth functions on the manifold.

The notion of locally $\phi$-symmetry was first introduced by Takahashi [10] on a Sasakian manifold. Again in a recent paper [2] De and Sarkar introduced the notion of locally $\phi$-Ricci symmetric Sasakian manifolds. Also $\phi$-Ricci symmetric Kenmotsu manifolds have been studied by Shukla and Shukla [4].

An LP-Sasakian manifold is said to be locally $\phi$-Ricci symmetric if

$$\begin{align*}
\phi^2(\nabla_X Q)(Y) &= 0,
\end{align*}$$

where $Q$ is the Ricci operator defined by $g(QX, Y) = S(X, Y)$ and $X, Y$ are orthogonal to $\xi$.

The Ricci tensor $S$ of an LP-Sasakian manifold is said to be $\eta$-parallel if it satisfies

$$\begin{align*}
(\nabla_X S)(\phi Y, \phi Z) &= 0,
\end{align*}$$

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for all vector fields $X, Y$ and $Z$. The notion of $\eta$-parallelity in a Sasakian manifold was introduced by Kon [4].

Let $M (\phi, \xi, \eta, g)$ be an almost contact metric manifold with $\dim M = m = 2n + 1$. The equation $\eta = 0$ defines an $(m - 1)$-dimensional distribution $D$ on $M$ [11]. By an $(m - 1)$-homothetic deformation or $D$-homothetic deformation [12] we mean a change of structure tensors of the form

$$
\tilde{\eta} = a \eta, \quad \tilde{\xi} = \frac{1}{a} \xi, \quad \tilde{\phi} = \phi, \quad \tilde{g} = ag + a(a - 1)\eta \otimes \eta,
$$

where $a$ is a positive constant. If $M(\phi, \xi, \eta, g)$ is an almost contact metric structure with contact form $\eta$, then $M(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is also an almost contact metric structure [12]. Denoting by $W^i_{jk}$ the difference $\Gamma^i_{jk} - \Gamma^i_{jk}$ of Christoffel symbols we have in an almost contact metric manifold [12]

$$
W(X, Y) = (1 - a)[\eta(Y)\phi X + \eta(X)\phi Y] + \frac{1}{2}(1 - \frac{1}{a})[(\nabla_X \eta)(Y) + (\nabla_Y \eta)(X)]\xi \tag{1.4}
$$

for all $X, Y \in \chi(M)$. If $R$ and $\tilde{R}$ denote respectively the curvature tensor of the manifold $M(\phi, \xi, \eta, g)$ and $M(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, then we have [12]

$$
\tilde{R}(X, Y)Z = R(X, Y)Z + (\nabla_X W)(Z, Y) - (\nabla_Y W)(Z, X) + W(W(Z, Y), X) - W(W(Z, X), Y) \tag{1.5}
$$

for all $X, Y, Z \in \chi(M)$.

A plane section in the tangent space $T_p(M)$ is called a $\phi$-section if there exists a unit vector $X$ in $T_p(M)$ orthogonal to $\xi$ such that $\{X, \phi X\}$ is an orthonormal basis of the plane section. Then the sectional curvature

$$
K(X, \phi X) = g(R(X, \phi X)X, \phi X)
$$

is called a $\phi$-sectional curvature. A para contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be of constant $\phi$-sectional curvature if at any point $p \in M$, the sectional curvature $K(X, \phi X)$ is independent of the choice of non-zero $X \in D_p$, where $D$ denotes the contact distribution of the para contact metric manifold defined by $\eta = 0$.

The present paper is organized as follows:

After preliminaries in section 3, we prove some important Lemmas. Section 4 deals with the study of $(2n + 1)$-dimensional $\eta$–Einstein $LP$-Sasakian manifolds and prove that these manifolds are invariant under a $D$-homothetic deformation. Also we study $\phi$-sectional curvature, locally $\phi$-Ricci symmetry and $\eta$-parallelity of the Ricci tensor in a $(2n + 1)$-dimensional $LP$-Sasakian manifold under a $D$-homothetic deformation. Finally in section 5, we cited an example of $LP$-Sasakian manifold which verifies some theorems of section 4.
2. Preliminaries

Let $M^{2n+1}$ be an $2n+1$-dimensional differentiable manifold endowed with a (1,1) tensor field $\phi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and a Lorentzian metric $g$ of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p: T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-,+,+,...,+)$, where $T_pM$ denotes the tangent space of $M$ at $p$ and $\mathbb{R}$ is the real number space which satisfies

\begin{equation}
\phi^2(X) = X + \eta(X)\xi, \eta(\xi) = -1,
\end{equation}

\begin{equation}
g(X, \xi) = \eta(X), g(\phi X, \phi Y) = g(X, Y) + \eta(\xi)\eta(Y)
\end{equation}

for all vector fields $X, Y$. Then such a structure $(\phi, \xi, \eta, g)$ is termed as Lorentzian almost paracontact structure and the manifold $M^{2n+1}$ with the structure $(\phi, \xi, \eta, g)$ is called Lorentzian almost paracontact manifold [5]. In the Lorentzian almost paracontact manifold $M^{2n+1}$, the following relations hold [5]:

\begin{equation}
\phi \xi = 0, \eta(\phi X) = 0,
\end{equation}

\begin{equation}
\Omega(X, Y) = \Omega(Y, X),
\end{equation}

where $\Omega(X, Y) = g(X, \phi Y)$.

Let $\{e_i\}$ be an orthonormal basis such that $e_1 = \xi$. Then the Ricci tensor $S$ and the scalar curvature $r$ are defined by

\[ S(X, Y) = \sum_{i=1}^{n} \epsilon_i g(R(e_i, X)Y, e_i) \]

and

\[ r = \sum_{i=1}^{n} \epsilon_i S(e_i, e_i), \]

where we put $\epsilon_i = g(e_i, e_i)$, that is, $\epsilon_1 = -1, \epsilon_2 = \cdots = \epsilon_n = 1$.

An Lorentzian almost paracontact manifold $M^n$ equipped with the structure $(\phi, \xi, \eta, g)$ is called Lorentzian paracontact manifold if

\[ \Omega(X, Y) = \frac{1}{2} \{ (\nabla_X \eta)Y + (\nabla_Y \eta)X \}. \]

An Lorentzian almost paracontact manifold $M^n$ equipped with the structure $(\phi, \xi, \eta, g)$ is called an $LP$-Sasakian manifold [5] if

\[ (\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X. \]
In an LP-Sasakian manifold the 1-form $\eta$ is closed. Also in [5], it is proved that if an $n$-dimensional Lorentzian manifold $(M^n, g)$ admits a timelike unit vector field $\xi$ such that the 1-form $\eta$ associated to $\xi$ is closed and satisfies

$$ (\nabla_X \nabla_Y \eta) Z = g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z), $$

then $M^n$ admits an LP-Sasakian structure.

Further, on such an LP-Sasakian manifold $M^n (\phi, \xi, \eta, g)$, the following relations hold [3]:

\begin{align}
(2.5) & \quad \eta(R(X,Y)Z) = [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \\
(2.6) & \quad S(X, \xi) = 2n\eta(X), \\
(2.7) & \quad R(X, Y)\xi = [\eta(Y)X - \eta(X)Y], \\
(2.8) & \quad R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \\
(2.9) & \quad (\nabla_X \phi)(Y) = [g(X, Y)\eta + 2\eta(X)\eta(Y)\xi + \eta(Y)X],
\end{align}

for all vector fields $X, Y, Z$, where $R, S$ denote respectively the curvature tensor and the Ricci tensor of the manifold. Also since the vector field $\eta$ is closed in an LP-Sasakian manifold, we have (4,5)

\begin{align}
(2.10) & \quad (\nabla_X \eta)Y = \Omega(X, Y), \\
(2.11) & \quad \Omega(X, \xi) = 0, \\
(2.12) & \quad \nabla_X \xi = \phi X,
\end{align}

for any vector field $X$ and $Y$.

3. Some Lemmas

In this section we shall state and prove some Lemmas which will be needed to prove the main results.

**Lemma 3.1.** [5] In an LP-Sasakian manifold, the following relation holds

\begin{align}
 g(R(\phi X, \phi Y)\phi Z, \phi W) = g(R(X, Y)Z, W) + g(X, W)\eta(Y)\eta(Z) \\
 - g(X, Z)\eta(W)\eta(Y) + g(Y, Z)\eta(X)\eta(W) \\
 - g(Y, W)\eta(X)\eta(Z).
\end{align}
Lemma 3.2. Let \((M^{2n+1}, g)\) be an \(LP\)-Sasakian manifold. Then the Ricci operator \(Q\) commutes with \(\phi\).

Proof. From (3.1), it follows that

\[
\phi R(\phi X, \phi Y)\phi Z = R(X, Y)Z - [\eta(Z)Y - g(Y, Z)\xi]\eta(X) + [X\eta(Z) - g(X, Z)\xi]\eta(Y).
\]

(3.2)

Let \(\{e_i, \phi e_i, \xi\}, i = 1, 2, \ldots, n\) be an orthonormal frame at any point of the manifold. Then putting \(Y = Z = e_i\) in (3.2) and taking summation over \(i\) and using \(\eta(e_i) = 0\), we get

\[
\sum_{i=1}^{n} \epsilon_i \phi R(\phi X, \phi e_i)\phi e_i = \sum_{i=1}^{n} \epsilon_i R(X, e_i)e_i - n\eta(X)\xi,
\]

(3.3)

where \(\epsilon_i = g(e_i, e_i)\).

Again setting \(Y = Z = \phi e_i\) in (3.2) and taking summation over \(i\) and using \(\eta, \phi = 0\), we get

\[
\sum_{i=1}^{n} \epsilon_i \phi R(\phi X, e_i)e_i = \sum_{i=1}^{n} \epsilon_i R(X, \phi e_i)\phi e_i - n\eta(X)\xi.
\]

(3.4)

Adding (3.3) and (3.4) and using the definition of the Ricci tensor, we obtain

\[
\phi(Q\phi X - R(\phi X, \xi)\xi) = QX - R(X, \xi)\xi - 2n\eta(X)\xi.
\]

Using (2.1) and \(\phi \xi = 0\) in the above relation, we have

\[
\phi(Q\phi X) = QX - 2n\eta(X)\xi.
\]

Operating both sides by \(\phi\) and using (2.1), symmetry of \(Q\) and \(\phi \xi = 0\) we get \(\phi Q = Q\phi\). This proves the lemma.

\[
\square
\]

Proposition 3.1. In an \(2n + 1\)-dimensional \(\eta\)-Einstein \(LP\)-Sasakian manifold, the Ricci tensor \(S\) is expressed as

\[
S(X, Y) = \left[\frac{r}{2n} - 1\right]g(X, Y) - \left[\frac{r}{2n} - 2n - 1\right]\eta(X)\eta(Y).
\]

(3.5)
4. main results

In this section we study $\eta$-Einstein $LP$-Sasakian manifolds, $\phi$- sectional curvature, locally $\phi$- Ricci symmetry and $\eta$- parallelity of the Ricci tensor of an odd dimensional $LP$-Sasakian manifold under a D- homothetic deformation.

In virtue of (2.10), the relation (1.4) reduces to

$$ W(X, Y) = (1 - a)[\eta(Y)\phi X + \eta(X)\phi Y] + (1 - \frac{1}{a})g(\phi X, Y)\xi. $$

In view of (2.9), (2.10) and (2.12), the relation (4.1) yields

$$ (\nabla_Z W)(X, Y) = (1 - a)[g(\phi Z, Y)\phi X + g(X, Z)\eta(Y)\xi + 2\eta(X)\eta(Y)\eta(Z)\xi + g(\phi Z, Y)\phi Y + \eta(X)g(Y, Z)\xi] $$

$$ + \frac{a - 1}{a}g(\phi X, Y)\phi Z. $$

Using (4.1) and (4.2) into (1.5), we obtain by virtue of (2.7) and (2.10) that

$$ R(X, Y)Z = R(X, Y)Z + (1 - a)[g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + 2\eta(X)\eta(Y)\eta(Z)X $$

$$ - 2\eta(X)\eta(Z)Y + g(\phi Z, X)\phi Y - g(\phi Y, Z)\phi X] $$

$$ + \frac{a - 1}{a}[g(\phi Z, Y)\phi X - g(\phi Z, X)\phi Y] $$

$$ + (1 - a)^2[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X $$

$$ - \frac{(1 - a)^2}{a}[g(\phi Z, X)\phi Y - g(\phi Z, Y)\phi X]. $$

Putting $Y = Z = \xi$ in (4.3) and using (2.11) we obtain

$$ R(\eta(X)\xi) = R(\eta(X)\xi) + 2(1 - a)[-X + \eta(X)\xi] - (1 - a)^2\phi^2 X. $$

Let $\{e_i, \phi e_i, \xi\}, i = 1, 2, \ldots, n$ be an orthonormal frame at any point of the manifold. Then putting $Y = Z = \phi e_i$ in (4.3) and taking summation over $i$ and using $\eta(e_i) = 0$, we get

$$ \sum_{i=1}^{n}\epsilon_i R(X, e_i)e_i = \sum_{i=1}^{n}\epsilon_i R(X, e_i)e_i - (1 - a)n\eta(X)\xi, $$

where $\epsilon_i = g(\epsilon_i, e_i)$.

Again setting $Y = Z = \phi e_i$ in (4.3) and taking summation over $i$ and using $\eta.\phi = 0$, we get

$$ \sum_{i=1}^{n}\epsilon_i R(X, \phi e_i)\phi e_i = \sum_{i=1}^{n}\epsilon_i R(X, \phi e_i)\phi e_i - (1 - a)n\eta(X)\xi. $$
Adding (4.5) and (4.6) and using the definition of Ricci operator we have

\[
QX - \bar{R}(X, \xi)\xi = QX - R(X, \xi)\xi - 2(1-a)n\eta(X)\xi.
\]

In view of (4.4) we get from (4.7)

\[
\bar{S}(X, Y) = S(X, Y) - [2(1-a) + (1-a)^2]g(X, Y) - [2(1-a)(n-1) + (1-a)^2]\eta(X)\eta(Y),
\]

which implies that

\[
QX = QX - [2(1-a) + (1-a)^2]X - [2(1-a)(n-1) + (1-a)^2]\eta(X)\xi.
\]

Operating \(\bar{\phi} = \phi\) on both sides of (4.3) from the left we have

\[
\bar{\phi}QX = \phi QX - [2(1-a) + (1-a)^2]\phi X.
\]

Again, putting \(\bar{\phi}X = \phi X\) in (4.3) from the right we have

\[
\bar{Q}\phi X = Q\phi X - [2(1-a) + (1-a)^2]\phi X.
\]

Subtracting (4.10) and (4.11) we get

\[
(\bar{\phi}Q - \bar{Q}\phi)X = (\phi Q - Q\phi)X.
\]

Therefore using Lemma 3.2 we can state the following:

**Theorem 4.1.** Under a D-homothetic deformation, the expression \(\bar{Q}\phi = \phi\bar{Q}\) holds in an \((2n+1)\)-dimensional \(LP\)-Sasakian manifold.

### 4.1. \(\eta\)-Einstein \(LP\)-Sasakian manifolds

Let \(M(\phi, \xi, \eta, g)\) be a \((2n+1)\)-dimensional \(\eta\)-Einstein \(LP\)-Sasakian manifold which reduces to \(M(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})\) under a D-homothetic deformation. Then from (4.3) it follows by virtue of (3.3) that

\[
\bar{S}(X, Y) = \bar{\lambda}\bar{g}(X, Y) + \bar{\mu}\bar{\eta}(X)\bar{\eta}(Y),
\]

where \(\bar{\lambda}, \bar{\mu}\) are smooth functions given by

\[
\bar{\lambda} = \frac{r}{2n} - (a - 2)^2
\]

and

\[
\bar{\mu} = \frac{r}{2n} - 4n + 2an - a^2.
\]

In view of the relation (4.3) we state the following:

**Theorem 4.2.** Under a D-homothetic deformation, a \((2n+1)\)-dimensional \(\eta\)-Einstein \(LP\)-Sasakian manifold is invariant.
4.2. \( \phi \)-sectional curvature of \( LP \)-Sasakian manifolds

In this section we consider the \( \phi \)-sectional curvature on a \((2n+1)\)-dimensional \( LP \)-Sasakian manifold.

From (4.3) it can be easily seen that

\[
(4.16) \quad \bar{K}(X, \phi X) - K(X, \phi X) = -2(a - 1)
\]

and hence we state the following theorem.

**Theorem 4.3.** Under a D-homothetic deformation, the \( \phi \)-sectional curvature of a \((2n+1)\)-dimensional \( LP \)-Sasakian manifold is not an invariant.

If a \((2n+1)\)-dimensional \( LP \)-Sasakian manifold \( M(\phi, \xi, \eta, \bar{g}) \) satisfies

\[
R(X, Y)\bar{\xi} = 0 \quad \text{for all } X, Y,
\]

then it can be easily seen that \( K(X, \phi X) = 0 \) and hence from (4.16) it follows that

\[
\bar{K}(X, \phi X) = -2(a - 1) \neq 0
\]

where \( X \) is a unit vector field orthogonal to \( \xi \) and \( K(X, \phi X) \) is the \( \phi \)-sectional curvature. This implies that the \( \phi \)-sectional curvature \( \bar{K}(X, \phi X) \) is non-vanishing. Therefore we state the following:

**Theorem 4.4.** There exists \((2n+1)\)-dimensional \( LP \)-Sasakian manifold with non-zero \( \phi \)-sectional curvature.

4.3. Locally \( \phi \)-Ricci symmetric \( LP \)-Sasakian manifolds

In this section we study locally \( \phi \)-Ricci symmetry on an \( LP \)-Sasakian manifold.

Differentiating (4.5) covariantly with respect to \( W \) we obtain

\[
(\nabla_W Q))(X) = (\nabla_W Q))(X)
- [2(1 - a)(n - 1) + (1 - a)^2](\nabla_W \eta)(X)\xi
- [2(1 - a)(n - 1) + (1 - a)^2]\eta(X)\nabla_W \xi.
\]

(4.17)

Operating \( \phi^2 \) on both sides of (4.17) and taking \( X \) as an orthonormal vector to \( \xi \) we obtain

\[
(4.18) \quad \phi^2(\nabla_W \bar{Q})(X) = \phi^2(\nabla_W Q))(X).
\]

In view of the relation (4.18) we state the following:

**Theorem 4.5.** Under a D-homothetic deformation a locally \( \phi \)-Ricci symmetry on an \( LP \)-Sasakian manifold is invariant.
4.4. $\eta$– parallel Ricci tensor of an $LP$-Sasakian manifolds

Let us consider the $\eta$- parallelicity of the Ricci tensor on an $LP$-Sasakian manifold.

Differentiating (4.8) covariantly with respect to $W$ and using (2.10) we obtain

$$\nabla_W \tilde{S}(X,Y) = \nabla_W S(X,Y) - [2(1-a)(n-1) + (1-a)^2] \left[ g(\phi W, X)\eta(Y) + g(\phi W, Y)\eta(X) \right].$$

(4.19)

In (4.19) replacing $X$ by $\phi X$, $Y$ by $\phi Y$ and using (2.3) we get

$$\nabla_W \tilde{S}(\phi X, \phi Y) = \nabla_W S(\phi X, \phi Y).$$

(4.20)

Hence we can state the following:

**Theorem 4.6.** Under a D-homothetic deformation $\eta$– parallelicity of the Ricci tensor on an $LP$-Sasakian manifold is invariant.

5. Example

We consider the 3-dimensional manifold $M = \{ (x, y, z) \in \mathbb{R}^3 \}$, where $(x, y, z)$ are standard coordinate of $\mathbb{R}^3$.

The vector fields

$$e_1 = e^z \frac{\partial}{\partial y}, \quad e_2 = e^z \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of $M$.

Let $g$ be the Lorentzian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = 1,$$

$$g(e_3, e_3) = -1.$$

Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$.

Let $\phi$ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = -e_1, \quad \phi(e_2) = -e_2, \quad \phi(e_3) = 0.$$

Then using the linearity of $\phi$ and $g$, we have

$$\eta(e_3) = -1,$$

$$\phi^2 Z = Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$.
Then for \( e_3 = \xi \), the structure \((\phi, \xi, \eta, g)\) defines a Lorentzian paracontact structure on \( M \).

Let \( \nabla \) be the Levi-Civita connection with respect to the Lorentzian metric \( g \) and \( R \) be the curvature tensor of \( g \). Then we have

\[
[e_1, e_2] = 0 \quad , [e_1, e_3] = -e_1 \quad \text{and} \quad [e_2, e_3] = -e_2.
\]

Taking \( e_3 = \xi \) and using Koszul’s formula for the Lorentzian metric \( g \), we can easily calculate

\[
\nabla_{e_1} e_3 = -e_1, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_1} e_1 = -e_3,
\]

\[
\nabla_{e_2} e_3 = -e_2, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_1 = 0,
\]

(5.1) \( \nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0. \)

From the above it can be easily seen that \( M^3(\phi, \xi, \eta, g) \) is an LP-Sasakian manifold. With the help of the above results it can be easily verified that

\[
R(e_1, e_2) e_3 = 0 \quad , \quad R(e_2, e_3) e_3 = -e_2 \quad , \quad R(e_1, e_3) e_3 = -e_1,
\]

\[
R(e_1, e_2) e_2 = e_1 \quad , \quad R(e_2, e_3) e_2 = -e_3 \quad , \quad R(e_1, e_3) e_2 = 0,
\]

\[
R(e_1, e_2) e_1 = -e_2 \quad , \quad R(e_2, e_3) e_1 = 0 \quad , \quad R(e_1, e_3) e_1 = -e_3.
\]

From the above expressions of the curvature tensor we obtain

\[
S(e_1, e_1) = g(R(e_1, e_2) e_2, e_1) - g(R(e_1, e_3) e_3, e_1) = 2.
\]

Similarly we have

\[
S(e_2, e_2) = 2
\]

and

\[
S(e_3, e_3) = -2.
\]

Therefore,

\[
r = S(e_1, e_1) + S(e_2, e_2) - S(e_3, e_3) = 6.
\]

From [3] we know that in a 3- dimensional LP-Sasakian manifold

\[
R(X, Y) Z = \left( \frac{r-4}{2} \right) [g(Y, Z)X - g(X, Z)Y] + \left( \frac{r-6}{2} \right) [g(Y, Z)\eta(X)\xi

- g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y].
\]

(5.2)

Now using (5.2) we get

\[
g(R(X, Y) Z, W) = \left( \frac{r-4}{2} \right) [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]

+ \left( \frac{r-6}{2} \right) [g(Y, Z)\eta(X)\eta(W) - g(X, Z)\eta(Y)\eta(W)

+ \eta(Y)\eta(Z)g(X, W) - \eta(X)\eta(Z)g(Y, W)].
\]

(5.3)
From (5.3), it follows that the ϕ- sectional curvature of the manifold is given by
\[ K(X, \phi X) = \frac{r - 4}{2} \]
for any vector field X orthogonal to ξ.

In view of the above relation we get
\[ K(e_1, \phi e_1) = K(e_2, \phi e_2) = \frac{r - 4}{2} \]
Again it can be easily shown from (6.3) that
\[ \bar{K}(e_1, \phi e_1) - K(e_1, \phi e_1) = -2(a - 1) \]
and
\[ \bar{K}(e_2, \phi e_2) - K(e_2, \phi e_2) = -2(a - 1) \]
Therefore Theorem 4.3 is verified.

References