On a two-variables fractional partial differential inclusion via Riemann-Liouville derivative

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Abstract. We investigate the existence of solution for a two-variables fractional partial differential inclusion via Riemann-Liouville derivative. Also, we provide an example to illustrate our main result.

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1. Introduction

There are many published works about fractional partial differential equations by using the notions delay or time-fractional (see for example, [1], [2], [9] and [10]). It is interesting to work on two variables fractional partial differential equations (see for example, [3], [4], [6] and [11]).

Let $\theta = (0,0)$ and $\alpha = (\alpha_1, \alpha_2)$ where $0 < \alpha_1, \alpha_2 \leq 1$. Also, put $J_a \times J_b = [0,a] \times [0,b]$ where $a$ and $b$ are positive constants. The Riemann-Liouville fractional partial integral of order $\alpha$ for a function $u \in L^1(J_a \times J_b)$ is defined by

$$(I_\theta^\alpha u)(x,y) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}u(s,t)dtds$$

whenever the integral exists (see for more details (see for example, [14], [15] and [16]).

The Riemann-Liouville partial derivative of fractional order $\alpha$ for a function $u \in L^1(J_a \times J_b)$ is defined by

$$(D_\theta^\alpha u)(x,y) = D_{xy}^2(I_\theta^{1-\alpha}u)(x,y) = \frac{\partial^2}{\partial x\partial y} \int_0^x \int_0^y \frac{(x-s)^{-\alpha_1}(y-t)^{-\alpha_2}}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)}dtds$$

(see for more details (see for example, [14], [15] and [16]). Note that,

$$(I_\theta^\alpha I_\theta^\beta u)(x,y) = (I_\theta^{\alpha+\beta} u)(x,y)$$

whenever $\beta = (\beta_1, \beta_2) > 0$ (16).

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Let \((X, d)\) be a metric space, \(\mathcal{P}(X)\) the class of all nonempty subsets of \(X\), \(\mathcal{P}_{cl}(X)\) the class of all closed subsets of \(X\), \(\mathcal{P}_{bd}(X)\) the class of all bounded subsets of \(X\), \(\mathcal{P}_{cp}(X)\) the class of all compact subsets of \(X\) and \(\mathcal{P}_{cv}(X)\) the class of all convex subsets of \(X\). A multi-valued map \(F : J_a \times J_b \rightarrow \mathcal{P}_{cl}(\mathbb{R})\) is measurable whenever the function \((x, y) \mapsto d(w, F(x, y)) = \inf\{||w - v|| : v \in F(x, y)\}\) is measurable for all \(w \in \mathbb{R}\), where \(J_a \times J_b = [0, a] \times [0, b]\) ([12]). Also, the Pompeiu-Hausdorff metric \(H_d : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow [0, \infty)\) is defined by

\[
H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},
\]

where \(d(A, b) = \inf_{a \in A} d(a, b)\) ([2]). Then \((\mathcal{P}_{cl, bd}(X), H_d)\) is a metric space and \((\mathcal{P}_{cl}(X), H_d)\) is a generalized metric space ([4]). Recall that a multifunction \(F : X \rightarrow \mathcal{P}(X)\) is said to be a contraction if there exists \(k \in (0, 1)\) such that \(H_d(F(u), F(v)) \leq kd(u, v)\) for all \(u, v \in X\) ([4]). An element \(u \in X\) is called endpoint of the multifunction \(F : \mathcal{P}(X) \rightarrow \mathcal{P}(X)\) whenever \(Fu = \{u\}\) ([8]). We say that the multifunction \(F\) has an approximate endpoint property whenever \(\inf_{u \in X} \sup_{w \in Fu} d(u, w) = 0\) ([8]). A real-valued function \(f\) on \(\mathbb{R}\) is called upper semi-continuous whenever \(\limsup_{n \to \infty} f(\lambda_n) \leq f(\lambda)\) for all sequence \(\{\lambda_n\}_{n \geq 1}\) with \(\lambda_n \to \lambda\). In this paper, by using main idea of [5], [6] and [17], we investigate the existence of solutions for the two-variables fractional partial differential inclusion

\[
(D_\alpha^\delta u)(x, y) \in F(x, y, u(x, y)),
\]

with the partial integral boundary value conditions

\[
(I_\delta^{1-\alpha} u)(x, 0) = \lambda_1 \varphi(x), \quad (I_\delta^{1-\alpha} u)(0, y) = \lambda_2 \gamma(y),
\]

where \(D_\delta^\alpha\) denotes the Riemann-Liouville fractional partial derivative of order \(\alpha\), \((x, y) \in J_a \times J_b\), \(0 < \alpha_i \leq 1\), \(\lambda_i \in \mathbb{R}^+\) \((i = 1, 2)\) and \(F : J_a \times J_b \times \mathbb{R} \to \mathcal{P}(\mathbb{R})\) is a compact valued multi-valued map. Here, the functions \(\varphi : J_a \to \mathbb{R}\) and \(\gamma : J_b \to \mathbb{R}\) are absolutely continuous with \(\varphi(0) = \gamma(0) = 0\). We need next endpoint result.

**Theorem 1.1.** ([8]) Suppose that \((X, d)\) is a complete metric space, \(\psi : [0, \infty) \to [0, \infty)\) is an upper semi-continuous function such that \(\psi(t) < t\) and \(\liminf_{t \to \infty} (t - \psi(t)) > 0\) for all \(t > 0\) and \(T : X \to CB(X)\) is a multifunction such that \(H_d(Tx, Ty) \leq \psi(d(x, y))\) for all \(x, y \in X\). Then \(T\) has a unique endpoint if and only if \(T\) has approximate endpoint property.

2. Main results

Now, we are ready to state and prove our main results. First, we give next key result.

**Lemma 2.1.** Let \(f \in L(J_a \times J_b)\) and \(\alpha = (\alpha_1, \alpha_2) \in (0, 1] \times (0, 1]\). Then the continuous function \(u_0 \in L(J_a \times J_b)\) is a solution for the fractional partial differential equation

\[
(D_\alpha^\delta u)(x, y) = f(x, y)
\]
with boundary conditions \((I_{\theta}^{1-\alpha}u)(x,0) = \lambda_1 \varphi(x) \) and \((I_{\theta}^{1-\alpha}u)(0,y) = \lambda_2 \gamma(y)\) if and only if \(u_0\) is a solution for the fractional integral equation

\[(2.2) u(x,y) = \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_{\theta}^{\alpha_2} \gamma)(y) + \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_{\theta}^{\alpha_1} \varphi)(x) + (I_{\theta}^{1} f)(x, y).\]

**Proof.** Let \(u_0\) be a solution for the fractional partial differential equation \((2.1)\). Then, we have \(D_{xy}^2 (I_{\theta}^{1-\alpha}u_0)(x,y) = f(x, y)\) and so

\[\begin{aligned}
(I_{\theta}^{1-\alpha}u_0)(x,y) - (I_{\theta}^{1-\alpha}u_0)(0,0) &= (I_{\theta}^{1} f)(x, y) \\
\int_0^x (x-s)^{\alpha_1-1} \varphi(s) dt &= \lambda_1 (\int_0^y (y-t)^{\alpha_2-1} \gamma(t) dt) + \lambda_2 (\int_0^y (y-t)^{\alpha_2-1} \gamma(t) dt)
\end{aligned}\]

By using the boundary conditions, we get

\[\begin{aligned}
(I_{\theta}^{1-\alpha}u_0)(x,y) &= \lambda_1 \varphi(x) - \lambda_2 \gamma(y) + \lambda_1 \varphi(0) = (I_{\theta}^{1} f)(x, y) \\
\text{and so } (I_{\theta}^{1-\alpha}u_0)(x,y) - (I_{\theta}^{1-\alpha}u_0)(0,0) &= \lambda_1 \varphi(x) + \lambda_2 \gamma(y).
\end{aligned}\]

Since \(\varphi(x) = \int_0^x \varphi(s) ds\) and \(\gamma(y) = \int_0^y \gamma(t) dt\), we obtain

\[\begin{aligned}
(I_{\theta}^{1} p)(x,y) &= \lambda_1 \left(\int_0^x \varphi(\int_0^s \varphi(t) dt) ds + \int_0^y \int_0^t \gamma(t) dt \varphi(s) ds \right) \\
&\quad + \lambda_2 \left(\int_0^x \varphi(\int_0^s \varphi(t) dt) ds + \int_0^y \int_0^t \gamma(t) dt \varphi(s) ds \right)
\end{aligned}\]
Since \((I^{\alpha_1}_\theta \phi)(x) \in L(J_a)\) and \((I^{\alpha_2}_\theta \gamma)(y) \in L(J_b)\), the functions \(\int_0^x (I^{\alpha_1}_\theta \phi)(s) ds\)
and \(\int_0^y (I^{\alpha_2}_\theta \gamma)(t) dt\) are absolutely continuous and so there exists \(D^2_{xy}(I^{\alpha}_\theta p)(x, y)\) for almost all \((x, y) \in J_a \times J_b\). By applying the operator \(D^2_{xy}\) on both sides of (2.3), we get \(D^2_{xy} \left[ I^{\alpha}_\theta \left( u_0(x, y) - (I^{\alpha}_\theta f)(x, y) \right) \right](x, y) = D^2_{xy} \left[ (I^{\alpha}_\theta p)(x, y) \right] \). Thus,

\[
\begin{align*}
    u_0(x, y) - (I^{\alpha}_\theta f)(x, y) &= D^2_{xy} \left[ \frac{\lambda_1 y^{\alpha_2}}{\Gamma(\alpha_2 + 1)} \int_0^x (I^{\alpha_1}_\theta \phi)(s) ds + \frac{\lambda_2 x^{\alpha_1}}{\Gamma(\alpha_1 + 1)} \int_0^y (I^{\alpha_2}_\theta \gamma)(t) dt \right] \\
    &= D_x \left[ \frac{\lambda_1 y^{\alpha_2 - 1}}{\Gamma(\alpha_2)} \int_0^x (I^{\alpha_1}_\theta \phi)(s) ds + \frac{\lambda_2 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (I^{\alpha_2}_\theta \gamma)(y) \right] \\
    &= \frac{\lambda_1 y^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I^{\alpha_1}_\theta \phi)(x) + \frac{\lambda_2 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (I^{\alpha_2}_\theta \gamma)(y).
\end{align*}
\]

Hence, \(u_0(x, y) = \frac{\lambda_1 y^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I^{\alpha_1}_\theta \phi)(x) + \frac{\lambda_2 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (I^{\alpha_2}_\theta \gamma)(y) + (I^{\alpha}_\theta f)(x, y)\). This shows that \(u_0\) is a solution for the fractional integral equation (2.2). Now, let \(u_0\) be a solution for the fractional integral equation (2.2). Then, \(I^{1-\alpha}_\theta u_0)(x, y) =
\]

\[
\begin{align*}
    I^{1-\alpha}_\theta \left[ \frac{\lambda_1 y^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I^{\alpha_1}_\theta \phi)(x) + \frac{\lambda_2 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (I^{\alpha_2}_\theta \gamma)(y) \right] (x, y) + (I^{1}_\theta f)(x, y).
\end{align*}
\]

On the other hand by using \(B(z, w) = \int_0^1 (1 - x)^{w-1} x^{z-1} dx = \frac{\Gamma(z)}{\Gamma(z + w)}\), we get

\[
\begin{align*}
    I^{1-\alpha}_\theta \left[ \frac{\lambda_2 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (I^{\alpha_2}_\theta \gamma)(y) \right](x, y) \\
    &= \lambda_2 \int_0^x \int_0^y (x - s)^{-\alpha_1}(y - t)^{-\alpha_2} \left( t^{-\alpha_1 - 1} \Gamma(\alpha_1)(I^{\alpha_2}_\theta \gamma)(t) \right) dt \, ds \\
    &= \frac{\lambda_2}{\Gamma(\alpha_1) \Gamma(1 - \alpha_1) \Gamma(1 - \alpha_2)} \int_0^x \int_0^y (x - s)^{-\alpha_1} s^{\alpha_1 - 1} (y - t)^{-\alpha_2} (I^{\alpha_2}_\theta \gamma)(t) dt \, ds \\
    &= \frac{\lambda_2}{\Gamma(\alpha_1) \Gamma(1 - \alpha_1) \Gamma(1 - \alpha_2)} \int_0^x (x - s)^{-\alpha_1} s^{\alpha_1 - 1} \left( \frac{1}{\Gamma(1 - \alpha_2)} \int_0^y (y - t)^{-\alpha_2} (I^{\alpha_2}_\theta \gamma)(t) dt \right) ds \\
    &= \frac{\lambda_2}{\Gamma(1)} (I^{1}_\theta \gamma)(y) = \lambda_2 (\gamma(y) - \gamma(0)) = \lambda_2 \gamma(y)
\end{align*}
\]

and similarly \(I^{1-\alpha}_\theta \left[ \frac{\lambda_1 y^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I^{\alpha_1}_\theta \phi)(x) \right](x, y) = \lambda_1 \phi(x)\). Thus,

\[
(2.3) \quad (I^{1-\alpha}_\theta u_0)(x, y) = \lambda_2 \gamma(y) + \lambda_1 \phi(x) + (I^{\alpha}_\theta f)(x, y).
\]
By applying the operator $D^2_{xy}$ on both sides of (2.3), we obtain
\[
D^2_{xy} \left[ (I_0^{-\alpha} u_0)(x, y) \right] = D^2_{xy} \left[ \lambda_2 \gamma(y) + \lambda_1 \phi(x) + (I_0^\gamma f)(x, y) \right]
\]
and so $(D^2_{xy} u_0)(x, y) = f(x, y)$. By using (2.3), we get
\[
(I_0^{-\alpha} u_0)(x, 0) = \lambda_2 \gamma(0) + \lambda_1 \phi(x) + (I_0^\gamma f)(x, 0) = \lambda_1 \phi(x)
\]
and $(I_0^{-\alpha} u_0)(0, y) = \lambda_2 \gamma(y) + \lambda_1 \phi(0) + (I_0^\gamma f)(0, y) = \lambda_2 \gamma(y)$. This completes the proof.

Consider the Banach space $X = C(J_a \times J_b, \mathbb{R})$ endowed with the norm $\|u\| = \sup_{(x, y) \in J_a \times J_b} |u(t)|$. For $u \in X$, define the set of selections of $F$ by
\[
S_{F, u} := \{ v \in L^1(J_a \times J_b, \mathbb{R}) : v(x, y) \in F(x, y, u(x, y)) \text{ for almost all } (x, y) \in J_a \times J_b \}.
\]

It has been proved that $S_{F, u} \neq \emptyset$ for all $u \in C(J_a \times J_b, X)$ (2.2). We say that $u \in X$ is a solution for the boundary value problem (2.1)-(2.2) whenever it satisfies the boundary value conditions (2.2) and also there is a function $v \in L^1(J_a \times J_b, \mathbb{R})$ such that $v(x, y) \in F(x, y, u(x, y))$ for all $(x, y) \in J_a \times J_b$ and
\[
u(x, y) = \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)} (I_{\alpha} \gamma)(y) + \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)} (I_{\alpha} \phi)(x) + \frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1} (y-t)^{\alpha_2-1} v(s, t) dt ds
\]
for almost all $(x, y) \in J_a \times J_b$. Define the multifunction $\mathcal{N} : X \to \mathcal{P}(X)$ by
\[
\mathcal{N}(u) = \{ h \in X : h(x, y) = \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)} (I_{\alpha} \gamma)(y) + \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)} (I_{\alpha} \phi)(x) + \frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1} (y-t)^{\alpha_2-1} v(s, t) dt ds \text{ for all } (x, y) \in J_a \times J_b \}.
\]
Here, we provide our main result.

**Theorem 2.2.** Suppose that $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing upper semi-continuous map such that $\liminf_{t \to \infty} (t - \psi(t)) > 0$ and $\psi(t) < t$ for all $t > 0$, $F : J_a \times J_b \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is an integrable bounded multifunction such that $F(\cdot, \cdot, u) : J_a \times J_b \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for all $u \in \mathbb{R}$. Assume that there exists $m \in C(J_a \times J_b, [0, \infty))$ such that
\[
H_d(F(x, y, u) - F(x, y, u')) \leq \frac{1}{\Lambda_1} m(x, y) \psi(|u - u'|)
\]
for all $(x, y) \in J_a \times J_b$ and $u, u' \in \mathbb{R}$, where $\Lambda_1 = \sup \{ \frac{m(x, y) \psi(|u - u'|)}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)} \}$. If the multifunction $\mathcal{N}$ has the approximate endpoint property, then the fractional partial differential inclusion problem (2.1)-(2.2) has a solution.
Proof. First, we prove that the multifunction $\mathcal{N}$ has at least one endpoint. Let $u \in X$. Since the multivalued map $(x, y) \mapsto F(x, y, u(x, y))$ is measurable and is closed-value, it has measurable selection and so $S_{F, u}$ is nonempty. Let $\{p_n\}_{n \geq 1}$ be a sequence in $\mathcal{N}(u)$ with $p_n \to p$. For each $n$, choose $v_n \in S_{F, u_n}$ such that

$$p_n(x, y) = \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_{\theta}^{\alpha_2}(y)) + \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_{\theta}^{\alpha_1}(x)) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v_n(s, t)dt\,ds$$

for all $(x, y) \in J_a \times J_b$. Since the operator $F$ is compact, the sequence $\{v_n\}_{n \geq 1}$ has a subsequence converging to some $v \in L^1(J_a \times J_b)$. We denote this subsequence again by $\{v_n\}_{n \geq 1}$. It is easy to see that $v \in S_{F, u}$ and

$$p(x, y) = \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_{\theta}^{\alpha_2}(y)) + \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_{\theta}^{\alpha_1}(x)) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v(s, t)dt\,ds$$

for all $(x, y) \in J_a \times J_b$. This shows that $p \in \mathcal{N}(u)$ and so $\mathcal{N}(u)$ is closed. Note that, $\mathcal{N}(u)$ is bounded because $F$ has a compact values. Now, we show that $H_d(\mathcal{N}(u), \mathcal{N}(w)) \leq \psi(||u - w||)$ for all $u, w \in X$. Let $u, w \in X$ and $h_1 \in \mathcal{N}(w)$. Choose $v_1 \in S_{F, w}$ such that

$$h_1(x, y) = \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_{\theta}^{\alpha_2}(y)) + \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_{\theta}^{\alpha_1}(x)) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v_1(s, t)dt\,ds$$

for almost all $(x, y) \in J_a \times J_b$. By using the hypothesis, we have

$$H_d(F(x, y, u(x, y)) - F(x, y, w(x, y))) \leq \frac{1}{\Lambda_1} m(x, y)\psi(||u(x, y) - w(x, y)||)$$

and so we can choose $z \in F(x, y, u(x, y))$ such that

$$|v_1(x, y) - z| \leq \frac{1}{\Lambda_1} m(x, y)\psi(||u(x, y) - w(x, y)||).$$

Define the multivalued map $U : J_a \times J_b \to \mathcal{P}(\mathbb{R})$ by

$$U(x, y) = \{ z \in \mathbb{R} : |v_1(x, y) - z| \leq \frac{1}{\Lambda_1} m(x, y)\psi(||u(x, y) - w(x, y)||) \}.$$ 

Since $v_1$ and $\eta = \frac{1}{\Lambda_1} m\psi(||u - w||)$ are measurable, the multifunction $U(\cdot, \cdot) \cap F(\cdot, \cdot, u(\cdot, \cdot))$ is measurable. Hence, there exists $v_2(x, y) \in F(x, y, u(x, y))$ such
that $|v_1(x, y) - v_2(x, y)| \leq \frac{1}{\Lambda_1} m(x, y) \psi(|u(x, y) - w(x, y)|)$. Now, consider the element $h_2 \in \mathcal{N}(u)$ defined by

$$h_2(x, y) = \frac{\lambda_2 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (I_{\theta}^{\alpha_2} \hat{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I_{\theta}^{\alpha_1} \hat{\varphi})(x)$$

$$+ \frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1} v_1(s, t) dt ds$$

for all $(x, y) \in J_a \times J_b$. Put $\sup_{(x,y) \in J_a \times J_b} |m(x, y)| = \|m\|$. Then, we have

$$|h_1(x, y) - h_2(x, y)| \leq \frac{\lambda_2 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (I_{\theta}^{\alpha_2} \hat{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I_{\theta}^{\alpha_1} \hat{\varphi})(x)$$

$$+ \frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1} v_1(s, t) dt ds$$

$$- \frac{\lambda_2 x^{\alpha_1 - 1}}{\Gamma(\alpha_1)} (I_{\theta}^{\alpha_2} \hat{\gamma})(y) - \frac{\lambda_1 y^{\alpha_2 - 1}}{\Gamma(\alpha_2)} (I_{\theta}^{\alpha_1} \hat{\varphi})(x)$$

$$- \frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1} v_2(s, t) dt ds$$

$$\leq \frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1} |v_1(s, t) - v_2(s, t)| dt ds$$

$$\leq \frac{1}{\Lambda_1} \|m\| \psi(\|u - w\|) \left\{ \frac{x^{\alpha_1} y^{\alpha_2}}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)} \right\}$$

$$= \frac{\Lambda}{\Lambda_1} \psi(\|u - w\|) = \psi(\|u - w\|)$$

and so $\|h_1 - h_2\| = \sup_{(x,y) \in J_a \times J_b} |h_1(x, y) - h_2(x, y)| = \psi(\|u - w\|)$. Hence, $H_d(\mathcal{N}(u), \mathcal{N}(w)) \leq \psi(\|u - w\|)$ for all $u, w \in X$. Since the multifunction $\mathcal{N}$ has approximate endpoint property by using Theorem 1.1, there exists $u^* \in X$ such that $\mathcal{N}(u^*) = \{u^*\}$. It is easy to check that $u^*$ is a solution for the fractional partial differential inclusion problem (1.2). \hfill \Box

For illustration of our main result, we give the following example.

**Example 2.3.** Consider the fractional partial differential inclusion

$$D_{\theta}^\alpha u(x, y) \in \left[0, \frac{0.03xy|\sin u(x, y)|}{1 + |\sin u(x, y)|}\right]$$

with boundary value conditions $(I_{\theta}^{1-\alpha} u)(x, 0) = 0.1(e^x - 1)$ and $(I_{\theta}^{1-\alpha} u)(0, y) = 0.01y^2$, where $(x, y) \in [0, 1] \times [0, 1]$. Let $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 \in (0, 1]$, $\lambda_1 = 0.1$ and $\lambda_2 = 0.01$. Define the multifunction $F : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ by $F(x, y, z) = \left[0, \frac{0.03xy|\sin z(t)|}{1 + |\sin z(t)|}\right]$. If $m : [0, 1] \times [0, 1] \to [0, \infty)$ is defined by
\[ m(x, y) = \frac{3}{100}xy, \text{ then } \|m\| = \frac{3}{100}. \]
Consider the map \( \psi(t) = \frac{t}{2} \). It is clear that 
\( \psi \) is nondecreasing, upper semi-continuous on \((0, 1]\), \( \liminf_{t \to \infty} (t - \psi(t)) > 0 \)
and \( \psi(t) < t \) for all \( t > 0 \). Since \( \Gamma(\alpha_i + 1) < \frac{1}{2} \), we get
\[ \Lambda_1 = \|m\| \left\{ \frac{a^{\alpha_1}b^{\alpha_2}}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)} \right\} = \frac{3}{100} \left( \frac{1}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)} \right) < 0.12. \]

One can easily check that
\[ H_d(F(x, y, u_1) - F(x, y, u_2)) \leq \frac{1}{\Lambda_1} m(x, y) \psi \left( |u_1 - u_2| \right). \]

Put \( X = C_\mathbb{R}([0, 1] \times [0, 1]) \). Define \( \mathcal{N} : X \to P(X) \) by
\[ \mathcal{N}(u) = \{ h \in X : \text{ there exists } v \in S_{F, u} \text{ such that } h(x, y) = w(x, y) \text{ for all } (x, y) \in [0, 1] \times [0, 1] \}, \]
where
\[ w(x, y) = \frac{0.01x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_{\theta}^{\alpha_2}y)(y) + \frac{0.1y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_{\theta}^{\alpha_1}y)(x) \]
\[ + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v(s, t)dtds. \]

Since \( \sup_{u \in \mathcal{N}(0)} \|u\| = 0, \inf_{u \in X} \sup_{v \in \mathcal{N}(u)} \|u - s\| = 0 \) and so \( \mathcal{N} \) has the approximate endpoint property. Now by using Theorem 2.2, we conclude that the above fractional partial differential inclusion problem has a solution.

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**References**


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