On groupoids with involutions and their cohomology

El-kaïoum M. Moutuou

Abstract. We extend the definitions and main properties of graded extensions to the category of locally compact groupoids endowed with involutions. We introduce Real Čech cohomology, which is an equivariant-like cohomology theory suitable for the context of groupoids with involutions. The Picard group of such a groupoid is discussed and is given a cohomological picture. Eventually, we generalize Crainic’s result, about the differential cohomology of a proper Lie groupoid with coefficients in a given representation, to the topological case.

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0. Introduction

A Real\(^1\) object in a category \(\mathcal{C}\) is a pair \((A,f)\) consisting of an object \(A \in \text{Ob}(\mathcal{C})\) together with an element \(f \in \text{Isom}_\mathcal{C}(A,A)\), called the Real structure, such that \(f^2 = 1_A\). For instance, an Atiyah Real space \((X,\tau)\) \([2]\) is nothing but a Real object in the category of locally compact spaces. We are particularly interested in the category \(\mathfrak{G}_s\) \([25]\) of locally compact Hausdorff groupoids with strict homomorphisms \([15, 16]\) as morphisms; we shall refer to Real objects in \(\mathfrak{G}_s\) as Real groupoids. For example, let \(\mathbb{W}P^n(a_1, \ldots, a_n)\) be the weighted projective orbifold \([1]\) associated to the pairwise coprime integers \(a_1, \ldots, a_n\); then together with the coordinate-wise complex conjugation, \(\mathbb{W}P^n(a_1, \ldots, a_n)\) is a Real groupoid.

A morphism of Real groupoids is a morphism in \(\mathfrak{G}_s\) intertwining the Real structures. We may also speak of a Real strict homomorphism. Real groupoids form a category \(\mathfrak{RG}_s\) in which morphisms are Real strict homomorphisms. Moreover, they are the objects of a 2-category \(\mathfrak{RG}_s(2)\) defined as follows. Let \((G,\rho),(\Gamma,\varrho)\) \(\in\) Ob(\(\mathfrak{RG}_s\)). A generalized homomorphism \([7, 9, 16, 25]\) \(\Gamma \rightarrow G\) is said to be Real if \(Z\) is given a Real structure \(\tau\) such that the moment maps and the groupoid actions respect some coherent compatibility conditions with respect to the Real structures. A morphism of Real generalized homomorphisms \((Z,\tau) \rightarrow (Z',\tau')\) is a morphism of generalized homomorphisms \(Z \rightarrow Z'\) intertwining the Real structures. Henceforth, 1-morphisms in \(\mathfrak{RG}(2)\) are Real generalized homomorphisms and 2-morphisms are morphisms of Real generalized homomorphisms. All functorial properties we deal with in this paper are however discussed in the category \(\mathfrak{RG}\) defined as \(\mathfrak{RG}(2) \text{ "up to 2-isomorphisms"}\).

In \([21]\), a Čech cohomology theory for topological groupoids is defined as the Čech cohomology of simplicial topological spaces, and it is shown that the well-known isomorphism between \(S^1\)-central extensions of a discrete groupoid \(G\) and the second cohomology group \([19, 11]\) of \(G\) with coefficients in the sheaf of germs of \(S^1\)-valued functions also holds in the general case; \(i.e.,\), \(\text{Ext}(G,S^1) \cong H^2(G,\mathcal{S})\). We define here an analogous theory \(\check{H}R^*\)

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\(^1\)Note the capitalization, used to avoid confusion with a module over \(\mathbb{R}\) or a real manifold.
that fits well the context of Real groupoids. This theory was motivated by the classification of groupoid C*-dynamical systems endowed with involutions [17]. These can be thought of as a generalization of continuous-trace C*-algebras with involutions. Specifically, it is known [20] that given such a C*-algebra A, its spectrum X admits a Real structure τ, and its Dixmier–Douady invariant δ(A) ∈ ˇH^2(X,S^1) is such that δ(A) = τ^*δ(A), where the "bar" is the complex conjugation in S^1. In fact, thinking of X as a Real groupoid, we will see that all 2-cocycles satisfying the latter relation are classified by ˇH^2 R(X,S^1), where S^1 is endowed with the complex conjugation. ˇH^2 R appears then to provide the right cohomological interpretation of C*-dynamical systems with involutions.

We try, to the extent possible, to make the present paper self-contained. We start by collecting, in Section 1, a number of notions and results about Real groupoids most of which are adapted from many sources in the literature [15, 19, 25]; specifically, we define the group ˇExt_R(G,S) of (equivalence classes of) Real graded S-central extensions over a Real groupoid G, by a given Real abelian group S. In Section 2, we introduce Real Čech cohomology, following closely [21]. While ˇH^2 R behaves almost like a Z_2-equivariant cohomology theory, we will see that it is actually not. Geometric interpretations of the cohomology groups ˇH^1 R(G,S) and ˇH^2 R(G,S), for a Real Abelian group S, are given. Finally, we generalize a result by Crainic [4] (on the differential cohomology groups of a proper Lie groupoid) to topological proper (Real) groupoid.

1. Real groupoids and Real graded extensions

Recall [19, 16, 25] that a strict homomorphism between two groupoids \( \mathcal{G} \rightarrow X \) and \( \Gamma \rightarrow Y \) is a functor \( \varphi : \Gamma \rightarrow \mathcal{G} \) given by a map \( Y \rightarrow X \) on objects and a map \( \Gamma^{(1)} \rightarrow \mathcal{G}^{(1)} \) on arrows, both denoted again by \( \varphi \), which preserve the groupoid structure maps, i.e., \( \varphi(s(\gamma)) = s(\varphi(\gamma)), \varphi(r(\gamma)) = r(\varphi(\gamma)), \varphi(1_\gamma) = 1_{\varphi(\gamma)} \) and \( \varphi(\gamma_1 \gamma_2) = \varphi(\gamma_1) \varphi(\gamma_2) \) (hence \( \varphi(\gamma^{-1}) = \varphi(\gamma)^{-1} \)), for all \( (\gamma_1, \gamma_2) \in \Gamma^{(2)} \) and \( y \in Y \). Unless otherwise specified, all our groupoids are topological groupoids which are supposed to be Hausdorff and locally compact.

1.1. Real groupoids.

Definition 1.1. A Real groupoid is a groupoid \( \mathcal{G} \rightarrow X \) together with a strict 2-periodic homeomorphism \( \rho : \mathcal{G} \rightarrow \mathcal{G} \). The homeomorphism \( \rho \) is called a Real structure on \( \mathcal{G} \). Such a groupoid will be denoted by a pair \( (\mathcal{G}, \rho) \).

Example 1.2. Any topological Real space \( (X, \rho) \) in the sense of Atiyah [2] can be viewed as a Real groupoid whose the unit space and the space of morphisms are identified with \( X \); i.e., the operations in this Real groupoid is defined by \( s(x) = r(x) = x, x \cdot x = x, x^{-1} = x \).
**Example 1.3.** Any group with involution can be viewed as a Real groupoid with unit space identified with the unit element. Such a group will be called Real.

**Lemma 1.4.** Let $G$ be an abelian group equipped with an involution $\tau: G \rightarrow G$ (i.e., a Real structure). Set
\[
\mathcal{R}(\tau) := \{ g \in G \mid \tau(g) = g \} = \mathbb{R} G, \quad \mathcal{S}(\tau) := \{ g \in G \mid \tau(g) = -g \}.
\]
Then,
\[
G \otimes \mathbb{Z} \left[ \frac{1}{2} \right] \cong (\mathcal{R}(\tau) \oplus \mathcal{S}(\tau)) \otimes \mathbb{Z} \left[ \frac{1}{2} \right].
\]
If $\tau$ is understood, we will write $G$ for $\mathcal{S}(\tau)$. We call $\mathcal{R}(\tau)$ and $\mathcal{S}(\tau)$ the Real part and the imaginary part of $G$, respectively.

**Proof.** For all $g \in G$, one has $g + \tau(g) \in \mathbb{R} G$, and $g - \tau(g) \in \mathbb{R} G$. Therefore, after tensoring $G$ with $\mathbb{Z}[1/2]$, every $g \in G$ admits a unique decomposition
\[
g = \frac{g + \tau(g)}{2} + \frac{g - \tau(g)}{2} \in \mathbb{Z}[1/2] \otimes \left( \mathbb{R} G \oplus \mathbb{R} G \right). \quad \Box
\]

**Example 1.5.** Let $n \in \mathbb{N}^*$. Suppose $\rho$ is a Real structure on the additive group $\mathbb{R}^n$. Then there exists a unique decomposition $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$ such that $\rho$ is determined by the formula
\[
\rho(x, y) = (1_p \oplus (-1_q))(x, y) := (x, -y),
\]
for all $(x, y) = (x_1, \ldots, x_p, y_1, \ldots, y_q) \in \mathbb{R}^p \oplus \mathbb{R}^q$.

For each pair $(p, q) \in \mathbb{N}$, we will write $\mathbb{R}^{p,q}$ for the additive group $\mathbb{R}^{p+q}$ equipped with the Real structure $(1_p \oplus (-1_q))$.

Define the Real space $\mathbb{S}^{p,q}$ as the invariant subset of $\mathbb{R}^{p,q}$ consisting of elements $u \in \mathbb{R}^{p+q}$ of norm 1. For $q = p$, $\mathbb{S}^{p,p}$ is clearly identified with the Real space $\mathbb{S}^p$ whose Real structure is given by the coordinate-wise complex conjugation. Notice that $\mathbb{S}^{p,q} = \mathbb{S}^{0,0}$.

**Example 1.6.** Let $(X, \rho)$ be a topological Real space. Consider the fundamental groupoid $\pi_1(X)$ over $X$ whose arrows from $x \in X$ to $y \in X$ are homotopy classes of paths (relative to end-points) from $x$ to $y$ and the partial multiplication given by the concatenation of paths. The involution $\rho$ induces a Real structure on the groupoid as follows: if $[\gamma] \in \pi_1(X)$, we set $\rho([\gamma])$ the homotopy classes of the path $\rho(\gamma)$ defined by $\rho(\gamma)(t) := \rho(\gamma(t))$ for $t \in [0, 1]$.\[\text{Two Real structures } \rho \text{ and } \rho' \text{ on } \mathcal{G} \text{ are said to be conjugate if there exists a strict homeomorphism } \phi: \mathcal{G} \rightarrow \mathcal{G} \text{ such that } \rho' = \phi \circ \rho \circ \phi^{-1}. \text{ In this case we say that the Real groupoids } (\mathcal{G}, \rho) \text{ and } (\mathcal{G}, \rho') \text{ are equivalent.}
\]

**Definition 1.7.** We write $\mathcal{G} \xrightarrow{\rho} X$ (or $\rho\mathcal{G}$ when there is a risk of confusion) for the the subgroupoid of $\mathcal{G} \xrightarrow{} X$ by $\rho$.\]
Lemma 1.8. Let $\mathcal{G}$ and $\Gamma$ be Real groupoids, and let $\phi: \Gamma \rightarrow \mathcal{G}$ be a Real groupoid homomorphism, then $\phi(\Gamma)$ is a full subgroupoid of $^r\mathcal{G} \xrightarrow{\sim} ^rX$. If in addition $\phi$ is an isomorphism, then $^r\Gamma \cong ^r\mathcal{G} \xrightarrow{\sim} ^rX$.

In particular, if $\rho_1$ and $\rho_2$ are two conjugate Real structures on $\mathcal{G}$, then $^r\rho_1 \mathcal{G} \cong ^r\rho_2 \mathcal{G}$.

Proof. This is obvious since $\phi(\gamma) = \overline{\phi(\gamma)}$ for all $\gamma \in \Gamma$. \qed

Remark 1.9. Note that the converse of the second statement of the above lemma is false in general. For instance, consider the Real group $\mathbb{S}^1$ whose Real structure is given by the complex conjugation, and the Real group $\mathbb{Z}_2$ (with the trivial Real structure). We have $^r\mathbb{S}^1 = \{\pm 1\} \cong \mathbb{Z}_2 = ^r\mathbb{Z}_2$.

The following is an example of groupoids with equivalent Real structures.

Example 1.10. Recall ([8, IV.3]) that a Riemannian manifold $X$ is called globally symmetric if each point $x \in X$ is an isolated fixed point of an involutory isometry $s_x : X \rightarrow X$; i.e., $s_x$ is a diffeomorphism verifying $s_x^2 = \text{Id}_X$ and $s_x(x) = x$. Moreover, for every two points $x, y \in X$, $s_x$ and $s_y$ are related through the formula $s_x \circ s_y \circ s_x = s_{s_x(y)}$. Given such a space, each point $x \in X$ defines a Real structure on $X$ which leaves $x$ fixed. However, let $x$ and $y$ be two different points in $X$ and let $z \in X$ be such that $y = s_z(x)$. Then, we get $s_z \circ s_x \circ s_z = s_y$ which means that the diffeomorphism $s_z : X \rightarrow X$ implements an equivalence $s_x \sim s_y$. But since $x$ and $y$ are arbitrary, it turns out that all of the Real structures $s_x$ are equivalent. Thus, all of the Real spaces $(X, s_x)$ are equivalent to each others.

Now, recall [8, IV. Theorem 3.3] that if $G$ denotes the identity component of $I(X)$, where the latter is the group of isometries on $X$, then the map $\sigma_{x_0} : g \mapsto s_{x_0}gs_{x_0}$ is an involutory automorphism in $G$, for any arbitrary $x_0 \in X$. It follows that all of the points of $X$ give rise to equivalent Real groups $(G, \sigma_x)$.

From now on, by a Real structure on a groupoid, we will mean a representative of a conjugation class of Real structures. Moreover, we will sometimes put $\bar{g} := \rho(g)$, and write $\mathcal{G}$ instead of $(\mathcal{G}, \rho)$ when $\rho$ is understood.

Definition 1.11 (Real covers). Let $(X, \rho)$ be a Real space. We say that an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $X$ is Real if $\mathcal{U}$ is invariant with respect to the Real structure $\rho$; i.e., $\rho(U_i) \in \mathcal{U}, \forall i \in I$. Alternatively, $\mathcal{U}$ is Real if $I$ is equipped with an involution $i \mapsto i'$ such that $U_i = \rho(U_{i'})$ for all $i \in I$.

Remark 1.12. Observe that Real open covers always exist for all locally compact Real space $X$. Indeed, let $\mathcal{V} = \{V_{i'}\}_{i' \in I'}$ be an open cover of the space $X$. Let $I := I' \times \{\pm 1\}$ be endowed with the involution $(i', \pm 1) \mapsto (i', \mp 1)$. Next, put $U_{(i', \pm 1)} := \rho^{(\pm 1)}(V_{i'})$, where $\rho^{(\pm 1)}(g) := g$, and $\rho^{-1}(g) := \rho(g)$ for $g \in \mathcal{G}$. 

Definition 1.13 (Real action). Let \((Z, \tau)\) be a locally compact Hausdorff Real space. A (continuous) right Real action of \((\mathcal{G}, \rho)\) on \((Z, \tau)\) is given by a continuous open map \(s : Z \to X\) (called the \textit{generalized source map}) and a continuous map \(Z \times_{s, X, r} \mathcal{G} \to Z\), denoted by \((z, g) \mapsto zg\), such that:

(a) \(\tau(zg) = \tau(z)\rho(g)\) for all \((z, g) \in Z \times_{s, X, r} \mathcal{G}\).

(b) \(\rho(s(z)) = s(\tau(z))\) for all \(z \in Z\).

(c) \(s(zg) = s(g)\).

(d) \(z(gh) = (zg)h\) for \((z, g) \in Z \times_{s, X, r} \mathcal{G}\) and \((g, h) \in \mathcal{G}^{(2)}\).

(e) \(zs(z) = z\) for any \(z \in Z\) where we identify \(s(z)\) with its image in \(\mathcal{G}\) by the inclusion \(X \hookrightarrow \mathcal{G}\).

If such a Real action is given, we say that \((Z, \tau)\) is a (right) Real \(\mathcal{G}\)-space.

Likewise a (continuous) left Real action of \((\mathcal{G}, \rho)\) on \((Z, \tau)\) is determined by a continuous Real open surjection \(\tau : Z \to X\) (the \textit{generalized range map} of the action) and a continuous Real map \(\mathcal{G} \times_{s, X, r} Z \to Z\) satisfying the appropriate analogues of conditions (a), (b), (c), (d) and (e) above.

Given a right Real action of \((\mathcal{G}, \rho)\) on \((Z, \tau)\) with respect to \(s\), let \(\Psi : Z \times_{s, X, r} \mathcal{G} \to Z \times Z\) be defined by the formula \(\Psi(z, g) = (z, zg)\). Then we say that the action is \textit{free} if this map is one-to-one (or in other words if the equation \(zg = z\) implies \(g = s(z)\)). The action is called \textit{proper} if \(\Psi\) is proper.

Notations 1.14. If we are given such a right (resp. left) Real action of \((\mathcal{G}, \rho)\) on \((Z, \tau)\), and if there is no risk of confusion, we will write \(Z * \mathcal{G}\) (resp. \(\mathcal{G} * Z\)) for \(Z \times_{s, X, r} \mathcal{G}\) (resp. for \(\mathcal{G} \times_{s, X, r} Z\)).

1.2. Real \(\mathcal{G}\)-bundles.

Definition 1.15. Let \((\mathcal{G}, \rho)\) be a Real groupoid. A Real (right) \(\mathcal{G}\)-bundle over a Real space \((Y, \varrho)\) is a Real (right) \(\mathcal{G}\)-space \((Z, \tau)\) with respect to a map \(s : Z \to X\), together with a Real map \(\pi : Z \to Y\) (the \textit{generalized range map} of the action) and a continuous Real map \(\mathcal{G} \times_{s, X, r} Z \to Z\) satisfying the relation \(\pi(zg) = \pi(z)\) for any \((z, g) \in Z \times_{s, X, r} \mathcal{G}\), and such that for any \(y \in Y\), the induced map \(\tau_y : Z_y \to Z_{\varrho(y)}\) on the fibres is \(\mathcal{G}\)-antilinear in the sense that for \((z, g) \in Z_y \times_{s, X, r} \mathcal{G}\) we have

\[\tau_y(zg) = \tau_y(z)\rho(g)\]

as an element in \(Z_{\varrho(y)}\).

Such a bundle \((Z, \tau)\) is said to be \textit{principal} if:

(i) \(\pi : Z \to Y\) is \textit{locally split} (i.e., it is surjective and admits local sections).

(ii) The map \(Z \times_{s, X, r} \mathcal{G} \to Z \times Y Z, (z, g) \mapsto (z, zg)\) is a Real homeomorphism.
Remarks 1.16.

1. **The unit bundle.** Given a Real groupoid \((\mathcal{G}, \rho)\), its space of arrows \(\mathcal{G}^{(1)}\) is a \(\mathcal{G}\)-principal Real bundle over \(X\). Indeed, the projection is the range map \(r: \mathcal{G}^{(1)} \rightarrow X\), the generalized source map is given by \(s\), and the action is just the partial multiplication on \(\mathcal{G}\). This bundle is denoted by \(U(\mathcal{G})\) and is called the *unit* bundle of \(\mathcal{G}\) (see [16]).

2. **Pull-back.** Let \(Z \xrightarrow{s} X\) be a \(\mathcal{G}\)-principal Real bundle and \(f: Y' \rightarrow Y\) be a Real continuous map. Then the pull-back \(f^*Z := Y' \times_Y Z\) equipped with the involution \((g', \tau)\) has the structure of a \(\mathcal{G}\)-principal Real bundle over \(Y'\). Indeed, the right Real \(\mathcal{G}\)-action is given by the \(\mathcal{G}\)-action on \(Z\) and the generalized source map is \(s'(y', z) := s(z)\).

3. **Trivial bundles.** From the previous two remarks, we see that if \((Z, \tau)\) is any Real space together with a Real map \(\varphi: Z \rightarrow X\), then we get a \(\mathcal{G}\)-principal Real bundle \(\varphi^*U(\mathcal{G})\) over \(Z\); its total space being the space \(Z \times_{\mathcal{G}, \tau_X, \tau} \mathcal{G}\). A Bundle of this form is called *trivial* while a \(\mathcal{G}\)-principal Real bundle which is locally of this form is called *locally trivial*.

1.3. Generalized morphisms of Real groupoids.

**Definition 1.17.** A generalized morphism from a Real groupoid \((\Gamma, \varrho)\) to a Real groupoid \((\mathcal{G}, \rho)\) consists of a Real space \((Z, \tau)\), two maps \(Y \xleftarrow{\tau} Z \xrightarrow{s} X\), a left (Real) action of \(\Gamma\) with respect to \(\tau\), a right (Real) action of \(\mathcal{G}\) with respect to \(s\), such that:

(i) The actions commute, i.e., if \((z, g) \in Z \times_{s, X, \tau} \mathcal{G}\) and \((\gamma, z) \in \Gamma \times_{s, Y, \tau} Z\) we must have \(s(\gamma z) = s(z)\), \(\tau(\gamma g) = \tau(z)\) so that \(\gamma (\gamma z) = (\gamma z) g\).

(ii) The maps \(s\) and \(\tau\) are Real in the sense that \(s(\tau(z)) = \rho(s(z))\) and \(\tau(\tau(z)) = g(\tau(z))\) for any \(z \in Z\).

(iii) \(\tau: Z \rightarrow Y\) is a locally trivial \(\mathcal{G}\)-principal Real bundle.

**Example 1.18.** Let \(f: \Gamma \rightarrow \mathcal{G}\) be a Real strict morphism. Let us consider the fibre product \(Z_f := Y \times_{f, X, \varrho} \mathcal{G}\) and the maps \(\tau: Z_f \rightarrow Y\), \((y, g) \mapsto y\) and \(s: Z_f \rightarrow X\), \((y, g) \mapsto s(g)\). For \((\gamma, (y, g)) \in \Gamma \times_{s, Y, \tau} Z_f\), we set \(\gamma.(y, g) := (\tau(\gamma), f(\gamma) g)\) and for \(((y, g), g') \in Z_f \times_{s, X, \rho} \mathcal{G}\) we set \((y, g).g' := (y, gg')\). Using the definition of a strict morphism, it is easy to check that these maps are well-defined and make \(Z_f\) into a generalized morphism from \(\Gamma\) to \(\mathcal{G}\). Furthermore, the map \(\tau\) on \(Z_f\) defined by \(\tau(y, g) := (\varrho(y), \rho(g))\) is a Real involution and then \(Z_f\) is a Real generalized morphism.
Remark 1.22. It is easy to check that the composition of Real generalized homomorphisms is associative. For instance, if \( s = s' \circ \varphi \) and \( r = \tau' \circ \varphi \). We say that the Real generalized homomorphism \((Z, \tau)\) and \((Z', \tau')\) are isomorphic if there exists such a \( \varphi \) which is at the same time a homeomorphism.

Compositions of Real generalized morphisms are defined by the following proposition.

**Proposition 1.20.** Let \((Z', \tau')\) and \((Z'', \tau'')\) be Real generalized homomorphisms from \((\Gamma, \varrho)\) to \((\mathcal{S}', \rho')\) and from \((\mathcal{S}', \rho')\) to \((\mathcal{S}, \rho)\) respectively. Then \[ Z = Z' \times g' Z'' := (Z' \times g', \mathcal{G}(0), \mathcal{G}) \mathcal{Z}'' / (z', z'') \sim (z''g', g'\gamma^{-1}z'') \]
with the obvious Real involution, defines a Real generalized morphism from \( \Gamma \longrightarrow \mathcal{Y} \to \mathcal{S} \longrightarrow X \).

**Proof.** Let us first describe the structure maps

\[ Y \xleftarrow{\tau} Z \xrightarrow{s} X \]

and the actions.

For \((z', z'') \in Z\) we set \( \tau(z', z'') := \tau'(z') \) and \( s(z', z'') := s''(z'') \). These are well-defined and since

\[
\begin{align*}
\tau(z'g', g'^{-1}z'') &= \tau'(z'g') = s'(z'), \\
s(z'g', g'^{-1}z'') &= s''(g'^{-1}z'') = s''(z''),
\end{align*}
\]

from Definition 1.17(i). The actions are defined by \( \gamma.(z', z'') := (\gamma z', z'') \) and \( (z', z'').g := (z', z''g) \) for \((\gamma, (z', z'')) \in \Gamma \times s, Y, \mathcal{Z} \) and \(((z', z''), g) \in \mathcal{Z} \times s, X, \mathcal{S} \) while the Real involution is the obvious one:

\[ \tau(z', z'') := (\tau'(z'), \tau''(z'')). \]

Now to show the local triviality of \( Z \), notice that from (3) of Remarks 1.16, \( Z' \) and \( Z'' \) are locally of the form \( U \times \varphi, \mathcal{G}(0), \mathcal{G}' \) and \( V \times \varphi'', X, \mathcal{S} \) respectively, where \( \varphi : U \longrightarrow \mathcal{G}(0) \) and \( \varphi'' : V \longrightarrow X \) are Real continuous maps, \( U \) and \( V \) subspaces of \( Y \) and \( \mathcal{G}(0) \) respectively. It turns out that by construction, \( Z \) is locally of the form \( W \times \varphi, \mathcal{G}(0), \mathcal{S} \) where \( W = U \times \varphi', \mathcal{G}(0), \mathcal{V} \).

**Definition 1.19.** A morphism between two such morphisms \((Z, \tau)\) and \((Z', \tau')\) is a \( \Gamma \)-\( \mathcal{S} \)-equivariant Real map \( \varphi : Z \longrightarrow Z' \) such that \( s = s' \circ \varphi \) and \( r = \tau' \circ \varphi \). We say that the Real generalized homomorphism \((Z, \tau)\) and \((Z', \tau')\) are isomorphic if there exists such a \( \varphi \) which is at the same time a homeomorphism.

**Definition 1.21.** Given two Real generalized morphisms \((\Gamma, \varrho) \xrightarrow{(Z, \tau)} (\mathcal{S}', \rho')\) and \((\mathcal{S}', \rho') \xrightarrow{(Z', \tau')} (\mathcal{S}, \rho)\), we define their composition

\[ (Z' \circ Z, \tau) : (\Gamma, \varrho) \longrightarrow (\mathcal{S}, \rho) \]

to be \((Z \times g', Z', \tau \times \tau')\).

**Remark 1.22.** It is easy to check that the composition of Real generalized homomorphisms is associative. For instance, if

\[ \Gamma \xrightarrow{(Z_1, \rho_1)} \mathcal{S}_1 \xrightarrow{(Z_2, \rho_2)} \mathcal{S}_2 \xrightarrow{(Z_3, \rho_3)} \mathcal{S} \]
are given Real generalized morphisms, we get two Real generalized morphisms $Z = Z_1 \times_{g_1} (Z_2 \times_{g_2} Z_3)$ and $Z' = (Z_1 \times_{g_1} Z_2) \times_{g_2} Z_3$ between $(\Gamma, g)$ and $(\mathcal{G}, \rho)$; notice that here $Z$ and $Z'$ carry the obvious Real involutions. Moreover, the map $Z \rightarrow Z'$, $(z_1, (z_2, z_3)) \mapsto ((z_1, z_2), z_3)$ is a $\Gamma$-$\mathcal{G}$-equivariant Real homeomorphism. Hence, there exists a category $\mathfrak{RG}$ whose objects are Real locally compact groupoids and morphisms are isomorphism classes of Real generalized homomorphisms.

**Lemma 1.23.** Let $f_1, f_2 : \Gamma \rightarrow \mathcal{G}$ be two Real strict homomorphisms. Then $f_1$ and $f_2$ define isomorphic Real generalized homomorphisms if and only if there exists a Real continuous map $\varphi : Y \rightarrow \mathcal{G}$ such that $f_2(\gamma) = \varphi(r(\gamma))f_1(\gamma)\varphi(s(\gamma))^{-1}$.

**Proof.** Let $\Phi : Z_{f_1} \rightarrow Z_{f_2}$ be a Real $\Gamma$-$\mathcal{G}$-equivariant homeomorphism, where $Z_{f_i} = Y \times_{f_i, X, r} \mathcal{G}$. Then from the commutative diagrams

$$
\begin{array}{ccc}
Y & \xleftarrow{pr_1} & Z_{f_1} \\
\downarrow{\Phi} & & \downarrow{sopr_2} \\
Z_{f_2} & \xrightarrow{pr_1} & X
\end{array}
$$

we have $\Phi(x, g) = (x, h)$ with $s(g) = s(h)$; and then there exists a unique element $\varphi(x) \in \mathcal{G}$ such that $h = \varphi(x)g$. To see that this defines a continuous map $\varphi : Y \rightarrow \mathcal{G}$, notice that for any $x \in Y$, the pair $(x, f_1(x))$ is an element in $Z_{f_1}$, then $\varphi(x)$ is the unique element in $\mathcal{G}$ such that

$$
\Phi(x, f_1(x)) = (x, \varphi(x)f_1(x)).
$$

Furthermore, since $\Phi$ is Real,

$$
\Phi(g(x), \rho(f_1(x))) = (\varphi(x), \rho(\varphi(x))\rho(f_1(x))),
$$

which shows that $\varphi(g(x)) = \rho(\varphi(x))$ for any $x \in Y$; i.e., $\varphi$ is Real.

Now for $\gamma \in \Gamma$, take $x = s(\gamma)$, then from the $\Gamma$-equivariance of $\Phi$, we have

$$
\Phi(\gamma \cdot (s(\gamma), f_1(s(\gamma)))) = \Phi(r(\gamma), f_1(\gamma)) = \gamma \cdot \Phi(s(\gamma), f_1(s(\gamma)));
$$

so that

$$
(r(\gamma), \varphi(r(\gamma))f_1(\gamma)) = (r(\gamma), f_2(\gamma)\varphi(s(\gamma)))
$$

and $f_2(\gamma) \cdot r(\varphi(s(\gamma))) = \varphi(r(\gamma))f_1(\gamma)\varphi(s(\gamma))$; but $r(\varphi(s(\gamma))) = s(f_2(\gamma))$ by definition of $\varphi$ and this gives the desired relation. The converse is easy to check by working backwards. \qed

### 1.4. Morita equivalence

Let $(\Gamma, g)$ and $(\mathcal{G}, \rho)$ be two Real groupoids. Suppose that $f : (\Gamma, g) \rightarrow (\mathcal{G}, \rho)$ is an isomorphism in the category $\mathfrak{RG}_s$. In this case, we say that $(\Gamma, g)$ and $(\mathcal{G}, \rho)$ are *strictly equivalent* and we write $(\Gamma, g) \sim_{\text{strict}} (\mathcal{G}, \rho)$. Now, consider the induced Real generalized morphisms $(Z_f, \tau_f) : (\Gamma, g) \rightarrow (\mathcal{G}, \rho)$ and $(Z_{f^{-1}}, \tau_{f^{-1}}) : (\mathcal{G}, \rho) \rightarrow (\Gamma, g)$. Define the inverse of $Z_f$ by $Z_f^{-1} := \mathcal{G} \times_{r, X, f} Y$ with the obvious Real structure also
denoted by \( \tau_f \). The map \( Z_{f^{-1}} \rightarrow Z_f^{-1} \) defined by \((x, \gamma) \mapsto (f(\gamma), f^{-1}(x))\)
is clearly a \( \mathcal{G} \)-\( \Gamma \)-equivariant Real homeomorphism; hence, \((Z_{f^{-1}}, \tau_f^{-1})\) and \((Z_f^{-1}, \tau_f)\) are isomorphic Real generalized morphisms from \((\mathcal{G}, \rho)\) to \((\Gamma, \varrho)\).

Notice that \( Z_f^{-1} \) is \( Z_f \) as space; thus, \((Z_f, \tau_f)\) is at the same time a Real generalized morphism from \((\Gamma, \varrho)\) to \((\mathcal{G}, \rho)\) and from \((\mathcal{G}, \rho)\) to \((\Gamma, \varrho)\). Furthermore, it is simple to check that \( Z_f \circ Z_f^{-1} \) and \( \text{Id}_{\mathcal{G}} \) define isomorphic Real generalized morphisms from \((\mathcal{G}, \rho)\) into itself, and likewise, \( Z_f^{-1} \circ Z_f \) and \( \text{Id}_{\Gamma} \) are isomorphic Real generalized morphisms from \((\Gamma, \varrho)\) into itself.

**Definition 1.24.** Two Real groupoids \((\Gamma, \varrho)\) and \((\mathcal{G}, \rho)\) are said to be **Morita equivalent** if there exists a Real space \((Z, \tau)\) that is at the same time a Real generalized morphism from \(\Gamma\) to \(\mathcal{G}\) and from \(\mathcal{G}\) to \(\Gamma\); that is to say that \(Y \xrightarrow{f} Z\) is a \(\mathcal{G}\)-principal Real bundle and \(Z \xrightarrow{\tilde{s}} X\) is a \(\Gamma\)-principal Real bundle.

**Remark 1.25.** Given a Morita equivalence \((Z, \tau) : (\Gamma, \varrho) \rightarrow (\mathcal{G}, \rho)\), its inverse, denoted by \((Z^{-1}, \tau)\), is \((Z, \tau)\) as Real space, and if \(b : (Z, \tau) \rightarrow (Z^{-1}, \tau)\) is the identity map, the left Real \(\mathcal{G}\)-action on \((Z^{-1}, \tau)\) is given by \(g \cdot b(z) := b(z \cdot g^{-1})\), and the right Real \(\Gamma\)-action is given by \(b(z) \cdot \gamma := b(\gamma^{-1} \cdot z)\); \((Z^{-1}, \tau)\) is the corresponding Real generalized morphism from \((\mathcal{G}, \rho)\) to \((\Gamma, \varrho)\).

The discussion before Definition 1.24 shows that the Real generalized morphism induced by a Real strict morphism is actually a Morita equivalence. However, the converse is not true. Moreover, there is a functor

\[(1.2) \quad \mathfrak{R}\mathcal{G}_s \rightarrow \mathfrak{R}\mathcal{G},\]

where \(\mathfrak{R}\mathcal{G}_s\) is the category whose objects are Real locally compact groupoids and whose morphisms are Real strict morphisms, given by

\[f \mapsto Z_f.\]

**Definition 1.26 (Real cover groupoid).** Let \(\mathcal{G} \xrightarrow{\pi} X\) be a Real groupoid. Let \(\mathcal{U} = \{U_j\}\) be a Real open cover of \(X\). Consider the disjoint union \(\bigsqcup_{j \in J} U_j = \{(j, x) \in J \times X : x \in U_j\}\) with the Real structure \(\rho^{(0)}\) given by \(\rho^{(0)}(j, x) := (j, \rho(x))\) and define a Real local homeomorphism given by the projection \(\pi : \bigsqcup_{j \in J} U_j \rightarrow X, (j, x) \mapsto x\). Then the set

\(\mathcal{G}[\mathcal{U}] := \{(j_0, g, j_1) \in J \times \mathcal{G} \times J : r(g) \in U_{j_0}, s(g) \in U_{j_1}\}\),

endowed with the involution \(\rho^{(1)}(j_0, g, j_1) := (\tilde{j}_0, \rho(g), \tilde{j}_1)\) has a structure of a Real locally compact groupoid whose unit space is \(\bigsqcup_{j \in J} U_j\). The range and source maps are defined by \(\tilde{r}(j_0, g, j_1) := (j_0, r(g))\) and \(\tilde{s}(j_0, g, j_1) := (j_1, s(g))\); two triples are composable if they are of the form \((j_0, g, j_1)\) and \((j_1, h, j_2)\), where \((g, h) \in \mathcal{G}^{(2)}\), and their product is given by \((j_0, g, j_1) \cdot (j_1, h, j_2) := (j_0, gh, j_2)\). The inverse of \((j_0, g, j_1)\) is \((j_1, g^{-1}, j_0)\).
It is a matter of simple verifications to check the following:

**Lemma 1.27.** Let $\mathcal{G} \rightarrow X$ be a Real groupoid, and $\mathcal{U}$ a Real open cover of $X$. Then the Real generalized morphism $Z_i : \mathcal{G}[\mathcal{U}] \rightarrow \mathcal{G}$ induced from the canonical Real morphism

$$\iota : \mathcal{G}[\mathcal{U}] \rightarrow \mathcal{G}, (j_0, g, j_1) \mapsto g,$$

is a Morita equivalence between $(\mathcal{G}[\mathcal{U}], \rho)$ and $(\mathcal{G}, \rho)$.

**Definition 1.28.** Let $Z \pi \downarrow \downarrow s \rightarrow X$ be a locally trivial $G$-principal Real bundle. A section $s : Y \rightarrow Z$ is said to be Real if $s \circ \varphi = \tau \circ s$. Moreover, given a Real open cover $\{U_j\}_{j \in J}$ of $Y$, we say that a family of local sections $s_j : U_j \rightarrow Z$ is globally Real if for any $j \in J$, we have

$$s_{\bar{j}} \circ \varphi = \tau \circ s_j.$$  

(1.3)

**Lemma 1.29.** Any locally trivial $G$-principal Real bundle $\pi : Z \rightarrow Y$ admits a globally Real family of local sections $\{s_j\}_{j \in J}$ over some Real open cover $\{U_j\}$.

**Proof.** Choose a local trivialization $(U_i, \varphi_i)_{i \in I}$ of $Z$; i.e., $\varphi_i : U_i \rightarrow X$ are continuous maps such that $\pi^{-1}(U_i) =: Z_{U_i} \cong U_i \times_{\varphi_i, X, \rho} \mathcal{G}$ with $\tau_{Z_{U_i}} = (\varphi, \rho)$. It turns out that $Z_{U(i, \epsilon)} \cong U_i \times_{\varphi_i, X, \rho} \mathcal{G}$, where

$$\varphi_{\epsilon}^i := \rho^\epsilon \circ \varphi_i \circ g^\epsilon : U_i \rightarrow X$$

is a well-defined continuous map and $U_{(i, \epsilon)} := g^\epsilon(U_i)$ for $(i, \epsilon) \in I \times \mathbb{Z}_2$. However, for $(i, \epsilon) \in I \times \mathbb{Z}_2$, there is a homeomorphism

$$U_{(i, \epsilon)} \times_{\varphi_i, X, \rho} \mathcal{G} \xrightarrow{\varphi_{\epsilon+1}^i \times_{\varphi_i, X, \rho} \mathcal{G}} U_{(i, \epsilon)} \times_{\varphi_{\epsilon}^i, X, \rho} \mathcal{G}.$$

Now, putting $s_{(i, \epsilon)} : U_{(i, \epsilon)} \rightarrow Z$, $x \mapsto (x, \varphi_{\epsilon}^i(x))$, we obtain the desired sections. \hfill \square

For the remainder of this subsection we will need the following construction.

Let $(Z, \tau)$ be a Real space and $(\Gamma, \varrho)$ a Real groupoid together with a continuous Real map $\varphi : Z \rightarrow Y$. Then we define an induced groupoid $\varphi^* \Gamma$ over $Z$ in which the arrows from $z_1$ to $z_2$ are the arrows in $\Gamma$ from $\varphi(z_1)$ to $\varphi(z_2)$; i.e.,

$$\varphi^* \Gamma := Z \times_{\varphi, Y, \rho} \Gamma \times_{\varphi, Y, \rho} Z,$$

and the product is given by $(z_1, \gamma_1, z_2), (z_2, \gamma_2, z_3) = (z_1, \gamma_1 \gamma_2, z_3)$ whenever $\gamma_1$ and $\gamma_2$ are composable, while the inverse is given by

$$(z, \gamma, z')^{-1} = (z', \gamma^{-1}, z).$$
Moreover, the triple \((\rho, \varrho, \rho)\) defines a Real structure \(\varphi^*\varrho\) on \(\varphi^*\Gamma\) making it into a Real groupoid \((\varphi^*\Gamma, \varphi^*\varrho)\) that we will call the pull-back of \(\Gamma\) over \(Z\) via \(\varphi\).

**Lemma 1.30.** Given a continuous locally split Real open map \(\varphi : Z \to Y\), then the Real groupoids \(\Gamma\) and \(\varphi^*\Gamma\) are Morita equivalent.

**Proof.** Consider the Real strict homomorphism

\[\tilde{\varphi} : \varphi^*\Gamma \ni (z_1, \gamma, z_2) \mapsto \gamma \in \Gamma.\]

Then by Example 1.18 we obtain a Real generalized homomorphism

\[Z \xleftarrow{\pi_1} Z\tilde{\varphi} \xrightarrow{s \circ \pi_2} Y\]

with \(Z\tilde{\varphi} := Z \times_{\tilde{\varphi}, Y, \Gamma} \pi_1 \times \pi_2\), the obvious projections, and where \(Z \ni \varphi^*\Gamma\) by \(z \mapsto (z, \varphi(z), z)\). Now using the constructions of Example 1.18, it is very easy to check that \(Z\tilde{\varphi}\) is in fact a Morita equivalence. \(\square\)

**Proposition 1.31.** Two Real groupoids \((\Gamma, \varrho)\) and \((\mathcal{G}, \rho)\) are Morita equivalent if and only if there exist a Real space \((Z, \tau)\) and two continuous Real maps \(\varphi : Z \to Y\) and \(\varphi' : Z \to X\) such that \(\varphi^*\Gamma \cong (\varphi')^*\mathcal{G}\) under a Real (strict) homeomorphism.

**Proof.** Let \(Y \xleftarrow{\tau} Z \xrightarrow{\varrho} X\) be a Morita equivalence. Let us define

\[\Gamma \ltimes Z \ast Z \ltimes \mathcal{G} := \{((\gamma, z_1, z_2, g) \in (\Gamma \times_{s, Y, \pi} Z) \times (Z \times_{s, X, r} \mathcal{G}) \mid z_1g = \gamma z_2\}.

This defines a Real groupoid over \(Z\) whose range and source maps are defined by the second and the third projection respectively, the product is given by

\[(\gamma, z_1, z_2, g) \cdot (\gamma', z_2, z_3, g') = (\gamma \gamma', z_1, z_3, gg'),\]

provided that \(\gamma, \gamma' \in \Gamma^{(2)}\) and \(g, g' \in \mathcal{G}^{(2)}\), and the inverse of \((\gamma, z_1, z_2, g)\) is \((\gamma^{-1}, z_2, z_1, g^{-1})\). Now, for a given triple \((z_1, \gamma, z_2) \in \tau^*\Gamma\), the relations \(t(z_1) = \tau(\gamma)\) and \(t(z_2) = s(\gamma)\) give \(t(\gamma z_2) = \tau(z_1)\); then since \(\tau : Z \to Y\) is a Real \(\mathcal{G}\)-principal bundle, there exists a unique \(g \in \mathcal{G}\) such that \(\gamma z_2 = z_1g\). This gives an injecive homomorphism

\[\Psi : \tau^*\Gamma \to \Gamma \ltimes Z \ast Z \ltimes \mathcal{G},\]

\[(z_1, \gamma, z_2) \mapsto (\gamma, z_1, z_2, g),\]

which respects the Real structures. In the other hand, the map

\[\Phi : \Gamma \ltimes Z \ast Z \ltimes \mathcal{G} \to \tau^*\Gamma,\]

\[(\gamma, z_1, z_2, g) \mapsto (z_1, \gamma, z_2),\]

is a well-defined Real homomorphism that is injecive and Real. Moreover, these two maps are, by construction, inverse to each other so that we have a Real homeomorphism \(\tau^*\Gamma \cong \Gamma \ltimes Z \ast Z \ltimes \mathcal{G}\). Furthermore, since \(s : Z \to X\) is a Real \(\Gamma\)-principal bundle, we can use the same arguments to show that \(s^*\mathcal{G} \cong \Gamma \ltimes Z \ast Z \ltimes \mathcal{G}\) under a Real homeomorphism.
Conversely, if \( \varphi : Z \to Y \) and \( \varphi' : Z \to X \) are given continuous Real maps and \( f : \varphi^*\Gamma \to (\varphi')^*X \) is a Real homeomorphism of groupoids, then the induced Real generalized homomorphism

\[
\varphi^*\Gamma \xrightarrow{Z_f} (\varphi')^*\mathcal{G}
\]

is a Morita equivalence and Lemma 1.30 completes the proof. □

The following example provides a characterization of groupoids Morita equivalent to a given Real space.

**Example 1.32.** Let \((X, \rho), (Y, \varrho)\) be a locally compact Hausdorff Real spaces, and let \( \pi : (Y, \varrho) \to (X, \rho) \) be a continuous locally split Real open map. Form the Real groupoid \( Y[2] \xrightarrow{\longrightarrow} Y \), where \( Y[2] \) is the fibered-product \( Y \times_{\pi, X, \pi} Y \) equipped with the obvious Real structure; the groupoid structure on \( Y[2] \) is:

\[
\begin{align*}
  s(y_1, y_2) &:= y_2; \\
  r(y_1, y_2) &:= y_1; \\
  (y_1, y_2)^{-1} &:= (y_2, y_1); \\
  (y_1, y_2) \cdot (y_2, y_3) &:= (y_1, y_3).
\end{align*}
\]

Then the Real groupoids \( Y[2] \xrightarrow{\longrightarrow} Y \) and \( X \xrightarrow{\longrightarrow} X \) are Morita equivalent. Indeed, we have \( \pi^*X \sim_{\text{Morita}} X \), thanks to Lemma 1.30; but \( \pi^*X \) clearly identifies with \( Y[2] \) as Real groupoids.

Conversely, suppose \((\Gamma, \varrho)\) is a Real groupoids Morita equivalent to \( X \). Then in view of Proposition 1.31, there is a Real space \((Z, \tau)\), two continuous locally split Real open maps \( s : Z \to X, r : Z \to Y \) such that \( s^*X \cong \tau^*\Gamma \) as Real groupoids over \( Z \). In particular, \( r : Z \to Y \) is a principal Real \( X \)-bundle, so that the Real space \( Y \) is homeomorphic to the quotient Real space \( Z/X = Z \). Thus, we have isomorphism of Real spaces

\[
\tau^*\Gamma = Z \times_Y \Gamma \times_Y Z \cong Y \times_Y \Gamma \times_Y Y \cong \Gamma.
\]

Moreover, we have \( \pi^*X \cong [Z[2]] \) as Real spaces. Therefore, the Real groupoids \( \Gamma \xrightarrow{\longrightarrow} Y \) and \( Z[2] \xrightarrow{\longrightarrow} Z \) as isomorphic.

**Proposition 1.33** (Cf. Proposition 2.3 [25]). Any Real generalized morphism

\[
Y \xleftarrow{\tau} Z \xrightarrow{s} X
\]

is obtained by composition of the canonical Morita equivalence between \((\Gamma, \varrho)\) and \((\Gamma[\mathcal{U}], \varrho)\), where \( \mathcal{U} \) is an open cover of \( Y \), with a Real strict morphism \( f_\mathcal{U} : \Gamma[\mathcal{U}] \to \mathcal{G} \) (i.e., its induced morphism in the category \( \text{RG} \)).

**Proof.** From Lemma 1.30, there is a Real Morita equivalence \( Z_\xi : \tau^*\Gamma \to \Gamma \) and the Real homeomorphism \( \tau^*\Gamma \cong \Gamma \rtimes Z \rtimes \mathcal{G} \) induces a Real strict homomorphism \( f : \tau^*\Gamma \to \mathcal{G} \) given by the fourth projection, and hence a Real generalized homomorphism \( Z_f : \tau^*\Gamma \to \mathcal{G} \). Furthermore, by using the construction of these generalized homomorphisms, it is easy to check that
the composition $Z_i \times \Gamma Z$ is $r^* \Gamma \mathcal{G}$-equivariently homeomorphic to $Z$ (under a Real homeomorphism); i.e., the diagram

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\cong} & r^* \Gamma \\
Z & \downarrow & \\
\mathcal{G} & \end{array}
$$

is commutative in the category $\mathcal{R}\mathcal{G}$.

Consider a Real open cover $\mathcal{U} = \{U_j\}$ of $Y$ together with a globally
Real family of local sections $s_j : U_j \to Z$ of $r : Z \to Y$. Then, setting
$(j_0, \gamma, j_1) \mapsto (s_{j_0}(r(\gamma)), \gamma, s_{j_1}(s(\gamma)))$ for
$(j_0, \gamma, j_1) \in \Gamma[1]$, we get a Real
strict homomorphism $\tilde{s} : \Gamma[\mathcal{U}] \to r^* \Gamma$ such that the composition
$\Gamma[\mathcal{U}] \to r^* \Gamma \to \Gamma$ is the canonical map $\iota$ described in Example 1.26. Then, $f \circ \tilde{s} : \Gamma[\mathcal{U}] \to \mathcal{G}$ is the desired Real strict homomorphism. $\square$

This proposition leads us to think of a Real generalized homomorphism
from a Real groupoid $(\Gamma, \rho)$ to a Real groupoid $(\mathcal{G}, \rho)$ as a Real strict mor-
phism $f_{\mathcal{U}} : (\Gamma[\mathcal{U}], \rho) \to (\mathcal{G}, \rho)$, where $\mathcal{U}$ is a Real open cover of $Y$.

To refine this point of view, given two Real groupoids $(\Gamma, \rho)$ and $(\mathcal{G}, \rho)$,
let $\Omega$ denote the collection of such pairs $(\mathcal{U}, f_{\mathcal{U}})$. We say that $\mathcal{U}$ and $\mathcal{U}'$
are isomorphic provided that $Z_{f_{\mathcal{U}}} \circ Z_{\mathcal{U}}^{-1} \cong Z_{f_{\mathcal{U}'}}, Z_{\mathcal{U}'}^{-1}$,
where $\mathcal{U} = (\Gamma[\mathcal{U}], \rho) \to (\Gamma, \rho)$ and $\mathcal{U}' = (\Gamma[\mathcal{U}'], \rho) \to (\Gamma, \rho)$ are the canoni-
cal morphisms; this clearly defines an equivalence relation. We denote by
$\Omega((\Gamma, \rho), (\mathcal{G}, \rho))$ the set of isomorphism classes of elements of $\Omega$.

Let $(\mathcal{U}, f_{\mathcal{U}}) : (\Gamma, \rho) \to (\mathcal{G}', \rho')$ be an equivalence class in $\Omega((\Gamma, \rho), (\mathcal{G}', \rho'))$
and let $(\mathcal{V}, f_{\mathcal{V}}) : (\mathcal{G}', \rho') \to (\mathcal{G}, \rho)$ be an element in $\Omega((\mathcal{G}', \rho'), (\mathcal{G}, \rho))$. Let
$\iota_{\mathcal{V}} : G[\mathcal{V}] \to G'$ be the canonical morphism, and let $Z_{\mathcal{G}'}^{-1} : (\mathcal{G}', \rho') \to
(\mathcal{G}[\mathcal{V}], \rho')$ be the inverse of $Z_{\mathcal{G}'}$. Next, we apply Proposition 1.33 to the Real
generalized morphism $Z_{\mathcal{G}'}^{-1} \circ Z_{f_{\mathcal{U}}} : \Gamma[\mathcal{U}] \to \mathcal{G}[\mathcal{V}]$ to get a Real open cover $\mathcal{U}'$
of $Y$ containing $\mathcal{U}$ and a Real strict morphism $\varphi_{\mathcal{U}'} : (\Gamma[\mathcal{U}], \rho) \to (\mathcal{G}[\mathcal{V}], \rho')$.
Then, we pose

$$(1.4) \quad (\mathcal{V}, f_{\mathcal{V}}) \circ (\mathcal{U}, f_{\mathcal{U}}) := (\mathcal{U}', f_{\mathcal{U}'})$$

with $f_{\mathcal{U}'} = f_{\mathcal{V}} \circ \varphi_{\mathcal{U}'}$; thus we get an element of $\Omega((\Gamma, \rho), (\mathcal{G}, \rho))$. It follows
that there exists a category $\mathcal{R}\mathcal{G}\mathcal{G}_s$ whose objects are Real groupoids, and in
which a morphism from $(\Gamma, \rho)$ to $(\mathcal{G}, \rho)$ is a class $(\mathcal{U}, f_{\mathcal{U}})$ in $\Omega((\Gamma, \rho), (\mathcal{G}, \rho))$.

**Example 1.34.** Any Real strict morphism $f : (\Gamma, \rho) \to (\mathcal{G}, \rho)$ can be
identified with the pair $(Y, f)$, by considering the trivial Real open cover $Y$
consisting of one set, and by viewing the groupoid $\Gamma$ as the cover groupoid
$\Gamma[Y]$. In particular, $\mathcal{R}\mathcal{G}_s$ is a subcategory of $\mathcal{R}\mathcal{G}\mathcal{Q}$.

**Example 1.35.** Suppose that $(Z, \tau) : (\Gamma, \rho) \to (\mathcal{G}, \rho)$ is a Real general-
ized morphism. Then, Proposition 1.33 provides a unique class $(\mathcal{U}, f_{\mathcal{U}}) \in
\Omega((\Gamma, \rho), (\mathcal{G}, \rho))$. 


Note that a class \((\mathcal{U}, f_{\mathcal{U}}) \in \Omega((\Gamma, \varrho), (\mathcal{S}, \rho))\) is an isomorphism in \(\mathcal{R}\mathfrak{S}\Omega\) if there exists \((\mathcal{V}, f_{\mathcal{V}}) \in \Omega((\mathcal{S}, \rho), (\Gamma, \varrho))\) such that
\[
Z_{f_{\mathcal{U}}} \circ Z_{\mathcal{U}}^{-1} \circ Z_{f_{\mathcal{V}}} \cong Z_{\mathcal{V}} \quad \text{and} \quad Z_{f_{\mathcal{U}}} \circ Z_{\mathcal{U}}^{-1} \circ Z_{f_{\mathcal{V}}} \cong Z_{\mathcal{V}},
\]
where \(\iota_{\mathcal{U}} : (\Gamma[\mathcal{U}], \varrho) \rightarrow (\Gamma, \varrho)\) and \(\iota_{\mathcal{V}} : (\mathcal{S}[\mathcal{U}], \rho) \rightarrow (\mathcal{S}, \rho)\) are the canonical morphisms.

**Proposition 1.37.** Define \(F : \mathcal{R}\mathfrak{S} \rightarrow \mathcal{R}\mathfrak{S}\Omega\) by
\[
F(Z, \tau) := (\mathcal{U}, f_{\mathcal{U}}),
\]
where, if \((Z, \tau) : (\Gamma, \varrho) \rightarrow (\mathcal{S}, \rho)\) is a class of Real generalized morphisms, \((\mathcal{U}, f_{\mathcal{U}})\) is the class of pairs corresponding to \((Z, \tau)\).

Then \(F\) is a functor; furthermore, \(F\) is an isomorphism of categories.

**Proof.** Suppose that \((Z, \tau) : (\Gamma, \varrho) \rightarrow (\mathcal{S}', \rho')\) and \((Z', \tau') : (\mathcal{S}', \rho') \rightarrow (\mathcal{S}, \rho)\) are morphisms in \(\mathcal{R}\mathfrak{S}\). Let
\[
\begin{align*}
F(Z' \circ Z, \tau \times \tau') &= (\mathcal{U}, f_{\mathcal{U}}) \in \Omega((\Gamma, \varrho), (\mathcal{S}, \rho)), \\
F(Z, \tau) &= (\mathcal{U}', f_{\mathcal{U}'}) \in \Omega((\Gamma, \varrho), (\mathcal{S}', \rho')), \\
F(Z', \tau') &= (\mathcal{V}, f_{\mathcal{V}}) \in \Omega((\mathcal{S}', \rho'), (\mathcal{S}, \rho)).
\end{align*}
\]
Consider a Real open cover \(\tilde{\mathcal{U}}\) of \(Y\) containing \(\mathcal{U}'\) and a Real morphism \(\varphi_{\tilde{\mathcal{U}}} : (\Gamma[\tilde{\mathcal{U}}], \varrho) \rightarrow (\mathcal{S}'[\mathcal{V}], \rho')\) such that \(Z_{\varphi_{\tilde{\mathcal{U}}}} \circ Z_{\mathcal{U}}^{-1} \cong Z_{\mathcal{V}} \circ Z_{\mathcal{U}}^{-1} \circ Z_{f_{\mathcal{V}}}\) as Real generalized morphisms from \((\Gamma[\tilde{\mathcal{U}}], \varrho)\) to \((\mathcal{S}'[\mathcal{V}], \rho')\), where
\[
i : (\Gamma[\tilde{\mathcal{U}}], \varrho) \rightarrow (\Gamma[\mathcal{U}'], \varrho) \quad \text{and} \quad \iota_{\mathcal{V}} : (\mathcal{S}'[\mathcal{V}], \rho') \rightarrow (\mathcal{S}', \rho')
\]
are the canonical morphisms. Note that if \(\iota_{\tilde{\mathcal{U}}} : (\Gamma[\tilde{\mathcal{U}}], \varrho) \rightarrow (\Gamma, \varrho)\) is the canonical morphism, then \(\iota_{\tilde{\mathcal{U}}} = \iota_{\mathcal{V}} \circ \iota\); hence, \(Z_{\mathcal{U}}^{-1} \cong Z_{\mathcal{V}}^{-1} \circ Z_{\mathcal{U}}^{-1}\) by functoriality.

On the other hand, \(F(Z', \tau') \circ F(Z, \tau) = (\mathcal{V}, f_{\mathcal{V}}) \circ (\mathcal{U}, f_{\mathcal{U}}) = (\tilde{\mathcal{U}}, f_{\tilde{\mathcal{U}}})\), where \(f_{\tilde{\mathcal{U}}} = f_{\mathcal{V}} \circ \varphi_{\tilde{\mathcal{U}}}\). Henceforth,
\[
Z_{f_{\tilde{\mathcal{U}}} \circ Z_{\mathcal{U}}^{-1}} \cong Z_{f_{\mathcal{V}}} \circ Z_{\varphi_{\tilde{\mathcal{U}}}} \circ Z_{\mathcal{U}}^{-1} \circ Z_{f_{\mathcal{V}}} \cong Z_{f_{\mathcal{V}}} \circ Z_{\mathcal{U}}^{-1} \circ Z_{f_{\mathcal{V}}} \circ Z_{\mathcal{U}}^{-1} \cong Z' \circ Z,
\]
which shows that \(F(Z' \circ Z, \tau \times \tau') \cong F(Z', \tau') \circ F(Z, \tau)\), and thus \(F\) is a functor.

Now, it is not hard to see that we get an inverse functor for \(F\) by defining
\[
Z : \mathcal{R}\mathfrak{S}\Omega \rightarrow \mathcal{R}\mathfrak{S}, (\mathcal{U}, f_{\mathcal{U}}) \mapsto (Z_{f_{\mathcal{U}}} \circ Z_{\mathcal{U}}^{-1}, \tau),
\]
where \(\tau\) is defined in an obvious way. \(\square\)

### 1.5. Real graded twists.

**Definition 1.38** (Cf. [11, §2]). Let \(\Gamma \rightarrow Y\) be a Real groupoid and let \(S\) be a Real Abelian group. A Real graded \(S\)-twist \((\tilde{\Gamma}, \delta)\) over \(\Gamma\) consists of the following data:

(i) a Real groupoid \(\tilde{\Gamma}\) whose unit space is \(Y\), together with a Real strict homomorphism \(\pi : \tilde{\Gamma} \rightarrow \Gamma\) that restricts to the identity in \(Y\),
(ii) a (left) Real action of $S$ on $\tilde{\Gamma}$ compatible with the partial product in $\tilde{\Gamma}$ making \( \tilde{\Gamma} \xrightarrow{\pi} \Gamma \) a (left) Real $S$-principal bundle,

(iii) a strict homomorphism $\delta : \Gamma \rightarrow \mathbb{Z}_2$, called the grading, such that $\delta(\tilde{\gamma}) = \delta(\gamma)$ for any $\gamma \in \Gamma$.

In this case we refer to the triple $(\tilde{\Gamma}, \Gamma, \delta)$ as a Real graded $S$-twist, and it is sometimes symbolized by the “extension”

\[ S \longrightarrow \tilde{\Gamma} \xrightarrow{\pi} \Gamma \quad \begin{array}{c} \downarrow \delta \end{array} \quad \mathbb{Z}_2 \]

**Example 1.39** (The trivial twist). Given Real groupoid $\Gamma$, we form the product groupoid $\Gamma \times S$ and we endow it with the Real structure $(\tilde{\gamma}, \tilde{\lambda}) := (\gamma, \lambda)$ for. Let $S$ act on $\Gamma \times S$ by multiplication with the second factor. Then $\mathcal{F}_0 := (\Gamma \times S, 0)$ is a Real graded twist of $\Gamma$, where $0 : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ is the zero map. This element is called the trivial Real graded $S$-twist over $\Gamma$.

**Example 1.40.** Let $Y$ be a locally compact Real space and $\{U_i\}_{i \in I \times \{\pm 1\}}$ be a good Real open. Let us consider the Real groupoid $Y[\mathbb{U}] \longrightarrow \bigsqcup_i U_i$, and the space $Y \times S$ together with the Real structure $(y, \lambda) \mapsto (y, \lambda)$ and the Real $S$-action given by the multiplication on the second factor. We write $x_{i_0i_1i_2}$ for $(i_1, x, i_1) \in Y[\mathbb{U}]$. There is a canonical Real morphism $\delta : Y[\mathbb{U}] \rightarrow \mathbb{Z}_2$ given by $\delta(x_{i_0i_1i_2}) := \varepsilon_0 + \varepsilon_1$ for $i_0 = (i_0', \varepsilon_0)$, $i_1 = (i_1', \varepsilon_1) \in I$. Then, a Real graded $S$-twist $(\tilde{\Gamma}, Y[\mathbb{U}], \delta) \equiv \Gamma_{ij} \cong U_{ij} \times S$ subject to the multiplication

\[(x_{i_0i_1}, \lambda_1) \cdot (x_{i_1i_2}, \lambda_2) = (x_{i_0i_2}, \lambda_1\lambda_2c_{i_0i_1i_2}(x)),\]

where $c = \{c_{i_0i_1i_2}\}$ is a family of continuous maps $c_{i_0i_1i_2} : U_{i_0i_1i_2} \rightarrow S$ which is a 2-cocycle such that $c_{i_0i_1i_2, i_3}(\bar{x}) = c_{i_0i_1i_3}(x)$ for all $x \in U_{i_0i_1i_2} = U_{i_0} \cap U_{i_1} \cap U_{i_2}$. The pair $(\delta, c)$ will be called the Dixmier–Douady class of $(\tilde{\Gamma}, Y[\mathbb{U}], \delta)$ (see Section 2.12).

**Example 1.41.** Let $\Gamma \longrightarrow Y$ be a Real groupoid, and let $J : \Lambda \rightarrow Y$ be a Real $S$-principal bundle. Then the tensor product $r^*\Lambda \otimes s^*\overline{\Lambda}$, which is a Real $S$-principal bundle over $\Gamma$, naturally admits the structure of Real groupoid over $Y$, so that $(r^*\Lambda \otimes s^*\overline{\Lambda}, 0)$ is a Real graded $S$-twist over $\Gamma$.

There is an obvious notion of strict morphism of Real graded $S$-twists. For instance, two Real graded $S$-twists $(\tilde{\Gamma}_1, \Gamma, \delta_1)$ and $(\tilde{\Gamma}_2, \Gamma, \delta_2)$ are isomorphic if there exists a Real $S$-equivariant isomorphism of groupoids $f : \tilde{\Gamma}_1 \rightarrow \tilde{\Gamma}_2$.
such that the diagram

\[
\begin{array}{ccc}
\tilde{\Gamma}_1 & \xrightarrow{\pi_1} & \Gamma \\
\downarrow{f} & & \downarrow{\pi_2} \\
\tilde{\Gamma}_2
\end{array}
\]

commutes in the category \(\mathfrak{R}\mathfrak{G}_s\). In particular, we say that \((\tilde{\Gamma}, \delta)\) is strictly trivial if it isomorphic to the trivial Real graded groupoid \((\Gamma, \delta, 0)\). By \(\text{TwR}(\Gamma, S)\) we denote the set of strict isomorphism classes of Real graded \(S\)-twists over \(\Gamma\). The class of \((\tilde{\Gamma}, \delta)\) in \(\text{TwR}(\Gamma, S)\) is denoted by \([\tilde{\Gamma}, \delta]\).

**Definition 1.42** (Cf. [11, 23, 6]). Given two Real graded \(S\)-twists \(\mathcal{I}_1 = (\tilde{\Gamma}_1, \delta_1)\) and \(\mathcal{I}_2 = (\tilde{\Gamma}_2, \delta_2)\) over \(\mathcal{G}\), we define their tensor product

\[\mathcal{I}_1 \otimes \mathcal{I}_2 = (\tilde{\Gamma}_1 \otimes \tilde{\Gamma}_2, \delta_1 + \delta_2)\]

by the Baer sum of \(\mathcal{I}_1\) and \(\mathcal{I}_2\) defined as follows. Define the groupoid \(\tilde{\Gamma}_1 \otimes \tilde{\Gamma}_2\) as the quotient

\[\tilde{\Gamma}_1 \times_{\Gamma} \tilde{\Gamma}_2/S := \{ (\gamma_1, \gamma_2) \in \tilde{\Gamma}_1 \times_{\pi_1, \Gamma, \pi_2} \tilde{\Gamma}_2 \}/(\gamma_1, \gamma_2) \sim (\lambda \gamma_1, \lambda^{-1} \gamma_2),\]

where \(\lambda \in S\), together with the obvious Real structure. The projection \(\pi_1 \otimes \pi_2\) is just \(\pi_i\) and \(\delta = \delta_1 + \delta_2\) is given by \(\delta(\gamma) = \delta_1(\gamma) + \delta_2(\gamma)\).

The product in the Real groupoid \(\tilde{\Gamma}_1 \otimes \tilde{\Gamma}_2\) is

\[\gamma_1 \gamma_2 := (-1)^{\delta_2(\gamma_2)} \delta_1(\gamma_1')(\gamma_1', \gamma_2', \gamma_1 \gamma_2),\]

whenever this does make sense and where \(\gamma_i = \pi_2(\tilde{\gamma}_i), i = 1, 2\).

**Lemma 1.43** ([23, p.4]). Given \([\tilde{\Gamma}_1, \delta_1] \in \text{TwR}(\Gamma, S), i = 1, 2\), set

\[\mathcal{I}_1 \otimes \mathcal{I}_2 := (\tilde{\Gamma}_1 \otimes \tilde{\Gamma}_2, \delta_1 + \delta_2).\]

Then, under this sum, \(\text{TwR}(\Gamma, S)\) is an Abelian group whose zero element is given by the class of the trivial element \(\mathcal{I}_0 = (\mathcal{G} \times S, 0)\).

**Proof.** The tensor product defined above is commutative in \(\text{TwR}(\Gamma, S)\). Indeed, the groupoid \(\tilde{\Gamma}_2 \otimes \tilde{\Gamma}_1 = \tilde{\Gamma}_2 \times_{\Gamma} \tilde{\Gamma}_1/S\) is endowed with the multiplication

\[(\gamma_2', \gamma_1')(\gamma_2', \gamma_1') = (-1)^{\delta_2(\gamma_2')\delta_1(\gamma_1')} (\gamma_2' \gamma_1', \gamma_2' \gamma_1', \gamma_2 \gamma_1').\]

Then the map

\[\tilde{\Gamma}_1 \otimes \tilde{\Gamma}_2 \rightarrow \tilde{\Gamma}_2 \otimes \tilde{\Gamma}_1, (\gamma_1, \gamma_2) \mapsto (-1)^{\delta_1(\gamma_1)\delta_2(\gamma_2)} (\gamma_2, \gamma_1)\]

is a Real \(S\)-equivariant isomorphism of groupoids.

Now define the inverse of \((\tilde{\Gamma}, \delta)\) is \((\tilde{\Gamma}^{op}, \delta)\) where \(\tilde{\Gamma}^{op}\) is \(\tilde{\Gamma}\) as a set but, together with the same Real structure, but the \(S\)-principal bundle structure is replaced by the conjugate one, i.e., \(\lambda \gamma^{op} = (\lambda \gamma)^{op}\), and the product \(*_{op}\) in \(\tilde{\Gamma}^{op}\) is

\[\gamma *_{op} \gamma' := (-1)^{\delta(\gamma)\delta(\gamma')} \gamma \gamma'.\]
Now it is easy to see that the map
\[ \Gamma \times S \longrightarrow \bar{\Gamma} \times_{\bar{\Gamma}} \Gamma^\text{op} / S, \ (\gamma, \lambda) \longmapsto (\lambda \bar{\gamma}, \bar{\gamma}), \]
where \( \bar{\gamma} \in \bar{\Gamma} \) is any lift of \( \gamma \in \Gamma \), is an isomorphism.

We have the following criteria of strict triviality; the proof is the same as in [25, Proposition 2.8].

**Proposition 1.44.** Let \( (\bar{\Gamma}, \delta) \) be a Real graded \( S \)-twist over the Real groupoid \( \Gamma \longrightarrow Y \). The following are equivalent:

(i) \( (\bar{\Gamma}, \delta) \) is strictly trivial.
(ii) \( \delta(\gamma) = 0, \forall \gamma \in \Gamma \), and there exists a Real strict homomorphism \( \sigma : \Gamma \longrightarrow \bar{\Gamma} \) such that \( \pi \circ \sigma = \text{Id} \).
(iii) \( \delta(\gamma) = 0, \forall \gamma \in \Gamma \), and there exists a Real \( S \)-equivariant groupoid homomorphism \( \varphi : \bar{\Gamma} \longrightarrow S \).

**Example 1.45.** Let \( J : \Lambda \longrightarrow Y \) be a Real \( S \)-principal bundle with a Real (left) \( \Gamma \)-action that is compatible with the \( S \)-action; in other words \( Y \longleftarrow \Lambda \longrightarrow \star \) is a Real generalized homomorphism from \( \Gamma \) to \( S \). Then, the Real \( \Gamma \)-action induces an \( S \)-equivariant isomorphism \( \Lambda \ni v \mapsto \gamma \cdot v \in \Lambda_{r(\gamma)} \) for every \( \gamma \in \Gamma \). Hence, there is a Real \( S \)-equivariant groupoid isomorphism \( \varphi : r^*\Lambda \otimes s^*\Lambda \longrightarrow \Gamma \times S \) defined as follows. If \( (v, b(w)) \in \Lambda_{r(\gamma)} \otimes \Lambda_{s(\gamma)} \), there exists a unique \( \lambda \in S \) such that \( \gamma \cdot w = v \cdot \lambda \). We then set
\[ \varphi([v, b(w)]) := (\gamma, \lambda). \]
The inverse of \( \varphi \) is \( \varphi'(\gamma, \lambda) := [v_\gamma, \gamma^{-1} \cdot v_\gamma] \), where for \( \gamma \in \Gamma \), \( v_\gamma \) is any lift of \( r(\gamma) \) through the projection \( J \).

Observe that the set of Real graded \( S \)-twists of the from \( (r^*\Lambda \otimes s^*\Lambda, 0) \) over \( \Gamma \) (see Example 1.41) is a subgroup of \( \text{TwR}(\Gamma, S) \). By \( \text{extR}(\Gamma, S) \) we denote the quotient of \( \text{TwR}(\Gamma, S) \) by this subgroup.

Let us show that \( \text{extR}(\cdot, S) \) is functorial in the category \( \mathcal{RG}_s \). Let \( \Gamma, \Gamma' \) be two Real groupoids, and let \( f : \Gamma' \longrightarrow \Gamma \) be a morphism in \( \mathcal{RG}_s \). Suppose that \( \mathcal{I} = (\bar{\Gamma}, \delta) \) is a Real graded \( S \)-twist over \( \Gamma \). Then, the pull-back
\[ f^*\bar{\Gamma} := \bar{\Gamma} \times_{\pi, \Gamma, f} \Gamma' \]
of the Real \( S \)-principal bundle \( \pi : \bar{\Gamma} \longrightarrow \Gamma \), on which the Real groupoid structure is the one induced from the product Real groupoid \( \bar{\Gamma} \times \Gamma' \), defines a Real graded twist
\[ f^*\mathcal{I} := f^*\bar{\Gamma} \longrightarrow f^*\bar{\Gamma} \longrightarrow \Gamma' \]

\[ \Downarrow f^*\delta \]
\[ \mathbb{Z}_2 \]
where \( f^*\pi(\tilde{\gamma}, \gamma') := \gamma', \) \( f^*\delta(\gamma') := \delta(f(\gamma')) \in \mathbb{Z}_2, \) and the Real left \( S \)-
action on \( f^*\Gamma \) being given by \( \lambda \cdot (\tilde{\gamma}, \gamma') = (\lambda \tilde{\gamma}, \gamma') \). Suppose now that
\( \mathcal{T}_i = (\tilde{\Gamma}_i, \delta_i), \ i = 1, 2 \) are representatives in \( \text{extR}(\Gamma, S) \). Then,
\[
f^*(\mathcal{T}_1 \otimes \mathcal{T}_2) = f^*\mathcal{T}_1 \otimes f^*\mathcal{T}_2;
\]
indeed,
\[
f^*(\tilde{\Gamma}_1 \otimes \tilde{\Gamma}_2) = \left( \tilde{\Gamma}_1 \times_{\Gamma} \tilde{\Gamma}_2 / S \right) \times_{\Gamma'} \Gamma' \cong \left( (\Gamma_1 \times_{\Gamma} \Gamma') \times_{\Gamma} (\tilde{\Gamma}_2 \times_{\Gamma} \Gamma') \right) / S
= f^*\tilde{\Gamma}_1 \otimes f^*\tilde{\Gamma}_2.
\]
Moreover, it is easily seen that if \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) are equivalent in \( \text{extR}(\Gamma, S) \),
then so are \( f^*\mathcal{T}_1 \) and \( f^*\mathcal{T}_2 \). Thus, \( f \) induces a morphism of Abelian groups \( f^* : \text{extR}(\Gamma, S) \to \text{extR}(\Gamma', S) \). We then have proved this:

**Lemma 1.46.** The correspondence
\[
(1.11) \quad \text{extR}(\cdot, S) : \mathcal{RG} \to \mathcal{Ab},
\]
\[
\Gamma \mapsto \text{extR}(\Gamma, S), \quad f \mapsto f^*,
\]
where \( \mathcal{Ab} \) is the category of Abelian groups, is a contravariant functor. In particular, \( \text{extR}(\cdot, S) \) is invariant under Real strict isomorphisms.

**1.6. Real graded central extensions.** In this subsection we introduce
Real graded central extensions of Real groupoids, by adapting [11, 12, 6, 23]
to our context.

**Definition 1.47.** Let \( (\tilde{\Gamma}_i, \Gamma_i, \delta_i), i = 1, 2 \), be Real graded \( S \)-twists. Then a
Real generalized homomorphism \( Z : \tilde{\Gamma}_1 \to \tilde{\Gamma}_2 \) is said to be \( S \)-equivariant if
there is a Real action of \( S \) on \( Z \) such that
\[
(\lambda \tilde{\gamma}_1) \cdot z \cdot \tilde{\gamma}_2 = \tilde{\gamma}_1 \cdot (\lambda z) \cdot \tilde{\gamma}_2 = \tilde{\gamma}_1 \cdot z \cdot (\lambda \tilde{\gamma}_2),
\]
for any \((\lambda, \tilde{\gamma}_1, z, \tilde{\gamma}_2) \in S \times \tilde{\Gamma}_1 \times Z \times \tilde{\Gamma}_2 \) such that these products make sense. We refer to \( Z : (\tilde{\Gamma}_1, \Gamma_1, \delta_1) \to (\tilde{\Gamma}_2, \Gamma_2, \delta_2) \) as a generalized morphism of
Real graded \( S \)-twists. In particular, if \( Z \) is an isomorphism, the two
Real graded \( S \)-twists are said to be *Morita equivalent*; in this case we write
\((\tilde{\Gamma}_1, \Gamma_1, \delta_1) \sim (\tilde{\Gamma}_2, \Gamma_2, \delta_2)\).

**Lemma 1.48.** Let \( Z : (\tilde{\Gamma}_1, \Gamma_1, \delta_1) \to (\tilde{\Gamma}_2, \Gamma_2, \delta_2) \) be a generalized morphism. Then the \( S \)-action on \( Z \) is free and the Real space \( Z/S \) (with the obvious involution) is a Real generalized homomorphism from \( \Gamma_1 \) to \( \Gamma_2 \).

**Proof.** Same as [25, Lemma 2.10].

**Definition 1.49.** Let \( \mathcal{G} \) be a Real groupoid and \( S \) an abelian Real group.
A *Real graded \( S \)-central extension* of \( \mathcal{G} \) consists of a triple \((\tilde{\Gamma}, \Gamma, \delta, P)\), where
\((\tilde{\Gamma}, \Gamma, \delta) \) is a Real graded \( S \)-twist, and \( P \) is a (Real) Morita equivalence \( \Gamma \to \mathcal{G} \).
**Definition 1.50.** We say that \((\tilde{\Gamma}_1, \Gamma_1, \delta_1, P_1)\) and \((\tilde{\Gamma}_2, \Gamma_2, \delta_2, P_2)\) are Morita equivalent if there exists a Morita equivalence \(Z : (\tilde{\Gamma}_1, \Gamma_1, \delta_1) \rightarrow (\tilde{\Gamma}_2, \Gamma_2, \delta_2)\) such that the diagrams

\[
\begin{array}{c}
\xymatrix{
\Gamma_1 \ar[r]^{Z/S} \ar[d]_{P_1} & \Gamma_2 \ar[d]^{P_2} \\
\mathcal{G} & 
}
\end{array}
\]

and

\[
\begin{array}{c}
\xymatrix{
\Gamma_1 \ar[r]^{Z/S} \ar[d]_{\delta_1} & \Gamma_2 \ar[d]^{\delta_2} \\
\mathbb{Z}_2 & 
}
\end{array}
\]

commute in the category \(\mathfrak{R}\mathfrak{G}\). Such a \(Z\) is also called an equivalence bimodule of Real graded \(S\)-central extensions. The set of Morita equivalence classes of Real graded \(S\)-central extensions of \(\mathcal{G}\) is denoted by \(\operatorname{ExtR}(\mathcal{G}, S)\).

The set \(\operatorname{ExtR}(\mathcal{G}, S)\) admits a natural structure of abelian group described in the following way. Assume that \(E_i = (\tilde{\Gamma}_i, \Gamma_i, \delta_i, P_i), \ i = 1, 2, \) are two given Real graded \(S\)-central extensions of \(\mathcal{G}\), then \(Y_1 \xrightarrow{r} Z \xrightarrow{s} Y_2\) is a Morita equivalence between \(\Gamma_1\) and \(\Gamma_2\), where \(Z = P_1 \times_{\mathcal{G}} P_2\). But from Proposition 1.31 there exists a Real homeomorphism \(f : s^*\Gamma_2 \rightarrow r^*\Gamma_1\). Now one can see that the maps \(\pi : r^*\tilde{\Gamma}_1 \rightarrow r^*\Gamma_1, (z, \gamma_1, z') \mapsto (z, \pi_1(\gamma_1), z')\) and \(\pi' : s^*\tilde{\Gamma}_2 \rightarrow r^*\Gamma_1(z, \gamma_2, z') \mapsto \pi \circ f(z, \gamma_2, z')\) define two Real \(S\)-principal bundles and then \((r^*\Gamma_1, \delta)\) and \((s^*\tilde{\Gamma}_2, \delta)\), where \(\delta := \delta_1 \circ pr_2\), define elements of \(\operatorname{extR}(r^*\Gamma_1, S)\). Therefore, we can form the tensor product \((r^*\tilde{\Gamma}_1 \otimes s^*\tilde{\Gamma}_2, \delta \otimes \delta)\) are Real graded \(S\)-groupoid over \(r^*\Gamma_1\). Moreover, \(r^*\Gamma_1 \sim_{\text{Morita}} \Gamma_1\); then, if \(P : r^*\Gamma_1 \rightarrow \mathcal{G}\) is a Real Morita equivalence, we obtain a Real graded \(S\)-central extension of \(\mathcal{G}\) by setting

\[
E_1 \hat{\otimes} E_2 := (r^*\tilde{\Gamma}_1 \hat{\otimes} s^*\tilde{\Gamma}_2, r^*\Gamma_1, \delta, P),
\]

that we will call the tensor product of \(E_1\) and \(E_2\). Thus, we define the sum

\[
[E_1] + [E_2] := [E_1 \hat{\otimes} E_2],
\]

which is easily seen to be well-defined in \(\operatorname{ExtR}(\mathcal{G}, S)\). The inverse \(\operatorname{Ext}^{op}\) of \(E\) is \((\tilde{\Gamma}^{op}, \Gamma, \delta, P)\). Notice that \(\operatorname{extR}(\mathcal{G}, S)\) is naturally a subgroup of \(\operatorname{ExtR}(\mathcal{G}, S)\) by identifying a Real graded \(S\)-twist \((\tilde{\Gamma}, \mathcal{G}, \delta)\) with the Real graded \(S\)-central extension \((\tilde{\Gamma}, \mathcal{G}, \delta, \mathcal{G})\). We summarize this in the next lemma.

**Lemma 1.51.** Under the sum defined above, \(\operatorname{ExtR}(\mathcal{G}, S)\) is an abelian group whose zero element is the class of the trivial Real graded \(S\)-central extension \((\mathcal{G} \times S, \mathcal{G}, 0, \mathcal{G})\).
When the Real structure is trivial, then we recover the usual definition of graded central extensions (see [6] for instance) of $\mathcal{G}$ by the group $\mathbb{Z}_2$.

**Proposition 1.52.** Suppose that $\mathcal{G} \xrightarrow{\rho} X$ is equipped with a trivial Real structure. Then

$$\widehat{\text{Ext}}^R(\mathcal{G}, \mathbb{S}^1) \cong \widehat{\text{Ext}}(\mathcal{G}, \mathbb{Z}_2).$$

**Example 1.53.** Suppose $\mathcal{G}$ reduces to a Real space $X$. Then following Example 1.32, a Real graded $S$-central extension of $X$ is a triple $(\tilde{\Gamma}, Y^{[2]}, \delta)$, where $Y$ is a Real space together with a continuous locally split Real open map $\pi : Y \to X$, and $\delta : Y^{[2]} \to \mathbb{Z}_2$ is a Real morphism.

In particular, suppose $\rho$ is trivial. Then, by Proposition 1.52, giving a Real graded $S^1$-central extension of $X$ amounts to giving a real bundle gerbe

\[
\begin{array}{ccc}
\mathbb{Z}_2 & \longrightarrow & \tilde{\Gamma} \\
 & & \downarrow \\
Y^{[2]} & \longrightarrow & Y \\
 & \downarrow^\pi & \\
 & & X
\end{array}
\]

in the sense of Mathai, Murray, and Stevenson [14], together with an augmentation $\delta : Y^{[2]} \to \mathbb{Z}_2$.

### 1.7. Functoriality of $\widehat{\text{Ext}}^R(\cdot, S)$

The aim of this subsection is to show that $\widehat{\text{Ext}}^R(\cdot, S)$ is functorial in the category $\mathfrak{R}\mathfrak{S}$, and hence that the group $\widehat{\text{Ext}}^R(\mathcal{G}, S)$ invariant under Morita equivalence. To do this, we will need the following:

**Proposition 1.54.** Let $\mathcal{G} \xrightarrow{\rho} X$ be a Real groupoid. Then, there is an isomorphism of abelian groups

\[
\widehat{\text{Ext}}^R(\mathcal{G}, S) \cong \lim_{\mathcal{U}} \widehat{\text{Ext}}^R(\mathcal{G}[\mathcal{U}], S).
\]

Before giving the proof of this proposition, we have to describe the sum in the inductive limit

$$\lim_{\mathcal{U}} \widehat{\text{Ext}}^R(\mathcal{G}[\mathcal{U}], S).$$

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two Real open covers of $X$, and let $\mathcal{T}_i = (\mathcal{G}_i, \mathcal{G}^i[\mathcal{U}_i], \delta_i)$ be Real graded $S$-groupoids over $\mathcal{G}[\mathcal{U}_i]$, $i = 1, 2$. Let $(\mathcal{V}, f_\mathcal{V}) \in \Omega(\mathcal{G}[\mathcal{U}_1], \mathcal{G}[\mathcal{U}_2])$ be the unique class corresponding to the Real Morita equivalence $Z_{\mathcal{U}_1}^{-1} \circ Z_{\mathcal{U}_2}$ from $\mathcal{G}[\mathcal{U}_1]$ to $\mathcal{G}[\mathcal{U}_2]$. $\mathcal{V}$ is a Real open cover of $X$ containing $\mathcal{U}_1$, and

$$f_\mathcal{V} : \mathcal{G}[\mathcal{V}] \to \mathcal{G}[\mathcal{U}_2]$$
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is a Real strict morphism. Denote by \( \iota_{V,U} \) the canonical Real morphism \( G[V] \to G[U] \). Then, the tensor product of \( T_1 \) and \( T_2 \) is

\[
T_1 \otimes T_2 := \iota_{V,U}^* T_1 \otimes f_V^* T_2,
\]

which defines a Real graded S-groupoids over the Real groupoid \( G[V] \).

**Proof of Proposition 1.54.** For a Real graded S-central extension \( E = (\tilde{\Gamma}, \Gamma, \delta, P) \) of \( G \), let \( (V, f_V) \in \Omega (G, \Gamma) \) be the isomorphism in \( \mathfrak{R}S_\Omega \) corresponding to the Morita equivalence \( P^{-1} : G \to \Gamma \). Setting

\[
T_E := S \longrightarrow f_V^* \tilde{\Gamma} \xrightarrow{f_V \pi} G[V] \xrightarrow{\delta \circ f_V} Z_2
\]

we get a Real graded S-groupoid over \( G[V] \). It is not hard to check that this provides us the desired isomorphism of abelian groups; the inverse is given by the formula

\[
E_T := (\tilde{\mathcal{G}}, G[U], \delta, Z_U)
\]

for a Real graded S-twist \( T = (\tilde{\mathcal{G}}, G[U], \delta) \). \( \square \)

From this proposition, it is now possible to define the pull-back of a Real graded S-central extension via a Real generalized morphism. More precisely, we have

**Definition and Proposition 1.55.** Let \( G \) and \( G' \) be Real groupoids, and let \( Z : G' \to G \) be a Real generalized morphism. Let \( E = (\tilde{\Gamma}, \Gamma, \delta, P) \) be a representative in \( \hat{\text{Ext}}R(G, S) \), and \( T_E = (f_V^* \Gamma, G[V], \delta \circ f_V) \) its image in \( \lim_{\rightarrow U} \hat{\text{ext}}R(G[U], S) \) (see the proof of Proposition 1.54). Let

\[
(\mathcal{W}, f_W) \in \Omega (G', G[V])
\]

be the morphism in \( \mathfrak{R}S_\Omega \) corresponding to the Real generalized morphism \( Z_{v'} \circ Z : G' \to G[V] \). Then

\[
Z^* E := E_{f_W T_E}
\]

is a Real graded S-central extension of the Real groupoid \( G' \); it is called the pull-back of \( E \) along \( Z \).

Now the following is straightforward.

**Corollary 1.56.** There is a contravariant functor

\[
\hat{\text{Ext}}R(\cdot, S) : \mathfrak{R}S \to \mathfrak{Ab},
\]

which sends a Real groupoid \( G \) to the abelian group \( \hat{\text{Ext}}R(G, S) \). In particular, \( \hat{\text{Ext}}R(G, S) \) is invariant under Morita equivalences.
2. Real Čech cohomology

2.1. Real simplicial spaces. We start by recalling some preliminary notions. For each zero integer \( n \in \mathbb{N} \), we set \([n] = \{0, \ldots, n\}\). Recall [21] that the simplicial (resp. pre-simplicial) category \( \Delta \) (resp. \( \Delta' \)) is the category whose objects are the sets \([n]\), and whose morphisms are the nondecreasing (resp. increasing) maps \( f : [m] \to [n] \). For \( n \in \mathbb{N} \), we denote by \( \Delta^{(N)} \) the \( N \)-truncated full subcategory of \( \Delta \) whose objects are those \([k]\) with \( k \leq N \).

Definition 2.1. A Real simplicial (resp. pre-simplicial, \( N \)-simplicial) topological space consists of a contravariant functor from \( \Delta \) (resp. \( \Delta' \), \( \Delta^{(N)} \)) to the category \( \mathcal{RTop} \) whose objects are topological Real spaces and morphisms are continuous Real maps. A morphism of Real simplicial (resp. pre-simplicial, . . . ) spaces is a morphism of such functors.

More concretely, a Real (pre-)simplicial space is given by a family \((X_\bullet, \rho_\bullet) = (X_n, \rho_n)_{n \in \mathbb{N}}\) of topological Real spaces, and for every map \( f : [m] \to [n] \) we are given a continuous Real map (called face or degeneracy map depending which of \( m \) and \( n \) is larger) \( \tilde{f} : (X_n, \rho_n) \to (X_m, \rho_m) \), satisfying the relation \( \tilde{f} \circ g = \tilde{g} \circ \tilde{f} \) whenever \( f \) and \( g \) are composable.

Definition 2.2. Let \((X_\bullet, \rho_\bullet)\) be a Real simplicial space. For any \( N \in \mathbb{N} \), the \( N \)-skeleton of \((X_\bullet, \rho_\bullet)\) is the Real simplicial space \((X_\bullet, \rho_\bullet)^N\) "of dimension \( N \)"; that is, \((X_n, \rho_n)^N = (X_n, \rho_n)\) for \( n \leq N \), and \((X_n, \rho_n)^N = (X_N, \rho_N)\) for all \( n \geq N + 1 \).

Let \( \varepsilon^n_i : [n-1] \to [n] \) be the unique increasing injective map that avoids \( i \), and let \( \eta^n_i : [n+1] \to [n] \) be the unique nondecreasing surjective map such that \( i \) is reached twice; that is,

\[
\varepsilon^n_i(k) = \begin{cases} 
  k, & \text{if } k \leq i - 1, \\
  k + 1, & \text{if } k \geq i,
\end{cases}
\]

\[
\eta^n_i(k) = \begin{cases} 
  k, & \text{if } k \leq i; \\
  k - 1, & \text{if } k \geq i + 1.
\end{cases}
\]

We will omit the superscript \( n \) if there is no ambiguity.

If \((X_\bullet, \rho_\bullet)\) is a Real simplicial space, it is straightforward to check that the face and degeneracy maps

\[
\varepsilon^n_i : (X_n, \rho_n) \to (X_{n-1}, \rho_{n-1}),
\]

\[
\eta^n_i : (X_n, \rho_n) \to (X_{n+1}, \rho_{n+1}),
\]

\( i = 0, \ldots, n \) satisfy the following simplicial identities:

\[
\varepsilon^{n-1}_i \varepsilon^n_j = \varepsilon^{n-1}_{i-j} \varepsilon^n_i \text{ if } i \leq j - 1,
\]

\[
\eta^{n+1}_i \eta^n_j = \eta^{n+1}_{j+1} \eta^n_i \text{ if } i \leq j.
\]
\[ \varepsilon^n_i \eta^n_j = \tilde{\eta}^{n-1}_j \varepsilon^n_i \text{ if } i \leq j - 1, \]
\[ \varepsilon^n_i \eta^n_j = \tilde{\eta}^{n-1}_j \varepsilon^n_{i-1} \text{ if } i \geq j + 2, \]
\[ \varepsilon^n_j \eta^n_j = \varepsilon^{n+1}_j \eta^n_j = \text{Id}_{X_n}. \]

Conversely, let \((X_n, \rho_n)_{n \in \mathbb{N}}\) be a sequence of topological Real spaces together with maps satisfying (2.2). Then thanks to [13, Theorem 5.2], there is a unique Real simplicial structure on \((X_\bullet, \rho_\bullet)\) such that \(\varepsilon_i\) and \(\tilde{\eta}_i\) are the face and degeneracy maps respectively.

**Example 2.3** (Cf. [24, §2.3]). Consider the pair groupoid

\[ [n] \times [n] \xrightarrow{\varepsilon} [n]; \]

that is, the product is \((i, j)(j, k) := (i, k)\) and the inverse of \((i, j)\) is \((j, i)\).

If \((G, \rho)\) is a topological Real groupoid, we define

\[ G_n := \text{Hom}([n] \times [n], G) \]

as the space of strict morphisms from the groupoid \([n] \times [n] \xrightarrow{\varepsilon} [n]\) to \(G \xrightarrow{\varepsilon} X\). We obtain a Real structure on \(G_n\) by defining \(\rho_n(\varphi) := \rho \circ \varphi\), for \(\varphi \in G_n\). Any \(f \in \text{Hom}_\Delta([m], [n])\) (or \(f \in \text{Hom}_\Delta([m], [n])\)) naturally gives rise to a strict morphism \(f \times f : [m] \times [m] \rightarrow [n] \times [n]\), which, in turn, induces a Real map \(\tilde{f} : (G_n, \rho_n) \rightarrow (G_m, \rho_m)\) given by \(\tilde{f}(\varphi) := \varphi \circ (f \times f)\) for \(\varphi \in G_n\). Hence, we obtain a Real simplicial space \((G_\bullet, \rho_\bullet)\).

Notice that the groupoid

\[ [n] \times [n] \xrightarrow{\varepsilon} [n] \]

is generated by elements \((i - 1, i), 1 \leq i \leq n\); indeed, given an element \((i, j) \in [n] \times [n]\), we can suppose that \(i \leq j\) (otherwise, we take its inverse \((j, i)\)), and then \((i, j) = (i, i + 1) \ldots (j - 1, j)\). It turns out that any strict morphism \(\varphi : [n] \times [n] \rightarrow G\) is uniquely determined by its images \(\varphi(i - 1, i) \in G\); hence, the well-defined Real map

\[ G_n \rightarrow G^{(n)}, \varphi \mapsto (g_1, \ldots, g_n), \]

where \(g_i := \varphi(i - 1, i), 1 \leq i \leq n\), and

\[ G^{(n)} := \{(h_1, \ldots, h_n) \mid s(h_i) = r(h_{i-1}), i = 1, \ldots, n\}, \]

identifies \((G_n, \rho_n)\) with \((G^{(n)}, \rho^{(n)})\), where \(\rho^{(n)}\) is the obvious Real structure on the fibred product \(G^{(n)}\). Therefore, using this identification, the face maps \(\varepsilon^n_i : (G_n, \rho_n) \rightarrow (G_{n-1}, \rho_{n-1})\) of \((G_\bullet, \rho_\bullet)\) are given by:

\[ \varepsilon^n_i(g_1, g_2, \ldots, g_n) = (g_2, \ldots, g_n), \]
\[ \varepsilon^n_i(g_1, g_2, \ldots, g_n) = (g_1, \ldots, g_i g_{i+1}, \ldots, g_n), 1 \leq i \leq n - 1, \]
\[ \varepsilon^n_i(g_1, g_2, \ldots, g_n) = (g_1, \ldots, g_{n-1}), \]

for \(i = 1, \ldots, n\) respectively.
and for \( n = 1 \), by \( \tilde{\varepsilon}_{0}^{1}(g) = s(g), \tilde{\varepsilon}_{1}^{1}(g) = r(g) \); while the degeneracy maps 
\[
\tilde{\eta}_{i}^{n} : (S_{n}, \rho_{n}) \rightarrow (S_{n+1}, \rho_{n+1})
\]
are given by:
\[
\tilde{\eta}_{i}^{n}(g_{1}, g_{2}, \ldots, g_{n}) = (r(g_{1}), g_{1}, \ldots, g_{n}),
\]
\[
\tilde{\eta}_{i}^{n}(g_{1}, g_{2}, \ldots, g_{n}) = (g_{1}, \ldots, s(g_{i}), g_{i+1}, \ldots, g_{n}), \quad 1 \leq i \leq n,
\]
and \( \tilde{\eta}_{0}^{0} : S_{0} \rightarrow S_{1} \) is the unit map of the Real groupoid.

Now for \( n \in \mathbb{N} \), we define the space \((E\mathcal{G})_{n}\) of \((n+1)\)-tuples of elements of \( \mathcal{G} \) that map to the same unit; i.e.,
\[
(E\mathcal{G})_{n} := \{(\gamma_{0}, \ldots, \gamma_{n}) \in S^{n+1} \mid r(\gamma_{0}) = \cdots = r(\gamma_{n})\}.
\]
Suppose we are given \((g_{1}, \ldots, g_{n}) \in \mathcal{G}_{n}\). Then we can choose an \((n+1)\)-tuple \((\gamma_{0}, \ldots, \gamma_{n}) \in (E\mathcal{G})_{n}\) such that \( g_{i} = \gamma_{i-1}^{-1}\gamma_{i} \) for each \( i = 1, \ldots, n \). If \((\gamma_{0}', \ldots, \gamma_{n}')\) is another \((n+1)\)-tuple satisfying these identities, then
\[
s(\gamma_{i}') = s((\gamma_{i-1}')^{-1}\gamma_{i}') = s(\gamma_{i-1}'\gamma_{i}) = s(\gamma_{i}),
\]
for all \( i = 1, \ldots, n \), and that means that there exists a unique \( g \in \mathcal{G} \), such that \( s(g) = r(\gamma_{i}) \) and \( \gamma_{i}' = g \cdot \gamma_{i} \). This hence gives us a well-defined injective map
\[
\mathcal{G}_{n} \rightarrow (E\mathcal{G})_{n}/\sim, \quad (g_{1}, \ldots, g_{n}) \mapsto [\gamma_{0}, \ldots, \gamma_{n}],
\]
where \( (\gamma_{0}, \ldots, \gamma_{n}) \sim (g \cdot \gamma_{0}, \ldots, g \cdot \gamma_{n}) \). Moreover, this map is surjective, for if \((\gamma_{0}, \ldots, \gamma_{n}) \in (E\mathcal{G})_{n}\), one can consider morphisms \( g_{i} \) from \( s(\gamma_{i}) \) to \( s(\gamma_{i-1}) \), \( i = 1, \ldots, n \), so that we have
\[
\gamma_{1} = \gamma_{0}g_{1}, \quad \gamma_{2} = \gamma_{1}g_{2} = \gamma_{0}g_{1}g_{2}, \ldots, \gamma_{n} = \gamma_{0}g_{1} \cdots g_{n},
\]
and then
\[
[\gamma_{0}, \ldots, \gamma_{n}] = [r(g_{1}), g_{1}, g_{1}g_{2}, \ldots, g_{1} \cdots g_{n}]
\]
which gives the inverse \((E\mathcal{G})_{n}/\sim \ni [\gamma_{0}, \ldots, \gamma_{n}] \mapsto (g_{1}, \ldots, g_{n}) \in \mathcal{G}_{n}\). It hence turns out that we can identify \( \mathcal{G}_{n} \) with the quotient \((E\mathcal{G})_{n}\). Note that the quotient space \((E\mathcal{G})_{n}/\sim \) naturally inherits the Real structure \( \rho_{n+1} \) and that the isomorphism defined above is compatible with the Real structures.

Henceforth, an element of \( \mathcal{G}_{n} \) will be represented by a vector
\[
\vec{g} = (g_{1}, \ldots, g_{n}),
\]
where we view \( \vec{g} \) as a morphism \([n] \times [n] \rightarrow \mathcal{G} \), and \( g_{i} = \vec{g}(i-1, i) \), \( i = 1, \ldots, n \), or \( \vec{g} = [\gamma_{0}, \ldots, \gamma_{n}] \) as a class in \((E\mathcal{G})_{n}/\sim\). For the first picture, if \( f \in \text{Hom}_{\mathcal{G}}([m], [n]) \), then the Real face/degeneracy map \( \vec{f} : (\mathcal{G}_{n}, \rho_{n}) \rightarrow (\mathcal{G}_{m}, \rho_{m}) \) is given by:
\[
\vec{f}(\vec{g}) = (\vec{g} (f(0), f(1)), \ldots, \vec{g} (f(m-1), f(m))).
\]
For instance, if \( f \) in injective, then
\[
\vec{g} (f(i-1), f(i)) = \vec{g} (f(i-1), f(i-1) + 1) \cdots \vec{g} (f(i) - 1, f(i))
\]
for \( f(i) \geq 1 \), and thus
\[
\vec{f}(\vec{g}) = (g_{f(0)+1} \cdots g_{f(1)}, \ldots, g_{f(m-1)+1} \cdots g_{f(m)}).
\]
However, the second picture offers a more general formula for the face and degeneracy maps; roughly speaking, for any \( f \in \text{Hom}_\Delta([m],[n]) \), we have \( g(i,j) = \gamma_{i}^{-1}\gamma_{j} \) for every \((i,j) \in [n] \times [n]\). In particular,
\[
g(f(k-1),f(k)) = \gamma_{f(k-1)}^{-1}\gamma_{f(k)},
\]
for every \( k \in [m] \); then (2.5) gives:
\[
\hat{f}(g) = [\gamma_{f(0)}, \ldots, \gamma_{f(m)}].
\]

2.2. Real sheaves on Real simplicial spaces. In this subsection we closely follow [21, §3] to study Real sheaves on Real (pre-)simplicial spaces. We start by introducing some preliminary notions.

Let \( \mathcal{C} \) be a topological category. We define the category \( \mathcal{C}_{R} \) by setting:
- \( \text{Ob}(\mathcal{C}_{R}) \) consists of triples \((A, \sigma_A, A')\), where \( A, A' \in \text{Ob}(\mathcal{C}) \) and \( \sigma_A \in \text{Hom}_{\mathcal{C}}(A,A') \);
- \( \text{Hom}_{\mathcal{C}_{R}}((A, \sigma_A, A'),(B, \sigma_B, B')) \) consists of pairs \((f, \hat{f})\) of morphisms \( f : A \rightarrow B, \hat{f} : A' \rightarrow B' \) in \( \mathcal{C} \) such that the diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\sigma_A} & & \downarrow{\sigma_B} \\
A' & \xrightarrow{\hat{f}} & B'
\end{array}
\]

commute.

Now, let \( \phi : \mathcal{C} \rightarrow \mathcal{C} \) be a functor. Then we define the subcategory \( \mathcal{C}_\phi \) of \( \mathcal{C}_{R} \) whose objects are pairs \((A, \phi(A))\), where \( A \in \text{Ob}(\mathcal{C}) \), and in which a morphism from \((A, \phi(A))\) to \((B, \phi(B))\) is a pair \((f, \hat{f})\) of morphisms \( f : A \rightarrow B, \hat{f} : \phi(A) \rightarrow \phi(B) \) such that \( \hat{f} \circ \phi = \phi \circ f \). A fundamental example of this is the category \( \mathfrak{OB}(X) \) of open subsets of a given topological Real space \((X, \rho)\). Recall that objects of this category are the collection of the open sets \( U \subset X \), and morphisms are the canonical injections \( V \hookrightarrow U \) when \( V \subset U \). Given such a Real space \((X, \rho)\), the map \( \rho \) induces a functor (which is an isomorphism) \( \rho : \mathfrak{OB}(X) \rightarrow \mathfrak{OB}(X) \) given by
\[
\begin{pmatrix} V \xleftarrow{\iota} U \end{pmatrix} \mapsto \begin{pmatrix} \rho(V) \xleftarrow{\rho \circ \iota \circ \rho} \rho(U) \end{pmatrix}.
\]

Definition 2.4 (Real presheaves). Let \((X, \rho)\) be a topological Real space, and let \( \mathcal{C} \) be a topological category. A Real presheaf \((\mathfrak{F}, \sigma)\) on \((X, \rho)\) with values in \( \mathcal{C} \) is a contravariant functor from \( \mathfrak{OB}(X)_{\rho} \) to \( \mathcal{C}_{R} \); a morphism of Real presheaves is a morphism of such functors.

Specifically, from the fact that \( \rho : X \rightarrow X \) is a homeomorphism and from the canonical properties of the injections \( V \hookrightarrow U \) of open sets \( V \subset U \subset X \), a Real presheaf on \((X, \rho)\) with values in \( \mathcal{C} \) assigns to each open subset \( U \subset X \) a triple \((\mathfrak{F}(U), \sigma_U, \mathfrak{F}(\rho(U)))\), where \( \mathfrak{F}(U), \mathfrak{F}(\rho(U)) \) are objects of \( \mathcal{C} \), and \( \sigma_U \in \text{Isom}_{\mathcal{C}}(\mathfrak{F}(U), \mathfrak{F}(\rho(U))) \), and for \( V \subset U \) we are given two morphisms
\( \varphi_{V,U} : \mathfrak{G}(U) \rightarrow \mathfrak{G}(V) \) and \( \varphi_{\rho(V),\rho(U)U} : \mathfrak{G}(\rho(U)) \rightarrow \mathfrak{G}(\rho(V)) \), called the restriction morphisms, such that:

- \( \varphi_{U,U} = \text{Id}_\mathfrak{G}(U) \).
- \( \sigma_V \circ \varphi_{V,U} = \varphi_{\rho(V),\rho(U)} \circ \sigma_U \).
- \( \varphi_{W,U} = \varphi_{W,V} \circ \varphi_{V,U}, \) and \( \varphi_{\rho(W),\rho(U)} = \varphi_{\rho(W),\rho(V)} \circ \varphi_{\rho(V),\rho(U)} \).

A morphism of Real presheaves \( \phi : (\mathfrak{G}, \sigma^\emptyset) \rightarrow (\mathfrak{G}, \sigma^\emptyset) \) is then a family of \( \phi_U \in \text{Hom}_C(\mathfrak{G}(U), \mathfrak{G}(U)) \) such that, for all pairs of open sets \( U, V \) with \( V \subset U \), the diagrams below commute:

\[
\begin{array}{c}
\mathfrak{G}(\rho(U)) \xrightarrow{\sigma^\emptyset_U} \mathfrak{G}(U) \xrightarrow{\varphi^\emptyset_U} \mathfrak{G}(V) \\
\downarrow{\phi_U} & & \downarrow{\phi_V} \\
\mathfrak{G}(\rho(U)) \xrightarrow{\sigma^\emptyset_U} \mathfrak{G}(U) \xrightarrow{\varphi^\emptyset_U} \mathfrak{G}(V).
\end{array}
\]

As in the standard case, if \( (\mathfrak{G}, \sigma) \) is a Real presheaf over \( X \), and if \( U \) is an open subset of \( X \), an element \( s \in \mathfrak{G}(U) \) is called a section of \( (\mathfrak{G}, \sigma) \) on \( U \), and for \( x \in X \). If \( V \) is an open subset of \( U \), and \( s \in \mathfrak{G}(U) \), one often writes \( s|_V \) for \( \varphi_{V,U}(s) \).

**Definition 2.5** ([10, Definition 2.2]). A Real sheaf over \( (X, \rho) \) with values in \( C \) is a Real presheaf \( (\mathfrak{G}, \sigma) \) satisfying the following conditions:

1. For any open set \( U \subset X \), any open cover \( U = \bigcup_{i \in I} U_i \), any section \( s \in \mathfrak{G}(U) \), \( s|_{U_i} = 0 \) for all \( i \) implies \( s = 0 \).
2. For any open set \( U \subset X \), any open cover \( U = \bigcup_{i \in I} U_i \), any family of sections \( s_i \in \mathfrak{G}(U_i) \) satisfying \( s_i|_{U_{ij}} = s_j|_{U_{ij}} \) for all nonempty intersection \( U_{ij} \), there exists \( s \in \mathfrak{G}(U) \) such that \( s|_{U_i} = s_i \) for all \( i \).

A morphism of Real sheaves is a morphism of the underlying presheaves. We denote by \( \mathfrak{C}_{\mathbb{R}}(X) \) (or simply by \( \mathfrak{S}_{\mathbb{R}}(X) \) if there is no risk of confusion) for the category of Real sheaves on \( (X, \rho) \) with values in \( C \).

Notice that if \( (\mathfrak{G}, \sigma) \) is a Real sheaf (resp. presheaf) on \( (X, \rho) \), then \( \mathfrak{G} \) is a sheaf (resp. presheaf) on \( X \) in the usual sense. Recall that the stalk of \( \mathfrak{G} \) at a point \( x \in X \), denoted by \( \mathfrak{G}_x \), is the direct limit of the direct system \((\mathfrak{G}(U), \varphi_{V,U})\) where \( U \) runs along the family of open neighborhoods of \( x \); i.e.,

\[
\mathfrak{G}_x := \lim_{x \in U} \mathfrak{G}(U).
\]

The image of a section \( s \in \mathfrak{G}(U) \) in \( \mathfrak{G}_x \) by the canonical morphism

\[
\mathfrak{G}(U) \rightarrow \mathfrak{G}_x
\]

(where \( x \in U \)) is called the germ of \( s \) at \( x \) and denoted by \( s_x \).

Note that if \( U \) is an open neighborhood of \( x, \rho(U) \) is an open neighborhood of \( \rho(x) \), and the isomorphism \( \sigma_U : \mathfrak{G}(U) \ni s \mapsto \sigma_U(s) \in \mathfrak{G}(\rho(U)) \) extends to an isomorphism \( \sigma_x : \mathfrak{G}_x \rightarrow \mathfrak{G}_{\rho(x)} \), defined by \( \sigma_x(s_x) = (\sigma_U(s))_{\rho(x)} \), whose
inverse is $\sigma_{\rho(x)}$. We thus have a well-defined 2-periodic isomorphism, also denoted by $\sigma$, on the topological \footnote{Recall that if $\mathcal{F}$ is a presheaf over $X$, any section $s \in \mathcal{F}(U)$ induces a map $[s] : U \to \bigsqcup_x \mathcal{F}_x$, $y \mapsto s_y$. We give $\mathcal{T} := \bigsqcup_x \mathcal{F}_x$ the largest topology such that all the maps $[s]$ are continuous. On the other hand, associated to $\mathcal{F}$, there is a sheaf $\tilde{\mathcal{F}}$ given by $\tilde{\mathcal{F}}(U) := \Gamma(U, \mathcal{T})$, and we have that $\mathcal{F}(U) \cong \mathcal{T}(U, \mathcal{T})$ if and only if $\mathcal{F}$ is a sheaf. Then, given a Real presheaf $(\mathcal{F}, \sigma)$, one can define its associated Real sheaf in the same fashion.} space $\mathcal{T} := \bigsqcup_x \mathcal{F}_x$, given by
\begin{equation}
\sigma : \mathcal{T} \to \mathcal{T}, \ (x, s_x) \mapsto (\rho(x), \sigma(x)(s_x))
\end{equation}
which gives a Real space $(\mathcal{F}, \sigma)$.

**Example 2.6.** Let $(X, \rho)$ be a Real space. Then the space $C(X)$ of continuous complex valued functions on $X$ defines a Real sheaf of abelian groups on $(X, \rho)$ by $(U, \rho(U)) \mapsto (C(U), \tilde{\rho}_U, C(\rho(U)))$, where $\tilde{\rho}_U(f)(\rho(x)) := \tilde{f}(x)$.

**Definition 2.7** (Pushforward, pullback). Let $(X, \rho)$, $(Y, \varrho)$ be topological Real spaces, $f : (Y, \varrho) \to (X, \rho)$ a continuous Real map. Suppose that $(\mathcal{F}, \sigma)$ and $(\mathcal{G}, \varsigma)$ are Real sheaves on $(X, \rho)$ and $(Y, \varrho)$ respectively, with values in the same category $\mathcal{C}$.

(i) The **pushforward** of $(\mathcal{G}, \varsigma)$ by $f$, denoted by $(f_\ast \mathcal{G}, f_\ast \varsigma)$, is the Real sheaf on $(X, \rho)$ defined by the contravariant functor:
\begin{equation}
\mathcal{DB}(X, \rho) \to \mathcal{C}_R, \ (U, \rho(U)) \mapsto (f_\ast \mathcal{G}(U), f_\ast \varsigma_U, f_\ast \mathcal{G}(\rho(U))),
\end{equation}
where $f_\ast \mathcal{G}(U) := \mathcal{G}(f^{-1}(U))$, $f_\ast \varsigma_U := \varsigma_{f^{-1}(U)}$, and $f_\ast \mathcal{G}(\rho(U)) = \mathcal{G}(f^{-1}(\rho(U))) \cong \mathcal{G}(\varrho(f^{-1}(U)))$.

(ii) The **pullback** of $(\mathcal{F}, \sigma)$ along $f$, denoted by $(f^\ast \mathcal{F}, f^\ast \sigma)$, is the Real sheaf on $(Y, \varrho)$ associated to the Real presheaf defined by:
\begin{equation}
\mathcal{DB}(Y, \varrho) \to \mathcal{C}_R, \ (V, \varrho(V)) \mapsto (f^\ast \mathcal{F}(V), f^\ast \sigma_V, f^\ast \mathcal{F}(\varrho(V))),
\end{equation}
where $f^\ast \mathcal{F}(V) := \lim_{f(V) \subset U \subset X} \mathcal{F}(U)$, and $f^\ast \sigma_V : f^\ast \mathcal{F}(V) \to f^\ast \mathcal{F}(\varrho(V))$ is the morphism in $\mathcal{C}$ extending functorially $\sigma_U : \mathcal{F}(U) \to \mathcal{F}(\rho(U))$ along the family of open neighborhoods of $f(V)$ in $X$.

It immediately follows from this definition that we have a covariant functor:
\begin{equation}
\mathfrak{NTop} \to \mathfrak{RSh},
\begin{pmatrix}
(Y, \varrho) \\
\downarrow f
\end{pmatrix}
\mapsto
\begin{pmatrix}
\mathcal{F}(Y) \\
\downarrow f_\ast
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
\mathcal{F}(X) \\
\downarrow f_\ast
\end{pmatrix},
\end{equation}
and a contravariant functor:
\begin{equation}
\mathfrak{NTop} \to \mathfrak{RSh},
\begin{pmatrix}
(Y, \varrho) \\
\downarrow f
\end{pmatrix}
\mapsto
\begin{pmatrix}
\mathcal{F}(X) \\
\downarrow f^\ast
\end{pmatrix}
\longrightarrow
\begin{pmatrix}
\mathcal{F}(Y) \\
\downarrow f^\ast
\end{pmatrix},
\end{equation}
where \( \mathfrak{RSh} \) is the category whose objects are the categories of Real sheaves on given Real spaces and morphisms are functors of such categories.

We will also need the following proposition.

**Proposition 2.8.** Let \( f : (Y, \varrho) \to (X, \rho) \) be a a continuous Real map. Suppose that \((\mathfrak{F}, \sigma)\) and \((\mathfrak{G}, \zeta)\) are Real sheaves on \((X, \rho)\) and on \((Y, \varrho)\) respectively, with values in the same category \( \mathcal{C} \). Then

\[
\text{Hom}_{\mathfrak{RSh}(X)}((\mathfrak{F}, \sigma), (f_\ast \mathfrak{G}, f_\ast \zeta)) \cong \text{Hom}_{\mathfrak{RSh}(Y)}((f^\ast \mathfrak{F}, f^\ast \sigma), (\mathfrak{G}, \zeta)).
\]

**Proof.** The proof is the same as in the general case where Real structures are not concerned (see for instance [10, Proposition 2.3.3]). \( \square \)

**Definition 2.9.** Given a continuous Real map \( f : (Y, \varrho) \to (X, \rho) \) and Real sheaves \((\mathfrak{F}, \sigma)\) and \((\mathfrak{G}, \zeta)\) as above, we define the set \( \text{Hom}_f(\mathfrak{F}, \mathfrak{G})_{\sigma, \zeta} \) of Real \( f \)-morphisms from \((\mathfrak{F}, \sigma)\) to \((\mathfrak{G}, \zeta)\) to be

\[
\text{Hom}_{\mathfrak{RSh}(X)}((\mathfrak{F}, \sigma), (f_\ast \mathfrak{G}, f_\ast \zeta)) = \text{Hom}_{\mathfrak{RSh}(Y)}((f^\ast \mathfrak{F}, f^\ast \sigma), (\mathfrak{G}, \zeta)).
\]

**Definition 2.10.** Let \((X_\bullet, \rho_\bullet)\) be a Real simplicial (resp. pre-simplicial) space. A Real sheaf on \((X_\bullet, \rho_\bullet)\) is a family \((\mathfrak{F}_n, \sigma_n)\) such that \((\mathfrak{F}_n, \sigma_n)\) is a Real sheaf on \((X_n, \rho_n)\) for all \( n \), and such that for each morphism \( f : [m] \to [n] \) in \( \Delta \) (resp. \( \Delta' \)) we are given Real \( \hat{f} \)-morphisms \( \hat{f}^\ast \in \text{Hom}_f(\mathfrak{F}_m, \mathfrak{F}_n)_{\sigma_m, \sigma_n} \) such that

\[
\hat{f}^\ast \circ \hat{g}^\ast = \hat{f}_\ast \circ \hat{g}^\ast,
\]
whenever \( f \) and \( g \) are composable.

One can use the definition of the push-forward to give a concrete interpretation of this definition. Roughly speaking, a sequence \((\mathfrak{F}_n, \sigma_n)\) is a Real sheaf on a Real simplicial (resp. pre-simplicial, \ldots) space \((X_\bullet, \rho_\bullet)\), if for a given morphism \( f : [m] \to [n] \) in \( \Delta \) (resp. \( \Delta' \), \ldots), then for any pair of open sets \( U \subset X_n \) and \( V \subset X_m \) such that \( \hat{f}(U) \subset V \) there is a restriction map \( \hat{f}^\ast : \mathfrak{F}_n(V) \to \mathfrak{F}_m(U) \) such that the diagram

\[
\begin{array}{ccc}
\mathfrak{F}_m(V) & \xrightarrow{\hat{f}^\ast} & \mathfrak{F}_n(U) \\
\sigma_m^\ast & \downarrow & \sigma_n^\ast \\
\mathfrak{F}_m(\rho(V)) & \xrightarrow{\hat{f}_\ast} & \mathfrak{F}_n(\rho(U))
\end{array}
\]

commutes, and \( \hat{f}^\ast \circ \hat{g}^\ast = \hat{f}_\ast \circ \hat{g}^\ast : \mathfrak{F}_k(W) \to \mathfrak{F}_n(U) \) whenever \( \hat{g}(V) \subset W \subset X_k \).

Morphisms of Real sheaves over \((X_\bullet, \rho_\bullet)\) are defined in the obvious way; we denote by \( \mathfrak{Sh}_{\rho_\bullet}(X_\bullet) \) for the category of Real sheaves over \((X_\bullet, \rho_\bullet)\).
2.3. Real \( \mathcal{G} \)-sheaves and reduced Real sheaves.

Definition 2.11.
(i) A Real space \((Y, \varrho)\) is said to be \(\text{étale} \) over \((X, \rho)\) if there exists an \(\text{étale} \) Real map \(f : (Y, \varrho) \rightarrow (X, \rho)\); that is to say, every point \(y \in Y\) has an open neighborhood \(V\) such that \(f_V : V \rightarrow U\) is homeomorphism, where \(U\) in an open neighborhood of \(f(y)\) in \(X\).
(ii) A Real groupoid \((\mathcal{G}, \rho)\) is \(\text{étale} \) if the range (equivalently the source) map is \(\text{étale} \).
(iii) A morphism \(\pi \bullet : (Y \bullet, \varrho \bullet) \rightarrow (X \bullet, \rho \bullet)\) of Real (pre-)simplicial spaces is \(\text{étale} \) if for all \(n\), \(\pi_n : (Y_n, \varrho_n) \rightarrow (X_n, \rho_n)\) is \(\text{étale} \).

Example 2.12. Any Real sheaf \((\mathcal{F}, \sigma)\) on \((X, \rho)\) can be viewed as an \(\text{étale} \) Real space over \((X, \rho)\). Indeed, considering the underlying topological Real space \((\mathcal{F}, \sigma)\), it is easy to check that the canonical projection \(\mathcal{F} \rightarrow X, (x, s_x) \mapsto x\) is an \(\text{étale} \) Real map.

Definition 2.13. Let \((\mathcal{G}, \rho)\) be a topological Real groupoid. A Real \(\mathcal{G}\)-sheaf (or an \(\text{étale} \) Real \(\mathcal{G}\)-space) is an \(\text{étale} \) Real space \((E_0, \nu_0)\) over \((X, \rho)\) equipped with a continuous Real \(\mathcal{G}\)-action.

We say that \((E_0, \nu_0)\) is an Abelian Real \(\mathcal{G}\)-sheaf if in addition it is an Abelian Real sheaf on \((X, \rho)\) such that the action \(\alpha_g : (E_0)_{s(g)} \rightarrow (E_0)_{r(g)}\) is a group homomorphism, for any \(g \in \mathcal{G}\).

A morphism of Real \(\mathcal{G}\)-sheaves \((E_0, \nu_0)\) and \((E'_0, \nu'_0)\) is a \(\mathcal{G}\)-equivariant continuous Real map \(\psi : (E_0, \nu_0) \rightarrow (E'_0, \nu'_0)\) such that \(p' \circ \psi = p\).

The category of Real \(\mathcal{G}\)-sheaves is denoted by \(\mathcal{B}_\rho \mathcal{G}\), and is called the classifying topos of \((\mathcal{G}, \rho)\).

Examples 2.14.
(1) Considering a Real space \((X, \rho)\) as a Real groupoid, a Real \(X\)-sheaf is the same thing as a Real sheaf over \((X, \rho)\); in other words we have \(\mathcal{B}_\rho X \cong \text{Sh}_\rho(X)\).
(2) If \((\mathcal{G}, \rho)\) is a Real group, then a Real \(\mathcal{G}\)-sheaf is just a Real space equipped with a continuous Real \(\mathcal{G}\)-action.

Lemma 2.15. Any generalized Real morphism \((Z, \tau) : (\Gamma, \varrho) \rightarrow (\mathcal{G}, \rho)\) induces a morphism of toposes
\[ Z^* : \mathcal{B}_\rho \mathcal{G} \rightarrow \mathcal{B}_\rho \Gamma. \]
Consequently, there is a contravariant functor
\[ \mathcal{B} : \mathcal{R}\mathcal{G} \rightarrow \mathcal{R}\mathcal{B}\mathcal{G}, \]
defined by
\[ ( (\Gamma, \varrho) \xrightarrow{(Z, \tau)} (\mathcal{G}, \rho) ) \mapsto ( \mathcal{B}_\rho \mathcal{G} \xrightarrow{Z^*} \mathcal{B}_\rho \Gamma ), \]
where $\mathcal{RG}$ is the category whose objects are classifying toposes of Real groupoids.

**Proof.** As noted in [15, 2.2] for the usual case, any Real morphism $f : (\Gamma, \varrho) \to (\mathcal{G}, \rho)$ gives rise to a functor $f^\ast : \mathcal{B}_\rho \mathcal{G} \to \mathcal{B}_\varrho \Gamma$. Indeed, if $(\mathcal{E}_0, \nu_0)$ is a Real $\mathcal{G}$-sheaf through an étale Real $\mathcal{G}$-map $p : (\mathcal{E}_0, \nu_0) \to (X, \rho)$, then we obtain a Real $\Gamma$-sheaf $(f^\ast \mathcal{E}_0, f^\ast \nu_0)$ by pulling back $(\mathcal{E}_0, \nu_0)$ along $f$; i.e., $f^\ast \mathcal{E}_0 = Y \times_f X, \mathcal{E}_0$, $f^\ast \nu_0 = \varrho \times \nu_0$, $f^\ast p(y, e) := y$, and the right Real $\Gamma$-action is $\gamma \cdot (s(\gamma), e) := (r(\gamma), f(\gamma) \cdot e)$ when $p(e) = s(f(\gamma))$. If $\psi : (\mathcal{E}_0, \nu_0) \to (\mathcal{E}_0', \nu_0')$ is a morphism of Real $\mathcal{G}$-sheaves, then the map $f^\ast \psi : (f^\ast \mathcal{E}_0, f^\ast \nu_0) \to (f^\ast \mathcal{E}_0', f^\ast \nu_0')$ defined by $f^\ast \psi(y, e) := (y, \psi(e))$ is obviously a morphism a Real $\Gamma$-sheaves. It follows that any $(U, f_U) \in \text{Hom}_{\mathfrak{RG}}((\Gamma, \varrho), (\mathcal{G}, \rho))$ gives rise to a covariant functor $f_U^\ast : \mathcal{B}_\rho \mathcal{G} \to \mathcal{B}_\varrho \Gamma[U]$. Now if $(Z, \tau)$ corresponds to $(U, f_U)$, and if as in the previous chapter, $\iota : \Gamma[U] \to \Gamma$ is the canonical Real morphism, then we can push forward $(f_U^\ast \mathcal{E}_0, f_U^\ast \nu_0)$ through $\iota$ to get a Real $\Gamma$-sheaf $(Z^\ast \mathcal{E}_0, Z^\ast \nu_0)$; i.e.,

$$ (2.17) \quad Z^\ast \mathcal{E}_0 := \iota_* f_U^\ast \mathcal{E}_0, $$

and the Real structure $Z^\ast \nu_0$ is the obvious one. \hfill $\Box$

**Lemma 2.16.** Let $(\mathcal{G}, \rho)$ be a topological Real groupoid. Then, a Real $\mathcal{G}$-sheaf canonically defines a Real sheaf over the Real simplicial space $(\mathcal{G}_n, \rho_n)_{n \in \mathbb{N}}$.

To prove this Lemma, we need some more preliminary notions.

**Definition 2.17.** ([21]) A morphism $\pi_* : (\mathcal{E}_*, \nu_*) \to (X_*, \rho_*)$ of Real simplicial spaces is called reduced if for all $m, n$ and for all $f \in \text{Hom}_\Delta([m], [n])$, the morphism $\tilde{f} : \mathcal{E}_m \to \mathcal{E}_n$ induces an isomorphism

$$ (\mathcal{E}_n, \nu_n) \cong (X_n \times_f X_m, \pi_m \mathcal{E}_m, \rho_n \times \nu_n). $$

In this case, we say that $(\mathcal{E}_*, \nu_*)$ is a reduced Real simplicial space over $(X_*, \rho_*)$.

Morphisms of reduced Real simplicial spaces over $(X_*, \rho_*)$ are defined in the obvious way.

**Definition 2.18** ([21]). We say that a Real sheaf $(\mathcal{T}_*, \sigma_*)$ over a Real simplicial space $(X_*, \rho_*)$ is reduced if for all $m, n$ and all $f \in \text{Hom}_\Delta([m], [n])$, $\tilde{f}^\ast \in \text{Hom} \left((\tilde{f}^\ast \mathfrak{T}_m, \tilde{f}^\ast \sigma m), (\mathfrak{T}_n, \sigma n)\right)$ is an isomorphism.

**Lemma 2.19** ([21, Lemma 3.5]). Let $(X_*, \rho_*)$ be a Real simplicial space. Then, there is a one-to-one correspondence between reduced Real sheaves over $(X_*, \rho_*)$ and reduced étale Real simplicial spaces over $(X_*, \rho_*)$.

**Proof.** Suppose that we are given a Real sheaf $(\mathcal{T}_*, \sigma_*)$ over the Real simplicial space $(X_*, \rho_*)$, and let $(\mathfrak{T}_n, \sigma_n)_{n \in \mathbb{N}}$ be its underlying sequence of topological Real spaces. We already know from Example 2.12 that each of the canonical projection maps $\pi_n : (\mathfrak{T}_n, \sigma_n) \to (X_n, \rho_n)$ is étale. Now suppose that
\((\mathfrak{S}^*, \sigma^*)\) is reduced; that is to say that for any morphism \(f \in \text{Hom}_\Delta([m],[n])\), and every open set \(V \subset X_m\), \(\tilde{f}^* : \mathfrak{S}^m(V) \rightarrow \mathfrak{S}^n(\tilde{f}^{-1}(V))\) is an isomorphism, so that we have a commutative diagram

\[
\begin{array}{ccc}
\mathfrak{S}^m(V) & \xrightarrow{\tilde{f}^*} & \mathfrak{S}^n(\tilde{f}^{-1}(V)) \\
\sigma^m_V & & \sigma^n_{\tilde{f}^{-1}(V)} \\
\mathfrak{S}^m(\rho^m(V)) & \xleftarrow{\tilde{f}^*} & \mathfrak{S}^n(\rho^m(\tilde{f}^{-1}(V))).
\end{array}
\]

Let \(x \in X_n\), \(y \in X_m\) such that \(\tilde{f}(x) = y\), and let \(U \subset X_n\) and \(V \subset X_m\) be open neighborhoods of \(x\) and \(y\) respectively such that \(\tilde{f}(U) \subset V\). Then, for a section \(s^m \in \mathfrak{S}^m(V)\), we have an element \((x,y,s^m) \in X_n \times_{f,X_m,\pi_m} \mathcal{F}_m\) to which we assign an element \((x,s^m \circ \tilde{f}) \in \mathcal{F}_n\) as follows: whenever \(U \subset \tilde{f}^{-1}(V)\), the section \(s^m \in \mathfrak{S}^m(V) \cong \mathfrak{S}^n(\tilde{f}^{-1}(V))\) has a restriction \(s^n := s^m |_{\tilde{f}^{-1}(V)} \in \mathfrak{S}^n(U)\). In this way we get a well-defined map \(X_n \times_{f,X_m,\pi_m} \mathcal{F}_m \rightarrow \mathcal{F}_n\). Moreover, it is easy to check that this map is an isomorphism; the inverse is the map

\[
\mathcal{F}_n \ni (x,s^n_x) \mapsto (x,(\tilde{f}(x),(\tilde{f}^*s^n)_{\tilde{f}(x)})) \in X_n \times_{f,X_m,\pi_m} \mathcal{F}_m,
\]

where if \(x \in U \subset X_n\) and \(\tilde{f}(U) \subset V \subset X_m\), \(\tilde{f}^*s^n\) is any section in \(\mathfrak{S}^m(V) \cong \mathfrak{S}^n(\tilde{f}^{-1}(V))\) that has the same class as \(s^n\) at the point \(x\) when restricted to \(\mathfrak{S}^n(U)\) through the restriction map \(\mathfrak{S}^n(\tilde{f}^{-1}(V)) \rightarrow \mathfrak{S}^n(U)\). Furthermore, for every \(f \in \text{Hom}_\Delta([m],[n])\), there is a face/degeneracy map \(\tilde{f} : (\mathcal{F}_n, \sigma_n) \rightarrow (\mathcal{F}_m, \sigma_m)\) given by \(\tilde{f}(x, s_x) := (\tilde{f}(x), (\tilde{f}^*s)_{f(x)})\); hence \((\mathcal{F}_n, \sigma_n)\) is a reduced \(\varepsilon\)étale Real simplicial space over \((X_\bullet, \rho_\bullet)\).

Conversely, if \(\pi_\bullet : (\mathcal{E}_\bullet, \nu_\bullet) \rightarrow (X_\bullet, \rho_\bullet)\) is a reduced \(\varepsilon\)étale morphism of Real simplicial spaces, we let \(\mathfrak{S}^n(U)\) be the space \(C(U, \mathcal{E}_\bullet)\) of continuous sections over \(U\) (where \(U\) is an open subset of \(X_n\)) of the projection \(\pi_n : (\mathcal{E}_n, \nu_n) \rightarrow (X_n, \rho_n)\). Next we define \(\sigma^n_U : \mathfrak{S}^n(U) \rightarrow \mathfrak{S}^n(\rho^n(U))\) by \(\sigma^n_U(s)(\rho^n(x)) := \nu_n(s(x))\). Notice that since the \(\pi_n\)'s are \(\varepsilon\)étale, one can recover the Real spaces \((\mathcal{E}_n, \nu_n)\) by considering the underlying Real spaces of the Real sheaves \((\mathfrak{S}^n, \sigma^n)\). Now for any \(f \in \text{Hom}_\Delta([m],[n])\) and for any open set \(V \subset X_m\), we have an isomorphism

\[
\tilde{f}^* : \mathfrak{S}^m(V) \rightarrow \mathfrak{S}^n(\tilde{f}^{-1}(V)),
\]

\[
s \mapsto \tilde{f}^*s,
\]

where \((\tilde{f}^*s)(x) = (x,s(\tilde{f}(x))) \in X_n \times_{f,X_m,\pi_m} \mathcal{E}_m \cong \mathcal{E}_m\).

Using the same construction as in the second part of this proof, we deduce the following:

**Lemma 2.20.** Any reduced Real simplicial space over \((X_\bullet, \rho_\bullet)\), \(\varepsilon\)étale or not, determines a Real sheaf over \((X_\bullet, \rho_\bullet)\).
Proof of Lemma 2.16. Let \((Z, \tau)\) be a Real \(\mathcal{G}\)-sheaf, and let

\[
\pi : (Z, \tau) \longrightarrow (X, \rho)
\]

be an étale Real map. Put for all \(n \geq 0\), \(\mathcal{E}_n := (\mathcal{G} \times Z)_n := \mathcal{G}_n \times \pi_n \times \pi \times Z\)

where \(\pi_n (g_1, \ldots, g_n) = \pi_n [\gamma_0, \ldots, \gamma_n] = s(\gamma_n) = s(g_n)\). Define \(\nu_n := \rho_n \times \tau\).

We thus obtain a Real simplicial space \((\mathcal{E}_n, \nu_n)\): the simplicial structure is given by

\[
(\gamma_0, \ldots, \gamma_n, z) \mapsto \left( (\gamma_{f(0)}, \ldots, \gamma_{f(m)}), \gamma_n^{-1} \gamma_{f(m)} \gamma_n \cdot z \right) \in \mathcal{E}_m,
\]

for \(f \in \text{Hom}_\Delta ([m], [n])\). Furthermore, it is straightforward to see that the projections \(\pi_n : \mathcal{E}_n \longrightarrow \mathcal{G}_n\) are compatible with the Real structures \(\nu_n\) and \(\rho_n\), and that they define a morphism of Real simplicial spaces. If \(f \in \text{Hom}_\Delta ([m], [n])\), then the assignment

\[
(\gamma_0, \ldots, \gamma_n, z) \mapsto \left( (\gamma_0, \ldots, \gamma_n), (\gamma_{f(0)}, \ldots, \gamma_{f(m)}), \gamma_{f(m)}^{-1} \gamma_n \cdot z \right)
\]

obviously defines a Real homeomorphism \(\mathcal{E}_n \cong \mathcal{G}_n \times_{f, \pi_n, \pi_m} \mathcal{E}_m\) which shows that \((\mathcal{E}_\bullet, \nu_\bullet)\) is a reduced Real simplicial space over \((\mathcal{G}_\bullet, \rho_\bullet)\). It follows from Lemma 2.20 that \((\mathcal{E}_\bullet, \nu_\bullet)\) determines an object of \(\mathbf{Sh}_{\rho_\bullet} (\mathcal{G}_\bullet)\). 

Remark 2.21. Notice that in the proof above we did not use the fact that \((Z, \tau)\) is étale. In fact, the Real \(\mathcal{G}\)-action suffices for \((Z, \tau)\) to give rise to a Real sheaf over \((\mathcal{G}_\bullet, \rho_\bullet)\). However, the property of being étale will be necessary to show that the Real sheaf obtained is reduced (as it is mentioned in the following corollary).

Corollary 2.22. Let \((\mathcal{G}, \rho)\) be a topological Real groupoid. Then there is a functor

\[
\mathcal{E} : \mathcal{B}_\rho \mathcal{G} \longrightarrow \mathbf{redSh}_{\rho_\bullet} (\mathcal{G}_\bullet),
\]

where \(\mathbf{redSh}_{\rho_\bullet} (\mathcal{G}_\bullet)\) is the full subcategory of \(\mathbf{Sh}_{\rho_\bullet} (\mathcal{G}_\bullet)\) consisting of all reduced Real sheaves over \((\mathcal{G}_\bullet, \rho_\bullet)\).

Proof. Let us keep the same notations as in the proof of Lemma 2.16. Since \(\pi\) is étale, so is \(\pi_n\) for all \(n\). The reduced Real simplicial space \((\mathcal{E}_\bullet, \nu_\bullet)\) is then étale over \((\mathcal{G}_\bullet, \nu_\bullet)\). Now, it suffices to apply Lemma 2.19.

\[\square\]

2.4. Real \(\mathcal{G}\)-modules.

Definition 2.23 (Cf. [21, Definition 3.9]). Let \((\mathcal{G}, \rho)\) be a topological Real groupoid. A Real \(\mathcal{G}\)-module is a topological Real groupoid \((\mathcal{M}, \_\_\_\_)\), with unit space \((X, \rho)\), and with source and range maps equal to a Real map \(\pi : (M, \_\_\_) \longrightarrow (X, \rho)\), such that:

- \(M_x (= M^x = M_\pi)\) is an abelian group for all \(x \in X\).
- For all \(x \in X\), the map \(\_\_\_ : M_x \longrightarrow M_{\rho(x)}\) is a group morphism.
- As a Real space, \((M, \_\_\_)\) is endowed with a Real \(\mathcal{G}\)-action

\[
\alpha : \mathcal{G} \times s, \pi \mathcal{M} \longrightarrow \mathcal{M}.
\]
• For each \( g \in \mathcal{G} \), the map \( \alpha_g : M_{s(g)} \to M_{r(g)} \) given by the action is a group morphism.

By Remark 2.21, any Real \( \mathcal{G} \)-module \((M, -)\) determines an abelian Real sheaf \((\mathcal{G}^\bullet, \sigma^\bullet)\) on \((\mathcal{G}, \rho^\bullet)\) constructed as follows: consider the reduced Real simplicial space \((E^\bullet, \nu^\bullet) = ((\mathcal{G} \times M)_n, \rho_n \times (-))\), where the Real simplicial structure is given by:

\[
\tilde{f}([\gamma_0, \ldots, \gamma_n], t) = \left([\gamma_{f(0)}, \ldots, \gamma_{f(m)}], \gamma_{f(m)}^{-1} \gamma_n \cdot t \right),
\]
for any \( f \in \text{Hom}_\Delta([m], [n]) \). Next, \((\mathcal{G}^\bullet, \sigma^\bullet)\) is defined as the sheaf of germs of continuous sections of the projections \( \pi^\bullet : (E^\bullet, \nu^\bullet) \to (\mathcal{G}^\bullet, \rho^\bullet) \).

**Example 2.24.** Let \((\mathcal{G}, \rho)\) be a topological Real groupoid. Let \( \mathcal{M} = X \times S^1 \) be endowed with the canonical Real structure \((x, \lambda) \mapsto (\rho(x), \tilde{\lambda})\), and Real \( \mathcal{G} \)-action \( g \cdot (s(g), \lambda) = (r(g), \lambda) \). Then \((\mathcal{M}, -)\) is a Real \( \mathcal{G} \)-module. The corresponding Real sheaf is called the constant sheaf of germs of \( S^1 \)-valued functions and denoted (abusively) \( \mathcal{G} \). More generally, if \( S \) is any Real group, \( X \times S \) is a Real \( \mathcal{G} \)-module, and the induced Real sheaf over \((\mathcal{G}, \rho^\bullet)\) is denoted by \( S \).

### 2.5. Pre-simplicial Real covers.

**Definition 2.25** (Cf. [21, Definition 4.1]). Let \((X^\bullet, \rho^\bullet)\) be a Real pre-simplicial space. A Real open cover of \((X^\bullet, \rho^\bullet)\) is a sequence \( \mathcal{U}^\bullet_{\text{nc}} = (U^\bullet_n)_{n \in \mathbb{N}} \) such that \( \mathcal{U}^\bullet_{\text{nc}} = (U^\bullet_n)_{n \in \mathbb{N}} \) is a Real open cover of \((X^\bullet, \rho^\bullet)\).

We say that \( \mathcal{U}^\bullet_{\text{nc}} \) is pre-simplicial if \((J^\bullet, -) = (J^\bullet_n, -)_{n \in \mathbb{N}} \) is a Real pre-simplicial set such that for all \( f \in \text{Hom}_\Delta([m], [n]) \) and for all \( j \in J^\bullet_n \), one has \( \tilde{f}(U^\bullet_m) \subseteq U^\bullet_f(J^\bullet_j) \). In the same way, one defines the notions of simplicial Real cover and \( N \)-simplicial Real cover.

We will use the same construction as in [21, §4.1] to show the following lemma.

**Lemma 2.26.** Any Real open cover \( \mathcal{U}^\bullet_{\text{nc}} \) of a Real (pre-)simplicial space \((X^\bullet, \rho^\bullet)\) gives rise to a pre-simplicial Real open cover \( \tilde{\mathcal{U}}^\bullet_{\text{nc}} \).

**Proof.** For each \( n \in \mathbb{N} \), let \( \mathcal{P}^\bullet_n = \bigcup_{k=0}^n \mathcal{P}^k_n \), where \( \mathcal{P}^k_n = \text{Hom}_\Delta([k], [n]) \). Let \( \mathcal{P}^\bullet = \bigcup_n \mathcal{P}^\bullet_n \), and let \( \Lambda_n \) (or \( \Lambda_n(J^\bullet) \) if there is a risk of confusion) be the set of maps

\[
\lambda : \mathcal{P}^\bullet \to \bigcup_k J^\bullet_k \text{ such that } \lambda(\mathcal{P}^k_n) \in J^\bullet_k \text{, for all } k.
\]

It is immediate to see that \( \Lambda_n \) is non-empty; indeed, for each \( k \in \mathbb{N} \), we fix a map \( \tilde{f}^k : [n] \to J^\bullet_k \) which can be written as \( \tilde{f}^k = (j^k_0, \ldots, j^k_n) \). Next, we define \( \tilde{f} = (\tilde{f}^k)_{k \in \mathbb{N}} \). Then the map \( \lambda : \mathcal{P}^\bullet \to \bigcup_k J^\bullet_k \) given by \( \lambda(\varphi) := \tilde{f} \circ \varphi \) lies in \( \Lambda_n \). Moreover, \( \Lambda_n \) has a Real structure defines as follows: if \( \varphi \in \mathcal{P}^k_n \), then we set

\[
\tilde{\lambda}(\varphi) := \lambda(\varphi) \in J^\bullet_k.
\]
Now, for all \( \lambda \in \Lambda_n \), we let

\[
U^n_{\lambda} := \bigcap_{k \leq n} \bigcap_{\varphi \in \mathcal{P}_n^k} \tilde{\varphi}^{-1}(U^k_{\lambda(\varphi))}).
\]

Let \( x \in X_n \). For each \( k \leq n \) and \( \varphi \in \mathcal{P}_n^k \), there is \( j^k_{\varphi} \in J_k \) such that \( \tilde{\varphi}(x) \in U^k_{j^k_{\varphi}} \subset X_k \). Define the map \( \lambda_{x} : \mathcal{P} \rightarrow \bigcup_k J_k \) by \( \lambda_{x}(\varphi) := (j^k_{\varphi})_k \).

Then one can see that \( x \) maps \( \lambda_{x} \) into \( \mathcal{P} \). Hence, \( (U^m_{\lambda(\varphi)})_{\lambda \in \Lambda_n} \) is a Real open cover of \((X_n, \rho_n)\). If for any \( f \in \text{Hom}_{\Delta^N}([m], [n]) \), we define a map \( \tilde{f} : \Lambda_n \rightarrow \Lambda_m \) by

\[
(\tilde{f}\lambda)(\varphi) := \lambda(f \circ \varphi), \text{ for all } \lambda \in \Lambda_n, \text{ and } \varphi \in \mathcal{P}_n^k,
\]

one sees that \( \tilde{f}(U^n_{\lambda}) \subseteq U^m_{\tilde{f}\lambda} \). Thus, \( \mathcal{U}_\bullet = ((U^m_{\lambda})_{\lambda \in \Lambda_n})_{n \in \mathbb{N}} \) is a pre-simplicial Real open cover of \((X_\bullet, \rho_\bullet)\).

In the same way, for \( N \in \mathbb{N} \) and \( n \leq N \), we denote by \( \Lambda_n^N \) the set of all maps

\[
\lambda : \bigcup_{k \leq n} \text{Hom}_{\Delta}([k], [n]) \rightarrow \bigcup_{k \leq n} J_k
\]

that satisfy \( \lambda(\text{Hom}_{\Delta}([k], [n])) \subseteq J_k \), and we set

\[
U^n_{\lambda} := \bigcap_{k \leq n} \bigcap_{\varphi \in \text{Hom}_{\Delta}([k], [n])} \tilde{\varphi}^{-1}(U^m_{\lambda(\varphi)}).
\]

Then we equip \( \Lambda_n^N \) with the Real structure defined in the same fashion, and we give it the \( N \)-simplicial structure defined as follows: for any \( f \in \text{Hom}_{\Delta^N}(\{m\}, \{n\}) \), the map \( \tilde{f} : \Lambda_n^N \rightarrow \Lambda_m^N \) is given by \( (\tilde{f}\lambda)(\varphi) := \lambda(f \circ \varphi) \).

We thus obtain a \( N \)-simplicial Real cover \( \mathcal{U}_\bullet = \mathcal{U}_n \mathcal{V}_\bullet = ((U^m_{\lambda})_{\lambda \in \Lambda_n})_{n \in \mathbb{N}} \) of the \( N \)-skeleton of \((X_\bullet, \rho_\bullet)\), where \( \mathcal{U}_n = (U^m_{\lambda})_{\lambda \in \Lambda_n^N} \).

We endow the collection of Real open covers of \((X_\bullet, \rho_\bullet)\) with the partial pre-order given by the following definition.

**Definition 2.27.** Let \( \mathcal{U}_\bullet \) and \( \mathcal{V}_\bullet \) be Real open covers of a Real simplicial space \((X_\bullet, \rho_\bullet)\), with \( \mathcal{U}_n = (U^m_{\lambda})_{\lambda \in \Lambda_n^N} \) and \( \mathcal{V}_n = (V^n_{\lambda})_{\lambda \in \Lambda_n^N} \). We say that \( \mathcal{V}_\bullet \) is finer than \( \mathcal{U}_\bullet \) if for each \( n \in \mathbb{N} \), there exists a Real map

\[
\theta_n : (I_n, \leq) \rightarrow (J_n, \leq)
\]

such that \( V^n_{\theta_n(i)} \subseteq U^n_{\lambda(i)} \) for every \( i \in I_n \). The Real map \( \theta_\bullet = (\theta_n)_{n \in \mathbb{N}} \) is required to be pre-simplicial (resp. \( N \)-simplicial) if \( \mathcal{U}_\bullet \) and \( \mathcal{V}_\bullet \) are pre-simplicial (resp. \( N \)-simplicial).

**2.6. “Real” Čech cohomology.**

**Definition 2.28** (Real local sections). Let \( (\mathcal{F}, \sigma) \) be an abelian Real (pre-) sheaf over \((X, \rho)\) and let \( \mathcal{U} = (U_j)_{j \in J} \) be a Real open cover of \((X, \rho)\). We say that a family \( s_j \in \mathcal{F}(U_j) \) is a globally Real family of local sections of \((\mathcal{F}, \sigma)\) over \( \mathcal{U} \) if for every \( j \in J \), \( s_j \) is the image of \( s_j \) in \( \mathcal{F}(U_j) \) by \( \sigma_{U_j} \).
We define $\text{CR}_{ss}(\mathcal{U}, \mathfrak{F})_{\rho, \sigma}$ to be the set of all globally Real families of local sections of $(\mathfrak{F}, \sigma)$ relative to $\mathcal{U}$; i.e.,

$$\text{CR}_{ss}(\mathcal{U}, \mathfrak{F})_{\rho, \sigma} := \left\{ (s_j)_{j \in J} \subset \prod_{j \in J} \mathfrak{F}(U_j) \mid s_j = \sigma_{U_j}(s_j), \forall j \in J \right\}.$$ 

To avoid irksome notations, we will write $\text{CR}_{ss}(\mathcal{U}, \mathfrak{F})$ or $\text{CR}_{ss}(\mathcal{U}, \mathfrak{F})_{\rho, \sigma}$ instead of $\text{CR}_{ss}(\mathcal{U}, \mathfrak{F})_{\rho, \sigma}$. It is clear that $\text{CR}_{ss}(\mathcal{U}, \mathfrak{F})$ is an abelian group.

Now let $(X, \rho_*)$ be a Real simplicial space, and let $\mathcal{U}_*$ be a pre-simplicial Real open cover of $(X, \rho_*)$. Suppose $(\mathfrak{F}^*, \sigma^*)$ is a (pre-simplicial) abelian Real (pre-)sheaf over $(X, \rho_*)$.

**Definition 2.29.** We define the complex $\text{CR}_{ss}^n(\mathcal{U}_*, \mathfrak{F}^*)_{\rho_*, \sigma^*}$ by

$$\text{CR}_{ss}^n(\mathcal{U}_*, \mathfrak{F}^*) := \text{CR}_{ss}(\mathcal{U}_n, \mathfrak{F}^n)_{\rho_n, \sigma^n},$$

for $n \in \mathbb{N}$. We will also write $\text{CR}_{ss}^n(\mathcal{U}_*, \mathfrak{F}^*)$ if there is no risk of confusion.

A Real $n$-cochain of $(X, \rho_*)$ relative to a pre-ssimplicial Real open cover $\mathcal{U}_*$ with coefficients in $(\mathfrak{F}^*, \sigma^*)$ is an element in $\text{CR}_{ss}^n(\mathcal{U}_*, \mathfrak{F}^*)$.

Let us consider again the maps $\varepsilon_k : [n] \to [n + 1]$ defined by (2.1), for $k = 0, \ldots, n + 1$. We have Real maps $\tilde{\varepsilon}_k : (J_{n+1}, -) \to (J_n, -)$, $\tilde{\varepsilon}_k : (X_{n+1}, \rho_{n+1}) \to (X_n, \rho_n)$, and $\tilde{\varepsilon}_k : (\mathfrak{F}^{n+1}, \sigma^{n+1}) \to (\mathfrak{F}^n, \sigma^n)$; and since $\tilde{\varepsilon}_k(U^1_j) \subseteq U^1_{\tilde{k}(j)}$ for every $j \in J_{n+1}$, we have a restriction map

$$\tilde{\varepsilon}_k^* : \mathfrak{F}^n(U^1_{\tilde{k}(j)}) \to \mathfrak{F}^{n+1}(U^1_j)$$

such that $\sigma^{n+1}_{U_{\tilde{k}(j)}} \circ \tilde{\varepsilon}_k^* = \tilde{\varepsilon}_k^* \circ \sigma^n_{U_{\tilde{k}(j)}}$.

**Definition 2.30.** Let $\mathcal{U}_*$ be a pre-simplicial Real open cover of $(X, \rho_*)$. For $n \geq 0$, we define the differential map

$$d^n : \text{CR}_{ss}^n(\mathcal{U}_*, \mathfrak{F}^*) \to \text{CR}_{ss}^{n+1}(\mathcal{U}_*, \mathfrak{F}^*)$$

also denoted by $d$, by setting for $c = (c_j)_{j \in J_n} \in \text{CR}_{ss}^n(\mathcal{U}_*, \mathfrak{F}^*)$ and for $j \in J_{n+1}$:

$$(dc)_j := \sum_{k=0}^{n+1} (-1)^k \tilde{\varepsilon}_k^*(c_{\tilde{k}(j)}).$$

**Remark 2.31.** The differential $d$ of (2.25) does indeed map $\text{CR}_{ss}^n(\mathcal{U}_*, \mathfrak{F}^*)$ to $\text{CR}_{ss}^{n+1}(\mathcal{U}_*, \mathfrak{F}^*)$; combining the fact that the $\tilde{\varepsilon}_k$ are Real maps and the discussion preceding the last definition, one has

$$(dc)_j = \sum_{k=0}^{n+1} (-1)^k \tilde{\varepsilon}_k^*(c_{\tilde{k}(j)}) = \sum_{k=0}^{n+1} (-1)^k \sigma^n_{U_{\tilde{k}(j)}}(c_{\tilde{k}(j)}) = \sigma^{n+1}_{U_{\tilde{k}(j)}}((dc)_j).$$

**Lemma 2.32.** The differential maps $d$ are group homomorphisms that satisfy $d^n \circ d^{n-1} = 0$ for $n \geq 1$. 

Proof. That for any \( n \in \mathbb{N} \), \( d^n \) is a group homomorphism is straightforward. Let \((c'_j)_{j' \in J_{n-1}} \in CR^{n-1}_{ss}(\mathbb{U}_*, \mathfrak{q}^*)\). Then, for \( j \in J_{n+1} \) one has

\[
(d^n d^{n-1} c)_j = \sum_{l=0}^{n+1} (-1)^l (\varepsilon^{n+1}_l)^* \left( \sum_{k=0}^{n} (-1)^k (\varepsilon^n_k)^* (c_{\varepsilon^n_k \varepsilon^{n+1}_l}(j)) \right)
\]

\[
= \sum_{l=0}^{n+1} \sum_{k=0}^{n} (-1)^{l+k} (\varepsilon^{n+1}_l)^* \circ (\varepsilon^n_k)^* (c_{\varepsilon^n_k \varepsilon^{n+1}_l}(j))
\]

\[
= \sum_{p=0}^{n} \sum_{k=0, k \leq 2p}^{n} (\varepsilon^{n+1}_{2p-k})^* (\varepsilon^n_k)^* (c_{\varepsilon^n_k \varepsilon^{n+1}_{2p-k}}(j)) - \sum_{p=0}^{n} \sum_{k=0, k \geq 2p+1}^{n} (\varepsilon^{n+1}_{2p+1-k})^* \circ (\varepsilon^n_k)^* (c_{\varepsilon^n_k \varepsilon^{n+1}_{2p+1-k}}(j)) = 0,
\]

since \( \varepsilon^{n+1}_r \circ \varepsilon^q = \varepsilon^{n+1}_{r+1} \circ \varepsilon^n_q \), for any \( r, q \leq n \).

We thus can give the following:

**Definition 2.33.** A Real n-cochain \( c \) in the kernel of \( d^n \) is called a Real n-cocycle relative to the pre-simplicial Real open cover \( \mathbb{U}_* \) with coefficients in \((\mathfrak{q}^*, \sigma^*)\); the Real n-cocycles form a subgroup \( ZR^n_{ss}(\mathbb{U}_*, \mathfrak{q}^*) \) of \( CR^n_{ss}(\mathbb{U}_*, \mathfrak{q}^*) \). The Real n-cochains belonging to the image of \( d^{n-1} \) are called Real n-coboundaries relative to \( \mathbb{U}_* \) and form a subgroup \( BR^n_{ss}(\mathbb{U}_*, \mathfrak{q}^*) \) (since \( d^2 = 0 \)). The \( n^{th} \) Real cohomology group of the pre-simplicial Real open cover \( \mathbb{U}_* \) with coefficients in \((\mathfrak{q}^*, \sigma^*)\) is defined by the \( n^{th} \) cohomology group of the complex

\[
\cdots \xrightarrow{d^{n-2}} CR^{n-1}_{ss}(\mathbb{U}_*, \mathfrak{q}^*) \xrightarrow{d^{n-1}} CR^n_{ss}(\mathbb{U}_*, \mathfrak{q}^*) \xrightarrow{d^n} CR^{n+1}_{ss}(\mathbb{U}_*, \mathfrak{q}^*) \xrightarrow{d^{n+1}} \cdots
\]

That is,

\[
HR^n_{ss}(\mathbb{U}_*, \mathfrak{q}^*) := \frac{ZR^n_{ss}(\mathbb{U}_*, \mathfrak{q}^*)}{BR^n_{ss}(\mathbb{U}_*, \mathfrak{q}^*)} := \frac{\ker d^n}{\im d^{n-1}}.
\]

**Example 2.34** (Cf. [21, Example 4.3]). Let \((X_*, \rho_*)\) be the constant Real simplicial space associated with a topological Real space \((X, \rho)\); that is \((X_n, \rho_n) = (X, \rho)\) for every \( n \geq 0 \). Suppose \( \mathbb{U} := \mathbb{U}_0 = (U^0_j)_{j \in J_0} \) is a Real open cover of \((X, \rho)\). Define \( J_n := J_0^{n+1} \) together with the obvious Real structure. Then \((J_n, \cdot)\) is admits a simplicial structure by

\[
\tilde{f}(j_0, \ldots, j_n) := (j_{f(0)}, \ldots, j_{f(m)}), \text{ for all } f \in \text{Hom}_\Delta([m], [n]).
\]

Let \( U^n_{(j_0, \ldots, j_n)} := U^0_{j_0} \cap \cdots \cap U^0_{j_n} \) and \( \mathbb{U}_n := (U^n_{j})_{j \in J_n} \). Of course \( \mathbb{U}_n \) is a Real open cover of \((X_*, \rho_0)\), and for any \( f \in \text{Hom}_\Delta([m], [n]) \) one has \( \tilde{f}(U^n_{(j_0, \ldots, j_n)}) = U^n_{\tilde{f}(j_0, \ldots, j_n)} \subseteq U^n_{f(0)} \cap \cdots \cap U^n_{f(m)} = U^n_{\tilde{f}(j_{f(0)}, \ldots, j_{f(m)})} \); hence \( \mathbb{U}_* \) is a simplicial Real open cover of \((X_*, \rho_*)\).
Let \((\mathcal{F}, \sigma)\) be an Abelian Real sheaf on \((X, \rho)\) and let \((\mathfrak{F}^n, \sigma^n) := (\mathfrak{F}, \sigma)\) for all \(n \geq 0\). Then, \(HR^*_{ss}(\mathcal{U}, \mathfrak{F}^*)\) can be viewed as the “Real” analogue of the usual (i.e., when all the Real structures are trivial) cohomology group \(H^*(\mathcal{U}, \mathfrak{F})\) and is denoted by \(HR^*(\mathcal{U}, \mathfrak{F})\). A Real 0-cochain is a globally Real family \((s_j)_{j \in J}\) of sections. Given such a family, the differential \(d^0\) gives: \((d^0s)(j_0, j_1) = s_{j_1|U_{j_0j_1}} - s_{j_0|U_{j_0j_1}}\); it hence defines a Real 0-cocycle if there exists a Real global section \(f \in \Gamma(X, \mathfrak{F})\) such that \(s_j = f_{U_j}\) for all \(j \in J\).

A Real 1-coboundary is then a family \((c_{j_0j_1})_{j_0, j_1 \in J}\) of sections \(c_{j_0j_1} \in \mathfrak{F}(U_{j_0j_1}) \cong \Gamma(U_{j_0j_1}, \mathcal{F})\) verifying \(c_{j_0j_1}(\rho(x)) = \sigma(c_{j_0j_1}(x))\) for every \(x \in U_{j_0j_1}\), and such that there exists a globally Real family \((s_j)_{j \in J}\) of sections \(s_j \in \Gamma(U, \mathcal{F})\) such that \(c_{j_0j_1} = s_{j_1} - s_{j_0}\) over all non-empty intersection \(U_{j_0j_1}\).

Finally, a Real 1-cochain \(c = (c_{j_0j_1}) \in CR^1_{ss}(\mathcal{U}, \mathfrak{F})\) can be seen as a family of sections \(c_{j_0j_1} \in \Gamma(U_{j_0j_1}, \mathcal{F})\) satisfying \(c_{j_0j_1}(\rho(x)) = \sigma(c_{j_0j_1}(x))\). Such a cocycle is 1-cocyclic if and only if one has \((dc)_{j_0j_1j_2} = 0\) for all \(j_0, j_1, j_2 \in J\); in other words, \(c_{j_0j_1} + c_{j_1j_2} = c_{j_0j_2}\) over all non-empty intersection \(U_{j_0j_1j_2}\).

We can apply Lemma 2.26 to generalize the definition of the Real cohomology groups relative to pre-simplicial Real open covers to arbitrary Real open covers of \((X, \rho^*)\).

**Definition 2.35.** Let \((X, \rho^*)\) be a Real (pre-)simplicial space and let \((\mathfrak{F}^*, \sigma^*) \in \text{Ob}(\text{Sh}_{\rho^*}(X^)))\). For any Real open cover \(\mathcal{U}^*\) of \((X, \mathfrak{F}^*)\), we let

\[CR^*(\mathcal{U}^*, \mathfrak{F}^*) := CR^*_{ss}(\mathfrak{F}, \mathfrak{F}^*)\]

and we define the **Real cohomology** groups of \(\mathcal{U}^*\) with coefficients in \((\mathfrak{F}^*, \sigma^*)\) by

\[HR^*(\mathcal{U}^*, \mathfrak{F}^*) := HR^*_{ss}(\mathfrak{F}, \mathfrak{F}^*)\]

We head now toward the definition of the **Real Čech cohomology**: roughly speaking, given an Abelian Real (pre-)sheaf \((\mathfrak{F}^*, \sigma^*)\) over a Real simplicial space \((X, \rho^*)\), we want to define the Real cohomology groups \(HR^n(X, \mathfrak{F}^*)\) as the inductive limit of the groups \(HR^n(\mathcal{U}^*, \mathfrak{F}^*)\) over some category of Real open covers of \((X, \rho^*)\). To do this, we need some preliminaries elements.

**Lemma 2.36.** Let \((X, \rho^*)\) and \((\mathfrak{F}^*, \sigma^*)\) be as above. Assume \(\mathcal{U}^* \subset \mathcal{V}^*\) be Real open covers of \((X, \rho^*)\), with \(\mathcal{U}^* = (U^n_j)_{j \in I_n}\) and \(\mathcal{V}^* = (V^n_i)_{i \in I_n}\). Then all refinements \(\theta^* : (I_n, -) \rightarrow (J_n, -)\) induces group homomorphisms

\[\theta_n^* : HR^n(\mathcal{U}^*, \mathfrak{F}^*) \rightarrow HR^n(\mathcal{V}^*, \mathfrak{F}^*)\]

**Proof.** In virtue of Lemma 2.26, one can assume that \(\mathcal{U}^* \subset \mathcal{V}^*\) are presimplicial, and so that \(\theta^*\) is a pre-simplicial Real map. Define

\[\theta_n^* : CR^n(\mathcal{U}^*, \mathfrak{F}^*) \rightarrow CR^n(\mathcal{V}^*, \mathfrak{F}^*)\]

as follows: for any \(c = (c_j)_{j \in J_n} \in CR^n(\mathcal{U}^*, \mathfrak{F}^*)\), we put

\[(\theta_n^*(c))_i := c_{\theta_n(i)|V^n_i};\]
i.e., \((\theta^*_n c)\): is the image of \(c_{\theta_n(i)}\) by the canonical restriction
\[
\mathfrak{F}^n(U^\alpha_{\theta_n(i)}) \rightarrow \mathfrak{F}^n(V^\alpha_n).
\]
A straightforward calculation shows that this does define an element in \(CR^n(V\bullet, \mathfrak{F}^\bullet)\). Moreover, it is clear that \(\theta^*_n\) is a group homomorphism for any \(n\). Moreover, since \(\theta\) is pre-simplicial, \(\tilde{\varepsilon}_k \circ \theta_{n+1} = \theta_n \circ \tilde{\varepsilon}_k\). Then, for \(i \in I_{n+1}\), one has
\[
(d\theta^*_n(c))_i = \sum_{k=0}^{n+1} (-1)^k \tilde{\varepsilon}_k^*(c_{\theta_n \circ \tilde{\varepsilon}_k(i)})|\mathfrak{F}^n_i
\]
then \(d^n \circ \theta^*_n = \theta^*_{n+1} \circ d^n\) for all \(n \in \mathbb{N}\). It turns out that \(\theta^*_n\) maps \(ZR^n(\mathfrak{U}\bullet, \mathfrak{F}^\bullet)\) into \(ZR^n(V\bullet, \mathfrak{F}^\bullet)\) and maps \(BR^n(\mathfrak{U}\bullet, \mathfrak{F}^\bullet)\) into \(BR^n(V\bullet, \mathfrak{F}^\bullet)\). Consequently, \(\theta^*_n\) passes through the quotients: \(\theta^*_n([c]) := [\theta_n(c)],\) for \(c \in ZR^n(\mathfrak{U}\bullet, \mathfrak{F}^\bullet)\).

As noted in [21], the map \(HR^n(\mathfrak{U}\bullet, \mathfrak{F}^\bullet) \rightarrow HR^n(V\bullet, \mathfrak{F}^\bullet)\) may depend on the choice of the given refinement.

**Definition 2.37.** Let \((X\bullet, \rho\bullet)\) and \((\mathfrak{F}^\bullet, \sigma^\bullet)\) be as previously. Let \(\mathfrak{U}\bullet\) and \(V\bullet\) be Real open covers of \((X\bullet, \rho\bullet)\). Let \(\phi_n, \psi_n: CR^n(\mathfrak{U}\bullet, \mathfrak{F}^\bullet) \rightarrow CR^n(V\bullet, \mathfrak{F}^\bullet)\) be two families of group homomorphisms commuting with \(d\). We say that \((\phi_n)_{n \in \mathbb{N}}\) and \((\psi_n)_{n \in \mathbb{N}}\) are *equivalent* (resp. *\(N\)-equivalent*, for a given \(N \in \mathbb{N}\) such that the \(N\)-keleton of \(V\bullet\) admits an \(N\)-simplicial Real structure) if for all \(n \in \mathbb{N}\) (resp. for all \(n \leq N\)), there exists a group homomorphism \(h^n: CR^n(\mathfrak{U}\bullet, \mathfrak{F}^\bullet) \rightarrow CR^{n-1}(\mathfrak{V}\bullet, \mathfrak{F}^\bullet)\), with the convention that \(CR^{-1}(\mathfrak{V}\bullet, \mathfrak{F}^\bullet) = \{0\}\) (and \(h^{N+1} = h^N\) in case of \(N\)-equivalence), such that
\[
(2.29) \quad \phi_n - \psi_n = d^{n+1} - h^n, \quad \forall n \in \mathbb{N} \text{ (resp. } \forall n \leq N\).
\]
Observe that such \(N\)-equivalent families \(\phi\) and \(\psi\) induces group homomorphisms
\[
HR^n(\mathfrak{U}\bullet, \mathfrak{F}^\bullet) \rightarrow HR^n(\mathfrak{V}\bullet, \mathfrak{F}^\bullet),
\]
also denoted by \(\phi_n\) and \(\psi_n\) respectively, and given by \(\phi_n([c]) := [\phi_n(c)],\) and \(\psi_n([c]) := [\psi_n(c)]\) for all \(c \in ZR^n(\mathfrak{U}\bullet, \mathfrak{F}^\bullet)\). Assume
\[
h^n: CR^n(\mathfrak{U}\bullet, \mathfrak{F}^\bullet) \rightarrow CR^{n-1}(\mathfrak{V}\bullet, \mathfrak{F}^\bullet)
\]
is such that (2.29) holds for all \(n \leq N\), then for all \(c \in ZR^n(\mathfrak{U}\bullet, \mathfrak{F}^\bullet)\), one has
\[
(\phi_n - \psi_n)([c]) = [d^{n+1} h^n(c)] + [h^{n+1} (d^n c)] = 0;
\]
in other words, \(\phi_n\) and \(\psi_n\) define the same homomorphism from \(HR^n(\mathfrak{U}\bullet, \mathfrak{F}^\bullet)\) to \(HR^n(\mathfrak{V}\bullet, \mathfrak{F}^\bullet)\) when \(n \leq N\).
It is clear that \((N\)-equivalence of morphisms\(\phi_n: CR^n(\mathfrak{U}\bullet, \mathfrak{F}^\bullet) \rightarrow CR^n(\mathfrak{V}\bullet, \mathfrak{F}^\bullet)\)
Lemma 2.39. Let $\mathcal{U}_\bullet, \mathcal{V}_\bullet \in \mathcal{N}$. We say that $\mathcal{V}_\bullet$ is $h$-finer than $\mathcal{U}_\bullet$ if $\mathcal{V}_\bullet$ is finer than $\mathcal{U}_\bullet$ in the sense of Definition 2.27, and if there exists $N \in \mathbb{N}$ such that the $N$-skeleton of $\mathcal{V}_\bullet$ admits an $N$-simplicial Real structure. In this case, we will write $\mathcal{U}_\bullet \preceq_N \mathcal{V}_\bullet$ or $\mathcal{U}_\bullet \preceq_h \mathcal{V}_\bullet$.

We refer to [21, Lemma 4.5]) for the proof of the following:

Lemma 2.39. Let $\mathcal{U}_\bullet$ and $\mathcal{V}_\bullet$ be Real open covers of $(X_\bullet, \rho_\bullet)$ such that $\mathcal{U}_\bullet \preceq_N \mathcal{V}_\bullet$. If $\theta_\bullet, \theta'_\bullet : (I_\bullet, -) \to (J_\bullet, -)$ are two arbitrary refinements, then their induced group homomorphisms $\theta^*_\bullet$ and $(\theta'_\bullet)^*$ are $N$-equivalent. Consequently, there is a canonical morphism

$$HR^n(\mathcal{U}_\bullet, \mathfrak{B}^\bullet) \to HR^n(\mathcal{V}_\bullet, \mathfrak{B}^\bullet)$$

for each $n \leq N$.

Example 2.40. By Lemma 2.26, from anyy Real open cover $\mathcal{U}_\bullet$ of $(X_\bullet, \rho_\bullet)$ and any $N \in \mathbb{N}$, one can form an $N$-simplicial Real open cover $\mathcal{U}^N_\bullet$ of the $N$-skeleton of $(X_\bullet, \rho_\bullet)$. Next, we define a new Real open cover $\mathcal{U}^N_\bullet$ by setting

$$\mathcal{U}^N_\bullet := \begin{cases} \mathcal{U}_n, & \text{if } n \leq N, \\ \mathcal{U}_n, & \text{if } n \geq N + 1. \end{cases}$$

It is clear that the $N$-skeleton of $\mathcal{U}^N_\bullet$ admits an $N$-simplicial Real structure. Recall that $\mathcal{U}^N_\bullet$ is indexed by $I_\bullet$, with $I_n = \Lambda^N_n$ if $n \leq N$ and $I_n = J_n$ if $n \geq N + 1$. Now we get a refinement $\eta_\bullet : (I_\bullet, -) \to (J_\bullet, -)$ by setting

$$\eta_\bullet := \begin{cases} \Lambda^N_n \to J_n, & \lambda \mapsto \lambda(\text{Id}_{[n]}), \\ \text{Id} : J_n \to J_n, & \text{if } n \geq N + 1. \end{cases}$$

hence $\mathcal{U}_\bullet \preceq_N \mathcal{U}^N_\bullet$ for all $N \in \mathbb{N}$. In particular, $\mathcal{U}_\bullet \preceq_0 \mathcal{U}_\bullet$.

We deduce from the example above that “$\preceq_h$” is a pre-order in the collection $\mathcal{N}$. Suppose that $\mathcal{U}_\bullet \preceq_h \mathcal{V}_\bullet \preceq_h \mathcal{W}$ and $K_\bullet \xrightarrow{\varphi_\bullet} I_\bullet \xrightarrow{\varphi'_\bullet} J_\bullet$ are refinements. Then it is easy to check that the maps $\varphi^*_\bullet$ and $(\varphi'_\bullet)^*$ defined by (2.28) verify the relation $(\varphi_n \circ \varphi'_n)^* = (\varphi'_n)^* \circ \varphi^*_n$ for all $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, we denote by $\mathcal{N}(n)$ the collection of all elements $\mathcal{U}_\bullet \in \mathcal{N}$ such that $\mathcal{U}_\bullet \preceq_N \mathcal{U}_\bullet$ for some $N \geq n + 1$; i.e., $\mathcal{U}_\bullet \in \mathcal{N}(n)$ if there is $N \geq n + 1$ such that the $N$-skeleton of $\mathcal{U}_\bullet$ admits an $N$-simplicial Real structure. It is obvious that “$\preceq_h$” is also a pre-order in $\mathcal{N}(n)$. Furthermore, Lemma 2.39, states that if $\mathcal{U}_\bullet \preceq_h \mathcal{V}_\bullet$ in $\mathcal{N}(n)$, there is a canonical map $HR^n(\mathcal{U}_\bullet, \mathfrak{B}^\bullet) \to HR^n(\mathcal{V}_\bullet, \mathfrak{B}^\bullet)$. It follows that for all $n \in \mathbb{N}$, the collection

$$\{HR^n(\mathcal{U}_\bullet, \mathfrak{B}^\bullet) \mid \mathcal{U}_\bullet \in \mathcal{N}(n)\}$$

is a directed system of groups; this allows us to give the following definition.
Definition 2.41. We define the $n$th Čech cohomology group $\check{H}R^n(X\bullet, \mathcal{F}^\bullet)$ to be the direct limit

$$\check{H}R^n(X\bullet, \mathcal{F}^\bullet) := \lim_{\longrightarrow} H^R_n(U\bullet, \mathcal{F}^\bullet).$$

Lemma 2.42. For every $U\bullet \in \mathcal{N}$, pre-simplicial or not, there is a canonical group homomorphism

$$\theta_{U\bullet} : H^R_n(U\bullet, \mathcal{F}^\bullet) \longrightarrow \check{H}R^n(X\bullet, \mathcal{F}^\bullet),$$

for all $n \in \mathbb{N}$.

Proof. For every $U\bullet \in \mathcal{N}$ (simplicial or not), and for every $n \in \mathbb{N}$, we define the map

$$\theta_{U\bullet} : H^R_n(U\bullet, \mathcal{F}^\bullet) \longrightarrow \check{H}R^n(X\bullet, \mathcal{F}^\bullet)$$

by composing the canonical homomorphism

$$N \theta^n : H^R_n(U\bullet, \mathcal{F}^\bullet) \longrightarrow H^R_n(\natural U\bullet, \mathcal{F}^\bullet)$$

with the canonical projection

$$p^N_{U\bullet} : H^R_n(\natural U\bullet, \mathcal{F}^\bullet) \longrightarrow \check{H}R^n(X\bullet, \mathcal{F}^\bullet),$$

for some $N \geq n + 1$; i.e., $\theta_{U\bullet} = p^N_{U\bullet} \circ N \theta^n$ (recall that $N \theta^n$ is defined by (2.31)).

Let $(\mathcal{G}^\bullet, \sigma^\bullet)$ and $(\mathcal{G}^\bullet, \varsigma^\bullet)$ be Abelian Real sheaves on a Real simplicial space $(X\bullet, \rho\bullet)$. Suppose that $\phi\bullet = (\phi_n)_{n \in \mathbb{N}} : (\mathcal{F}^\bullet, \sigma^\bullet) \longrightarrow (\mathcal{G}^\bullet, \varsigma^\bullet)$ is a morphism of Abelian Real (pre)shaves, and that $U\bullet$ is a Real open cover of $(X\bullet, \rho\bullet)$. Consider the pre-simplicial Real open cover $\natural U\bullet$ associated to $U\bullet$. Then for any $n \in \mathbb{N}$, and any $\lambda \in \Lambda_n$, there is a morphism of Abelian groups

$$\tilde{\phi}_n : \mathcal{G}^n(U\lambda) \longrightarrow \mathcal{G}^n(U\lambda), \ s_\lambda \longrightarrow \phi_n|U\lambda(s_\lambda),$$

satisfying $\varsigma^n_{U\lambda} \circ \tilde{\phi}_n = \tilde{\phi}_n \circ \sigma_{U\lambda}$. This gives a group homomorphism

$$\tilde{\phi}_n : CR^n_{ss}(\natural U\bullet, \mathcal{F}^\bullet) \longrightarrow CR^n_{ss}(\natural U\bullet, \mathcal{G}^\bullet).$$

Moreover, for any $\lambda \in \Lambda_{n+1}$ and any $k \in [n+1]$, one has a commutative diagram

$$\begin{array}{ccc}
\mathcal{G}^n(U\lambda) & \xrightarrow{\phi_n|U\lambda(\lambda)} & \mathcal{G}^n(U\lambda) \\
\tau_k & \downarrow & \downarrow \tau_k \\
\mathcal{G}^n(U\lambda+1) & \xrightarrow{\phi_{n+1}|U\lambda+1} & \mathcal{G}^n(U\lambda+1).
\end{array}$$
Thus, \( d^n \circ \tilde{\phi}_n = \tilde{\phi}_{n+1} \circ d^n \); i.e., one has a commutative diagram
\[
\begin{array} {c}
CR^n_{ss}(\natural \mathcal{U}_*, \mathcal{G}^*) \sigma \times \downarrow \tilde{\phi}_n \\
\downarrow \tilde{\phi}_{n+1}
\end{array}
\]
that shows that \( \phi \) gives rise to a homomorphism of Abelian groups
\[
(\phi_{\mathcal{U}})_* : HR^n(\mathcal{U}_*, \mathcal{G}^*) \sigma \times \longrightarrow HR^n(\mathcal{U}_*, \mathcal{G}^*) \sigma,
\]
and therefore a group homomorphism
\[
\phi_* : \tilde{HR}^n(X_*, \mathcal{G}^*) \sigma \times \longrightarrow \tilde{HR}^n(X_*, \mathcal{G}^*) \sigma,
\]
defined in the obvious way. We thus have shown that \( \tilde{HR}^* \) is functorial in the category \( \mathcal{Sh}_{\rho_\mathcal{S}}(X_*) \).

**Proposition 2.43.** Suppose \((X_*, \rho_\mathcal{S})\) is a Real simplicial space such that each \( X_n \) is paracompact. If
\[
0 \longrightarrow (\mathcal{G}^*, \sigma^*) \xrightarrow{\phi^*} (\tilde{\mathcal{G}}^*, \sigma^*) \xrightarrow{\phi^*} (\tilde{\mathcal{G}}^*, \sigma^*) \longrightarrow 0
\]
is an exact sequence of Real (pre-)sheaves over \((X_*, \rho_\mathcal{S})\), then there is a long exact sequence of Abelian groups
\[
0 \longrightarrow \tilde{HR}^0(X_*, \mathcal{G}^*) \xrightarrow{\phi^*} \tilde{HR}^0(X_*, \tilde{G}^*) \xrightarrow{\phi^*} \tilde{HR}^0(X_*, \tilde{G}^*)
\]
\[
\xrightarrow{\partial} \tilde{HR}^1(X_*, \mathcal{G}^*) \xrightarrow{\phi^*} \cdots
\]

The proof of this proposition is almost the same as in [21, §4].

**2.7. Comparison with usual groupoid cohomologies.** In this subsection we compare our cohomology with the usual cohomology theory in some special cases, especially with that developed in [21].

**Proposition 2.44.** Suppose \( S \) is an Abelian Real group. Let \( \ast S \) be the fixed point subgroup of \( S \). Let \((\mathcal{S}, \rho_\mathcal{S})\) be a Real groupoid. Then if \( \rho \) is trivial, we have
\[
\tilde{HR}^*(\mathcal{S}_*, S) = \tilde{HR}^* (\mathcal{S}_*, \ast S).
\]
In particular, if \( S \) has no non-trivial fixed point, we have \( \tilde{HR}^*(\mathcal{S}_*, S) = 0 \).

Notice that this result generalizes easily to the Real cohomology with coefficients in a Real sheaf induced from a Real \( \mathcal{S} \)-module.

**Proof.** Let \((c_\lambda) \in ZR^n(\mathcal{U}_*, S_*)\). Since \( \rho = \text{Id} \), we may take the involution on \( J_* \) to be trivial. For every \( G \in U_\lambda^n \), we have
\[
c_\lambda(G) = c_\lambda(G) = c_\lambda(G) \in \ast S.
\]
Thus \( c_\lambda \in ZR^n(\mathcal{U}_*, \ast S) \).
Conversely, we obviously have \( \check{H}^n(S \ast, rS) \subset \check{H}R^n(S \ast, S) \) since \( \rho \) is trivial.

\[ \square \]

**Corollary 2.45.** If \( \rho \) and the Real structure of \( S \) are trivial, then

\[ \check{H}^n(S \ast, S) = \check{H}^n(S \ast, S). \]

Focus now on the case where \( S \) reduces to a Real space \((X, \tau)\) and \( S = \mathbb{Z}^{0,1} \). Then \( \tau \) induces an action of \( \mathbb{Z}_2 \) on \( X \) by \((-1) \cdot x := \tau(x), (+1) \cdot x := x.\)

**Proposition 2.46.** We have the following group isomorphisms:

(i) \( \check{H}^n(X, \mathbb{Z}^{0,1}) \cong \check{H}^n_{(\mathbb{Z}_2, -)}(X, \mathbb{Z}), \) where the sign “\(-\)" stands for the \( \mathbb{Z}_2 \)-equivariant cohomology with respect to the action of \( \mathbb{Z}_2 \) on \( \mathbb{Z} \) given by \((-1) \cdot n := -n, (+1) \cdot n := n.\)

(ii) \( \check{H}^n(X, \mathbb{Z}) \cong \check{H}^n_{(\mathbb{Z}_2, +)}(X, \mathbb{Z}) \oplus \check{H}^n_{(\mathbb{Z}_2, +)}(X, \mathbb{Z}), \) where the sign “\(+\)" means the trivial \( \mathbb{Z}_2 \)-action on \( \mathbb{Z} \).

**Proof.** (i) Let \( c \in \check{H}R^n(X, \mathbb{Z}^{0,1}) \) be represented on the Real open cover \((U_j)\) of \( X \). Then \( c_{j_0...j_n}(\tau(x)) = -c_{j_0...j_n}(x) \) implies \( \tau^* c_{j_0...j_n}(x) = -c_{j_0...j_n}(x), \) \( \forall x \in X; \) in other words, \( c \) is \( \mathbb{Z}_2 \)-equivariant with respect to the \( \mathbb{Z}_2 \)-action “\(-\)" on \( \mathbb{Z} \). The converse is easy to check.

(ii) We define the involution \( \tilde{\tau} \) on \( \check{H}^n(X, \mathbb{Z}) \) by \( \tilde{\tau}(c) := -\tau^* c. \) Then it is straightforward that the Real part \( \check{\hat{H}}^n(X, \mathbb{Z}) \cong \check{H}R^n(X, \mathbb{Z}^{0,1}), \) while the imaginary part \( \check{\hat{H}}^n(X, \mathbb{Z}) \) is exactly \( \check{H}^n_{(\mathbb{Z}_2, +)}(X, \mathbb{Z}). \)

\[ \square \]

2.8. The group \( \check{\hat{H}}R^0 \). We shall recall the notations of [21, Section 4] that we will use throughout the rest of the section. Let \( \mathfrak{U}_\ast \) be a Real open cover of a Real simplicial space \((X_\ast, \rho_\ast)\) and let \( \mathfrak{U}_\ast \) be its associated pre-simplicial Real open cover. Recall that any \( \varphi \in \mathfrak{P}_n^k \) is represented by its image in \([n]; \) i.e., \( \varphi = \{\varphi(0), \ldots, \varphi(k)\}. \) Then \( \mathfrak{P}_n \) is nothing but the collection of all non empty subsets of \([n]\). Henceforth, any subset \( S = \{i_0, \ldots, i_k\} \subseteq [n], \) with \( i_0 \leq \cdots \leq i_k, \) designates the maps \( \varphi \in \mathfrak{P}_n^k \) such that \( \varphi(0) = i_0, \ldots, \varphi(k) = i_k. \)

**Notations 2.47.** With the above observations, any element \( \lambda \in \Lambda_n \) is represented by a \((2^{n+1} - 1)\)-tuple \((\lambda_S)_{0 \neq S \subseteq [n]}, \) where the subsets \( S \) are ordered first by cardinality, then by lexicographic order; i.e.,

\[
S \in \{\{0\}, \ldots, \{n\}, \{0, 1\}, \ldots, \{0, n\}, \{1, 2\}, \ldots, \{1, n\}, \ldots, \{0, 1, 2\}, \ldots, \{0, \ldots, n\}\},
\]

and \( \lambda_S := \lambda(S). \) For instance, any element \( \lambda \in \Lambda_1 \) is represented by a triple \((\lambda_0, \lambda_1, \lambda_01), \) with \( \lambda_0 = \lambda(\{0\}), \lambda_1 = \lambda(\{1\}) \) and \( \lambda_01 = \lambda(\{0, 1\}). \)

Recall that if \((\mathfrak{F}_\ast, \sigma_\ast)\) is an abelian Real sheaf over \((X_\ast, \rho_\ast)\), we are given two “restriction” maps on the space of global Real sections

\[
\varepsilon_0^\ast, \varepsilon_1^\ast : \mathfrak{F}_0^0(X_0)_{\sigma^0} \longrightarrow \mathfrak{F}_1^1(X_1)_{\sigma^1}.
\]
Let us set
\[
\Gamma_{\text{inv}}(\mathcal{F}^\bullet)_{\sigma^\bullet} := \ker \left( \mathcal{F}^0(X_0)_{\sigma^0} \xrightarrow{\bar{\varepsilon}_1} \mathcal{F}^1(X_1)_{\sigma^1} \right) = \left\{ s \in \mathcal{F}^0(X_0)_{\sigma^0} \mid \bar{\varepsilon}_1(s) = \bar{\varepsilon}_1^*(s) \right\}.
\]

**Proposition 2.48** ([21, Proposition 5.1]). Let \((\mathcal{F}^\bullet, \sigma^\bullet)\) be an abelian Real sheaf over \((X_\bullet, \rho_\bullet)\) and let \(\mathcal{U}_\bullet\) be a Real open cover of \((X_\bullet, \rho_\bullet)\). Then
\[
(\mathcal{F}^\bullet, \sigma^\bullet) \cong \mathcal{H}R^0(\mathcal{U}_\bullet, \mathcal{F}^\bullet)_{\sigma^\bullet} \cong \Gamma_{\text{inv}}(\mathcal{F}^\bullet)_{\sigma^\bullet}.
\]

**Proof.** One identifies \(\Lambda_0\) with \(J_0\). Note that \(P_1 = \{\varepsilon_0^1, \varepsilon_1^1, \text{Id}_{[1]}\}\), and that for any \(\lambda = (\lambda_0, \lambda_1, \lambda_{01})\) in \(\Lambda_1\) one has \(\varepsilon_0^1(\lambda) = \lambda(\varepsilon_0) = \lambda_1, \varepsilon_1^1(\lambda) = \lambda(\varepsilon_1) = \lambda_0\). We thus have \(U^1_{\lambda} = U^1_{\lambda_0} \cap U^0_{\lambda_1} \cap U^1_{\lambda_{01}} \cap U^0_{\lambda_0}\). Now, let \((s_{\lambda_0})_{\lambda_0 \in J_0} \in ZR^0(\mathcal{U}_\bullet, \mathcal{F}^\bullet)_{\sigma^\bullet}\). Then
\[
(\mathcal{F}^\bullet, \sigma^\bullet) \cong \mathcal{H}R^0(\mathcal{U}_\bullet, \mathcal{F}^\bullet)_{\sigma^\bullet} \cong \Gamma_{\text{inv}}(\mathcal{F}^\bullet)_{\sigma^\bullet}.
\]

2.9. \(\mathcal{H}R^1\) and the Real Picard group. Let us consider the same data as in the previous subsection. Let \(\mathcal{U}_\bullet\) be a Real open cover of \((X_\bullet, \rho_\bullet)\). For \(\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_0_1, \lambda_0_2, \lambda_1_2, \lambda_{01_2}) \in \Lambda_2\), one has
\[
U^2_{\lambda} = \bar{\varphi}_{00}(U^0_{\lambda_0}) \cap \bar{\varphi}_{01}(U^0_{\lambda_1}) \cap \bar{\varphi}_{02}(U^0_{\lambda_2}) \cap \bar{\varphi}_{03}(U^0_{\lambda_{01}}) \cap \bar{\varphi}_{10}(U^0_{\lambda_1}) \cap \bar{\varphi}_{11}(U^0_{\lambda_2}) \cap \bar{\varphi}_{12}(U^0_{\lambda_{01}}) \cap \bar{\varphi}_{13}(U^0_{\lambda_{12}}) \cap \bar{\varphi}_{20}(U^0_{\lambda_2}) \cap \bar{\varphi}_{21}(U^0_{\lambda_{01}}) \cap \bar{\varphi}_{22}(U^0_{\lambda_{12}}) \cap \bar{\varphi}_{23}(U^0_{\lambda_{21}}),
\]
where \(\varphi_{00} = \varepsilon_0^1 \circ \varepsilon_1^1, \varphi_{01} = \varepsilon_0^2 \circ \varepsilon_0^1\) and \(\varphi_{02} = \varepsilon_0^1 \circ \varepsilon_1^1\). Let \(c = (c_{\lambda})_{\lambda \in \Lambda_1} \in ZR^1(\mathcal{U}_\bullet, \mathcal{F}^\bullet)_{\sigma^\bullet}\). Then
\[
0 = (dc)_{\lambda_0 \lambda_1 \lambda_2 \lambda_0_1 \lambda_0_2 \lambda_1_2 \lambda_{01_2}} = \varepsilon_0^1 c_{\lambda_1 \lambda_2 \lambda_{12}} - \varepsilon_0^2 c_{\lambda_0 \lambda_2 \lambda_{02}} + \varepsilon_0^2 c_{\lambda_0 \lambda_1 \lambda_{01_2}},
\]
on \(U^2_{\lambda}\), and of course we get a similar identities for \((dc)_{\lambda_0 \lambda_1 \lambda_2 \lambda_0_1 \lambda_0_2 \lambda_1_2 \lambda_{01_2}}\) on \(U^2_{\lambda}\). Now applying \(\bar{\gamma}_1\) to \((2.39)\), we obtain
\[
c_{\lambda_0 \lambda_1 \lambda_{01}} = c_{\lambda_0 \lambda_1 \lambda_{02}} - c_{\lambda_1 \lambda_2 \lambda_{12}}
\]
on \(\varepsilon_1^1(U^0_{\lambda_0}) \cap \varepsilon_0^1(U^0_{\lambda_1}) \cap \varepsilon_0^1(U^0_{\lambda_2}) \cap U^1_{\lambda_0} \cap U^1_{\lambda_1} \cap U^1_{\lambda_2} \cap U^1_{\lambda_{12}} \cap \bar{\gamma}_1(U^2_{\lambda_{12}})\), which means that for any \(\lambda_0, \lambda_1, \lambda_0_1 \in J_0\), \(s_{\lambda_0 \lambda_1 \lambda_01}\) does not depend on the choice of \(\lambda_0\). Therefore, there exists a Real family
\[
(f_{\lambda_0 \lambda_1}) \in \prod_{\lambda_0, \lambda_1 \in \Lambda_0} \mathcal{F}^1(\varepsilon_1^1(U^0_{\lambda_0}) \cap \varepsilon_0^1(U^0_{\lambda_1})).
\]
such that \( f_{\lambda_0\lambda_1}[U_{\lambda_0\lambda_1\lambda_0\lambda_1}] = c_{\lambda_0\lambda_1\lambda_0} \) for any \((\lambda_0, \lambda_1, \lambda_0) \in \Lambda_1\). Now, the cocycle relation (2.40) becomes

\[
(2.40) \quad \tilde{\varepsilon}_2 f_{\lambda_1\lambda_2} - \tilde{\varepsilon}_1 f_{\lambda_0\lambda_1} + \tilde{\varepsilon}_0 f_{\lambda_0\lambda_1}
\]
on \(U_{\lambda_0\lambda_1\lambda_0\lambda_1} \cap U_{\lambda_0\lambda_2} \cap U_{\lambda_1\lambda_2}\).

Let \((\mathcal{G}, \rho)\) be a locally compact Hausdorff Real groupoid. We are interested in the 1st Real Čech cohomology group of \((\mathcal{G}, \rho)\) with coefficients in the Abelian Real sheaf \((\mathcal{S}\bullet, \sigma\bullet) = (\mathcal{S}, \sigma)\) over \((\mathcal{G}, \rho)\) associated to the Real \(\mathcal{G}\)-module \((X \times S, \rho \times -)\), where \((\mathcal{S}, -)\) is an Abelian group endowed with the trivial \(\mathcal{G}\)-action. Note that in this case, for any pre-simplicial Real open cover \(\mathcal{U} \in \mathcal{N}(n)\) of \((\mathcal{G}, \rho)\), elements of the group \(C \mathcal{R}^n(\mathcal{U}, \mathcal{S}\bullet)\) are of the form \((c_\lambda)_{\lambda \in \Lambda_n}\), where \(c_\lambda \in \Gamma(U^n_\lambda, \mathcal{S})\) are such that \(c_\lambda(\rho_n(\tilde{g})) = c_\lambda(\tilde{g}) \in \mathcal{S}\) for any \(\tilde{g} \in U^n_\lambda \subset \mathcal{G}_n\).

**Proposition 2.49.** With the above notations, the Real Čech cohomology group \(\tilde{H} \mathcal{R}^1(\mathcal{G}, \mathcal{S})\) is isomorphic to the group \(\text{Hom}_{\mathcal{R}\mathcal{G}}(\mathcal{G}, \mathcal{S})\) of isomorphism classes of Real generalized homomorphisms \((\mathcal{G}, \rho) \rightarrow (\mathcal{S}, -)\).

**Proof.** The operations in \(\text{Hom}_{\mathcal{R}\mathcal{G}}(\mathcal{G}, \mathcal{S})\) are defined as follows. If \((Z, \tau), (Z', \tau') : (\mathcal{G}, \rho) \rightarrow (\mathcal{S}, -)\) are Real generalized homomorphisms, their sum is

\[
(2.41) \quad (Z, \tau) + (Z', \tau') := Z \times_X Z'/\sim
\]
where \((z, z') \sim (z \cdot t^{-1}, z' \cdot t)\) for all \(t \in \mathcal{S}\), together with the obvious Real structure \(\tau \times \tau'\). The inverse of \((Z, \tau)\) is \((Z^{-1}, \tau)\), where \(Z^{-1}\) is \(Z\) as a topological space, and if \(b : Z \rightarrow Z^{-1}\) is the identity map, then the \(\mathcal{S}\)-action on \(Z^{-1}\) is defined by \(b(z) \cdot t := b(z \cdot t^{-1})\) and the \(\mathcal{G}\)-action is defined as follows: \((g, b(z)) \in \mathcal{G} \times Z^{-1}\) if and only if \((g, z) \in \mathcal{G} \times Z\), in which case we set

\[
g \cdot b(z) := b(g \cdot z).
\]
Finally, the Real structure on \(Z^{-1}\) is \(\tau(b(z)) := b(\tau(z))\). Then we define the sum in \(\text{Hom}_{\mathcal{R}\mathcal{G}}(\mathcal{G}, \mathcal{S})\) by \([Z, \tau] + [Z', \tau'] := [(Z, \tau) + (Z', \tau')]\), and we put \([Z, \tau]^{-1} := [(Z^{-1}, \tau)]\). It is not hard to check that subject to these operations, \(\text{Hom}_{\mathcal{R}\mathcal{G}}(\mathcal{G}, \mathcal{S})\) is an Abelian group.

Now, suppose we are given a Real open cover \(\mathcal{U}_0 = (U^0_j)_{j \in J_0}\) of \((X, \rho)\) trivializing the Real generalized homomorphism \((Z, \tau) : (\mathcal{G}, \rho) \rightarrow (\mathcal{S}, -)\). Let \(s_j \in J_0\) be a Real family of local sections of the \(\mathcal{S}\)-principal Real bundle \(\tau : (Z, \tau) \rightarrow (X, \rho)\). Form a pre-simplicial Real open cover \(\mathcal{U}_s\) of the Real simplicial space \((\mathcal{G}_s, \rho_s)\) by setting \(J_n := J_0^{n+1}, \mathcal{U}_n := (U^n_{J_0, \ldots, J_n})_{(J_0, \ldots, J_n) \in J_n}\), where

\[
(2.42) \quad U^n_{J_0, \ldots, J_n} := \left\{ (g_1, \ldots, g_n) \in \mathcal{G}_n \mid r(g_1) \in U^0_{J_0}, \ldots, r(g_n) \in U^0_{J_{n-1}}, s(g_n) \in U^0_{J_n} \right\}.
\]
Then, for all \( g \in U_{(j_0,j_1)}^1 \), \( \tau(g \cdot s_{j_1}(s(g))) = r(g) = r(s_{j_0}(r(g))) \); hence, there exists a unique element \( c_{j_0j_1}(g) \in S \) such that \( g \cdot s_{j_1}(s(g)) = s_{j_0}(r(g)) \cdot c_{j_0j_1}(g) \).

We then obtain a family of continuous functions \( c_{j_0j_1} : U_{(j_0,j_1)}^1 \to S \) such that

\[
g \cdot s_{j_1}(s(g)) = s_{j_0}(r(g)) \cdot c_{j_0j_1}(g), \quad \forall g \in U_{(j_0,j_1)}^1.
\]

Note further that \( U_{(j_0,j_1)}^1 = \bar{\varepsilon}_0^{-1}(U_{j_0}^0) \cap \bar{\varepsilon}_1^{-1}(U_{j_0}^0) \). Let \( (g_1,g_2) \in U_{(j_0,j_1,j_2)}^2 \).

Then

\[
(g_1g_2) \cdot s_{j_2}(s(g_2)) = g_1 \cdot s_{j_1}(r(g_2)) \cdot c_{j_1j_2}(g_2) = g_1 \cdot s_{j_1}(s(g_1)) \cdot c_{j_1j_2}(g_2)
\]

hence \( c_{j_0j_2}(g_1g_2) = c_{j_0j_1}(g_1) \cdot c_{j_1j_2}(g_2) \). In other words,

\[
\bar{\varepsilon}_0^*c_{\varepsilon_0(j_0,j_1,j_2)} \cdot (\bar{\varepsilon}_1^*c_{\varepsilon_1(j_0,j_1,j_2)})^{-1} \cdot \bar{\varepsilon}_2^*c_{\varepsilon_2(j_0,j_1,j_2)} = 1
\]

over all \( U_{(j_0,j_1,j_2)}^2 \). Moreover, we clearly have \( c_{j_0j_1}(\rho(g)) = \overline{c_{j_0j_1}(g)} \in S \). This gives us a Real 1-cocycle \( (c_{j_0j_1})_{(j_0,j_1)\in J_1} \in ZR^1(\mathcal{U}_*,S^*). \)

Suppose \( f : (Z,\tau) \to (Z',\tau') \) is an isomorphism of Real generalized morphisms (see chapter 2). Up to a refinement, we can choose \( \mathcal{U}_0 \) in such a way that we have two Real families \( (s_j)_{j \in J_0} \), \( (s'_j)_{j \in J_0} \) of local sections of the Real projections \( \tau : (Z,\tau) \to (X,\rho) \) and \( \tau' : (Z',\tau') \to (X,\rho) \) respectively. Since for all \( j \in J_0 \) and \( x \in U_j \), \( \tau'(f_{U_j}(s_j(x))) = \tau(s_j(x)) = x = \tau'(s'_j(x)) \), there exists a unique element \( \varphi_j(x) \in S \) such that \( s'_j(x) = f_{U_j}(s_j(x)) \cdot \varphi_j(x) \), and this gives a Real family of continuous functions \( \varphi_j : U_j \to S \). It follows that if \( c = (c_{j_0j_1}) \) and \( c' = (c'_{j_0j_1}) \) are the Real 1-cocycle associated to \( (Z,\tau) \) and \( (Z',\tau') \) respectively. Then, over \( U_{(j_0,j_1,j_2)}^1 \), one has

\[
g \cdot f_{U_{j_1}}(s_{j_1}(s(g))) \cdot \varphi_{j_1} = f_{U_{j_0}}(s_{j_0}(r(g))) \cdot \varphi_{j_0}(r(g)) \cdot c'_{j_0j_1}(g);
\]

But, since \( f \) is \( S \)-equivariant, we get

\[
f_{U_{j_0}}(s_{j_0}(r(g))) \cdot c_{j_0j_1}(g) \cdot \varphi_{j_1}(s(g)) = f_{U_{j_0}}(s_{j_0}(r(g))) \cdot \varphi_{j_0}(r(g)) \cdot c'_{j_0j_1}(g);
\]

thus \( c'_{j_0j_1}(g) \cdot c_{j_0j_1}(g) = \varphi_{j_1}(s(g)) \cdot \varphi_{j_0}(r(g))^{-1} \), or \( c' : c^{-1} \) over \( (j_0,j_1) \in J_1 \). This shows that \( c' : c^{-1} \in BHR^1(\mathcal{U}_*,S) \). We then deduce a well-defined group homomorphism

\[
c_1 : \text{Hom}_R(S,S) \to \bar{HR}^1(S_,S), \quad c_1([Z,\tau]) := [c_{j_0j_1}] \in HR^1(\mathcal{U}_*,S),
\]

where \( \mathcal{U}_* \) is the Real open cover defined from any Real local trivialization of \( (Z,\tau) \).

Conversely, given a Real Čech 1-cocycle \( c = (c_{\lambda_0\lambda_1}) \) over a pre-simplicial Real open cover \( \mathcal{U}_* \in \mathfrak{R}(1) \), we let \( Z := \coprod_{\lambda_0 \in \Lambda_0} U_{\lambda_0} \times S \) with the Real structure \( \nu \) defined by \( \nu(x,t) := (\rho(x),t) \), and equipped with the Real \( S \)-action \( g \cdot (s(g),t) := (r(g),c_{\lambda_0\lambda_1}(g) \cdot t) \) for any \( g \in U_{\lambda_0\lambda_1\lambda_1}^1, t \in S \), and the obvious Real \( S \)-action. It is easy to see that the canonical projections define
a Real generalized morphism \((Z, \nu) : (\mathcal{G}, \rho) \to (S, \tau)\). One can check that if \([c] = [c']\) then \((Z, \tau) \cong (Z', \tau')\) by working backwards. \(\square\)

**Remark 2.50.** Suppose that \((S, \sigma)\) is a non-abelian Real group. Then we still can talk about \(\check{\text{Cech}}\) Real 1-cocycles on \((\mathcal{G}, \rho)\) with coefficients on the non-Abelian Real sheaf \((\mathcal{S}, \sigma^\bullet)\), and then form in the same way \(HR^1(\mathcal{G}, \mathcal{S})\) as a set. However, there is no reason for \(HR^1(\mathcal{G}, S)\) to be an Abelian group, it is not even a group since the sum of a Real 1-cocycle is not necessarily a Real 1-cocycle. Nevertheless, the result above remains valid in the sense that there is a bijection between the set \(\text{Hom}_{\mathcal{H}\mathcal{R}\mathcal{E}}(\mathcal{G}, S)\) of isomorphism classes of generalized Real morphism \((\mathcal{G}, \rho) \to (S, \sigma)\) and the set \(HR^1(\mathcal{G}, S)\).

A particular example of Proposition 2.49 is when \(S = \mathbb{S}^1\) together with the complex conjugation as Real structure; in this case, the associated Real sheaf is denoted by \(\mathcal{S}^1\) as mentioned earlier. It is well known that the Picard group \(\text{Pic}(X)\) of a locally compact topological space \(X\) is isomorphic to the \(1^{\text{st}}\) sheaf cohomology group \(H^1(X, \mathbb{S}^1_X)\) (see for instance [3, Chap. 2]). In the Real case, we shall introduce the Real Picard group \(\text{PicR}(\mathcal{G})\) of a Real groupoid, and we will apply Proposition 2.49 to get an analogous result.

**Definition 2.51** (Real line \(\mathcal{G}\)-bundle).

1. By a Real line \(\mathcal{G}\)-bundle we mean a Real \(\mathcal{G}\)-space \((\mathcal{L}, \nu)\), and a continuous surjective Real map \(\pi : (\mathcal{L}, \nu) \to (X, \rho)\) such that \(\pi : \mathcal{L} \to X\) is a complex vector bundle of rank 1, and such that for every \(x \in X\), the induced isomorphism \(\nu_x : \mathcal{L}_x \to \mathcal{L}_{\rho(x)}\) is \(\mathbb{C}\)-anti-linear in the sense that \(\nu_x(v \cdot z) = \nu_x(v) \cdot \bar{z}\).

2. A homomorphism from a Real line \(\mathcal{G}\)-bundle \((\mathcal{L}, \nu)\) to a Real line \(\mathcal{G}\)-bundle \((\mathcal{L}', \nu')\) is a homomorphism of complex vector bundles \(\phi : \mathcal{L} \to \mathcal{L}'\) intertwining the Real structures and which is \(\mathcal{G}\)-equivariant; i.e., \(\phi(g \cdot v) = g \cdot \phi(v)\) for any \((g, v) \in \mathcal{G} \ltimes \mathcal{L}\).

3. We say that a Real line \(\mathcal{G}\)-bundle \((\mathcal{L}, \nu)\) is locally trivial if there exists a Real open cover \(\mathcal{U}\) of \((X, \rho)\), and a family of isomorphisms of complex vector bundles \(\varphi_j : U_j \times \mathbb{C} \to \mathcal{L}|_{U_j}\) such that:
   - \(\varphi_j(\rho(x), z) = \nu_{U_j}(\varphi_j(x, z))\) for all \(x \in U_j\) and \((x, z) \in U_j \times \mathbb{C}\).
   - If \(r(g) \in U_{j_0}\) and \(s(g) \in U_{j_1}\), then one has \(g \cdot \varphi_{j_1}(s(g), z) = \varphi_{j_0}(r(g), z)\).

**Example 2.52.** The trivial action \(\mathcal{G}\) on \(X \times \mathbb{C}\) (i.e., \(g \cdot (s, z) := (r(g), z)\)) is Real; moreover, the canonical projection \(X \times \mathbb{C} \to X\) defines a Real line \(\mathcal{G}\)-bundle that we call trivial.

**Definition 2.53** (Real hermitian \(\mathcal{G}\)-metric). Let \((\mathcal{L}, \nu)\) be a locally trivial Real line \(\mathcal{G}\)-bundle. A Real hermitian \(\mathcal{G}\)-metric on \((\mathcal{L}, \nu)\) is a continuous function \(h : \mathcal{L} \to \mathbb{R}_+\) such that:
   - \(h(\nu(v)) = h(v)\), and \(h(v \cdot z) = h(v) \cdot |z|^2\), for all \(v \in \mathcal{L}\), \(z \in \mathbb{C}\).
   - \(h(g \cdot v) = h(v)\), for all \((g, v) \in \mathcal{G} \ltimes \mathcal{L}\).
\[ \bullet h(v) > 0 \text{ whenever } v \in \mathcal{L}^+ := \mathcal{L} \setminus o, \text{ where } o : X \hookrightarrow \mathcal{L} \text{ is the zero-section.} \]

If such \( h \) exists, \((\mathcal{L}, \nu, h)\) is called a hermitian Real line \( \mathcal{S} \)-bundle (we will often omit the metric).

**Definition 2.54** (The Real Picard group). The Real Picard group of \((\mathcal{S}, \rho)\) is defined as the set of isomorphism classes of locally trivial hermitian Real line \( \mathcal{S} \)-bundles. This “group” is denoted by \( \text{PicR}(\mathcal{S}) \).

**Theorem 2.55.** (compare with [3, Theorem 2.1.8]). Let \((\mathcal{S}, \rho)\) be a locally compact Hausdorff Real groupoid. Then \( \text{PicR}(\mathcal{S}) \) is an Abelian group. Furthermore,

\[ \text{PicR}(\mathcal{S}) \cong \hat{H}R^1(\mathcal{S}, \mathbb{S}^1). \]

**Proof.** Associated to any hermitian Real line \( \mathcal{S} \)-bundle \( \pi : (\mathcal{L}, \nu) \to (X, \rho) \), there is a Real generalized morphism \((\mathcal{L}^1, \nu) : (\mathcal{S}, \rho) \to (\mathbb{S}^1, -)\) obtained by setting

\[ (\mathcal{L}^1, \nu) := \{ v \in \mathcal{L} \mid h(v) = 1 \}. \]

\( \pi : (\mathcal{L}^1, \nu) \to (X, \rho) \) is indeed an \( \mathbb{S}^1 \)-principal Real bundle, and \( \mathcal{L}^1 \) is invariant under the action of \( \mathcal{S} \). Hence \((\mathcal{L}^1, \nu)\) is indeed a Real generalized morphism. Conversely, if \((\hat{\mathcal{L}}, \hat{\nu}) : (\mathcal{S}, \rho) \to (\mathbb{S}^1, -)\) is a Real generalized morphism, define \( \mathcal{L} := \hat{\mathcal{L}} \times_{\mathbb{S}^1} \mathbb{C} \), where \( \mathbb{S}^1 \) acts by multiplication on \( \mathbb{C} \); \( \nu(v, z) := (\hat{\nu}(v), \bar{z}), g \cdot (v, z) := (g \cdot v, z) \) for \((g, v) \in \mathcal{S} \times \hat{\mathcal{L}}\), and \( h(v, z) := |z|^2 \).

Then \((\mathcal{L}, \nu, h)\) is a hermitian Real line \( \mathcal{S} \)-bundle. Moreover, it is not hard to check that if \((\mathcal{L}, \nu, h)\) and \((\mathcal{L}', \nu', h')\) are isomorphic hermitian Real line \( \mathcal{S} \)-bundles, then their associated Real generalized homomorphisms \((\mathcal{L}^1, \nu)\) and \((\mathcal{L}'^1, \nu')\) are isomorphic. We then have a map

\[ \text{PicR}(\mathcal{S}) \to H^1(\mathcal{S}, \mathbb{S}^1) \rho, ((\mathcal{L}, \nu, h)) \mapsto [\mathcal{L}^1, \nu] \]

which is clearly an isomorphism of Abelian groups. Now, applying Proposition 2.49, we get the desired result. \( \square \)

### 2.10. \( \hat{H}R^2 \) and ungraded Real extensions

Let us consider the subgroup \( \text{ExtR}^+ (\Gamma, \mathcal{S}) \) of ungraded Real \( \mathcal{S} \)-twists of the Real groupoid \( \Gamma \); that is \((\tilde{\Gamma}, \delta)\) is ungraded if \( \delta = 0 \). Similarly, we define the subgroup \( \hat{\text{ExtR}}^+ (\mathcal{S}, \mathcal{S}) \) of \( \hat{\text{ExtR}}(\mathcal{S}, \mathcal{S}) \) of ungraded Real \( \mathcal{S} \)-central extensions over \( \mathcal{S} \). Elements of \( \hat{\text{ExtR}}^+ (\mathcal{S}, \mathcal{S}) \) will then be denoted by pairs of the form \((\tilde{\Gamma}, \Gamma)\).

Let \( \mathcal{T} = \mathcal{S} \twoheadrightarrow \hat{\mathcal{S}} \xrightarrow{\pi} \mathcal{S}[\mathcal{U}_0] \in \text{ExtR}^+ (\mathcal{S}[\mathcal{U}_0], \mathcal{S}) \) be an ungraded Real \( \mathcal{S} \)-twist, for a fixed Real open cover \( \mathcal{U}_0 = (U^0_j)_{j \in J_0} \). Consider again the pre-simplicial Real open cover \( \mathcal{U}_\bullet \) of \((\mathcal{S}_\bullet, \rho_\bullet)\) defined by (2.42). Recall that the groupoid \( \mathcal{S}[\mathcal{U}_0] \) is defined by

\[ \mathcal{S}[\mathcal{U}_0] = \left\{ (j_0, g, j_1) \in J_0 \times \mathcal{S} \times J_0 \mid g \in U^1_{(j_0, j_1)} \right\}. \]
Suppose that the $S$-principal Real bundle $\pi : (\tilde{G}, \tilde{\rho}) \to (\mathcal{U}_0, \rho)$ admits a Real family of local continuous sections $s_{j_0j_1}$ relative to the Real open cover $\mathcal{V}_1$ of $(\mathcal{U}_0, \rho)$ given by $\mathcal{V}_1 = \{ V_{(j_0,j_1)} \}_{(j_0,j_1) \in J_1}$, where

$$V_{(j_0,j_1)}^1 := \{ j_0 \} \times U_{(j_0,j_1)}^1 \times \{ j_1 \}.$$  

Then, for any $(g_1, g_2) \in U_{(j_0,j_1,j_2)}^2$, we have that

$$\pi(s_{j_0j_1}(j_0, g_1, j_1) \cdot s_{j_1j_2}(j_1, g_2, j_2)) = \pi(s_{j_0j_1}(j_0, g_1, j_1)) \cdot \pi(s_{j_1j_2}(j_1, g_2, j_2))$$
$$= (j_0, g_1 g_2, j_2) = \pi(s_{j_0j_2}(j_0, g_1 g_2, j_2));$$

thus, there exists a unique element $\omega_{(j_0,j_1,j_2)}(g_1, g_2) \in S$ such that

$$(2.47) \ s_{j_0j_2}(j_0, g_1 g_2, j_2) = \omega_{(j_0,j_1,j_2)}(g_1, g_2) \cdot s_{j_0j_1}(j_0, g_1, j_1) \cdot s_{j_1j_2}(j_1, g_2, j_2).$$

This provides a family of continuous functions $\omega_{(j_0,j_1,j_2)} : U_{(j_0,j_1,j_2)}^2 \to S$ determined by $(2.47)$ and that clearly verifies

$$\omega_{(j_0,j_1,j_2)}(\rho(g_1), \rho(g_2)) = \omega_{(j_0,j_1,j_2)}(g_1, g_2), \forall (g_1, g_2) \in U_{(j_0,j_1,j_2)}^2 \subset \mathcal{G}_2.$$

It is straightforward that the family $(\omega_{(j_0,j_1,j_2)})$ verifies the cocycle condition; hence we obtain a Real Čech 2-cocycle

$$(2.48) \ \omega(\mathcal{F}) := (\omega_{(j_0,j_1,j_2)})_{(j_0,j_1,j_2) \in J_2} \in ZR^2(\mathcal{U}_0, S)$$

associated to $\mathcal{F}$.

In fact, this construction generalizes to arbitrary Real open covers $\mathcal{U}_0$ of $(\mathcal{G}_0, \rho_0)$.

**Lemma 2.56** (Cf. Proposition 5.6 in [21]). Let $(\mathcal{G}, \rho)$ be a topological Real groupoid. Given a Real open cover $\mathcal{U}_0$ of $(\mathcal{G}_0, \rho_0)$, let $\text{ext}^R_{\mathcal{U}_0}^+(\mathcal{G}[\mathcal{U}_0], S)$ denote the subgroup of all twists $S \to \tilde{G} \xrightarrow{\pi} \mathcal{G}[\mathcal{U}_0] \in \text{ext}^R_{\mathcal{U}_0}^+(\mathcal{G}[\mathcal{U}_0], S)$ such that $\pi$ admits a Real family of local continuous sections

$$s_\ell : \{ \lambda_0 \} \times U_\ell \times \{ \lambda_1 \} \to \tilde{G}$$

relative to the Real open cover

$$\mathcal{V}_1 := \{ \{ \lambda_0 \} \times U_{(\lambda_0,\lambda_1,\lambda_01)}^1 \times \{ \lambda_1 \} \}_{(\lambda_0,\lambda_1,\lambda_01) \in \Lambda_1}$$

of $(\mathcal{G}[\mathcal{U}_0], \rho)$. Then the canonical map

$$(2.49) \ \text{ext}^R_{\mathcal{U}_0}^+(\mathcal{G}[\mathcal{U}_0], S) \to HR^2(\mathcal{U}_0, S), \ \left[ \mathcal{F} \right] \mapsto [\omega(\mathcal{F})],$$

is a group isomorphism.

**Proof.** First, we prove $\text{ext}^R_{\mathcal{U}_0}^+(\mathcal{G}[\mathcal{U}_0], S)$ is a subgroup of $\text{ext}^R_{\mathcal{U}_0}^+(\mathcal{G}[\mathcal{U}_0], S)$. Let

$$\mathcal{F} = ( S \to \tilde{G} \xrightarrow{\pi} \mathcal{G}[\mathcal{U}_0] ), \ \mathcal{F}' = ( S \to \tilde{G}' \xrightarrow{\pi'} \mathcal{G}[\mathcal{U}_0] )$$

be two such twists. We want to show that if $\mathcal{F}$ is a Real family of local continuous sections, then $\mathcal{F}'$ is also a Real family of local continuous sections, and that $[\omega(\mathcal{F})] = [\omega(\mathcal{F}')]$.
be representatives in $\text{extR}^+_U(\mathcal{G}[U_0], S)$. Then their tensor product (see (1.8)) is

$$\mathcal{T} \otimes \mathcal{T}' := ( S \longrightarrow \tilde{G} \otimes \tilde{G}' \longrightarrow \mathcal{G}[U_0] , 0),$$

where $\tilde{G} \otimes \tilde{G}' = \tilde{G} \times_{\mathcal{G}[U_0]} \tilde{G}'/S$. Let

$$f_\lambda : \{\lambda_0\} \times U^1_\lambda \times \{\lambda_1\} \longrightarrow \tilde{G},$$

$$f'_\lambda : \{\lambda_0\} \times U^1_\lambda \times \{\lambda_1\} \longrightarrow \tilde{G}'$$

be Real families of continuous local sections of $\pi$ and $\pi'$ respectively. Then we get a Real family of continuous local sections

$$s_\lambda : \{\lambda_0\} \times U^1_\lambda \times \{\lambda_1\} \longrightarrow \tilde{G} \otimes \tilde{G}'$$

for $\pi$ by setting

$$s_\lambda(\lambda_0, g, \lambda_1) := [(f_\lambda(\lambda_0, g, \lambda_1), f'_\lambda(\lambda_0, g, \lambda_1))],$$

which implies that $\mathcal{T} \otimes \mathcal{T}' \in \text{extR}^+_U(\mathcal{G}[U_0], S)$.

Now let $\mathcal{T}$ be an (ungraded) Real twist of $(\mathcal{G}[U_0], \rho)$ such that $\pi$ verifies the condition of the lemma. Assume that $\mathcal{T}'$ is any Real twist of $(\mathcal{G}[U_0], \rho)$ isomorphic to $\mathcal{T}$. Let $f : \tilde{G} \longrightarrow \tilde{G}'$ be a Real $S$-equivariant isomorphism that makes the following diagram

$$\begin{array}{ccc}
\tilde{G} & \longrightarrow & \mathcal{G}[U_0] \\
\downarrow f & & \downarrow \pi' \\
\tilde{G}' & \longrightarrow & \mathcal{G}[U_0]
\end{array}$$

commute. Thus, given a Real family $s_\lambda : \{\lambda_0\} \times U^1_\lambda \times \{\lambda_1\} \longrightarrow \tilde{G}$, the maps $f \circ s_\lambda : \{\lambda_0\} \times U^1_\lambda \times \{\lambda_1\} \longrightarrow \tilde{G}'$ define a Real family of local continuous sections for $\pi'$; hence the class $[\mathcal{T}] \in \text{extR}^+_U(\mathcal{G}[U_0], S^1)$.

Suppose we are given a representative

$$\mathcal{T} = S \longrightarrow \tilde{G} \longrightarrow \mathcal{G}[U_0]$$

in $\text{extR}^+_U(\mathcal{G}[U_0], S)$. Recall that for $(\lambda_0, \lambda_1, \lambda_{01}) \in \Lambda_1$,

$$U^1_{\lambda_0\lambda_1\lambda_{01}} = U^1_{\lambda_0\lambda_1} \cap r^{-1}(U^0_{\lambda_0}) \cap s^{-1}(U^0_{\lambda_1}),$$

and for $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_{01}, \lambda_{02}, \lambda_{12}, \lambda_{012}) \in \Lambda_2$, we have from (2.38) that

$$U^2_{\lambda} = \varepsilon^{-1}_0 \circ r^{-1}(U^0_{\lambda_0}) \cap \varepsilon^{-1}_0 \circ s^{-1}(U^0_{\lambda_1}) \cap \varepsilon^{-1}_1 \circ s^{-1}(U^0_{\lambda_2})$$

$$\cap \varepsilon^{-1}_2(U^1_{\lambda_{01}}) \cap \varepsilon^{-1}(U^1_{\lambda_{02}}) \cap \varepsilon^{-1}(U^1_{\lambda_{12}}) \cap U^2_{\lambda_{012}}.$$

Then, for all $(g_1, g_2) \in U^2_{\lambda}$, one has:

- $g_1g_2 = \varepsilon_1(g_1, g_2) \in r^{-1}(U^0_{\lambda_0}) \cap s^{-1}(U^0_{\lambda_2}) \cap U^1_{\lambda_{02}} = U^1_{\lambda_0\lambda_2\lambda_{02}}$.
It turns out that the and hence

\[ g_1 \in r^{-1}(U_0^0) \cap s^{-1}(U_1^0) \cap U^1_{\lambda_0 \lambda_1 \lambda_{01}}, \]
\[ g_2 \in r^{-1}(U_0^0) \cap s^{-1}(U_2^0) \cap U^1_{\lambda_0 \lambda_1 \lambda_{12}}. \]

Then as in the discussion before the lemma (see (2.48)), there exists a Real family of functions \( \omega_\lambda : U^2_2 \to S^1 \) such that

\[ s_{\lambda_0 \lambda_2 \lambda_{02}}(\lambda_0, g_{12}, \lambda_2) = \omega_\lambda(g_1, g_2) \cdot s_{\lambda_0 \lambda_1 \lambda_{01}}(\lambda_0, g_1, \lambda_1) \cdot s_{\lambda_1 \lambda_2 \lambda_{12}}(\lambda_1, g_2, \lambda_2) \]

and \( \omega_\lambda(\rho(g_1), \rho(g_2)) = \omega_\lambda(g_1, g_2) \), for all \((g_1, g_2) \in U^2_2\). Moreover, it is easy to verify by a routine calculation that \((\omega_\lambda)_\lambda \in \Lambda_2\) verify the cocycle condition on

\[ U^3_{\lambda_0 \lambda_2 \lambda_3 \lambda_{023} \lambda_{12} \lambda_{13} \lambda_{0123}} \subset \mathcal{S}_2; \]

thus, we have constructed a Real Čech 2-cocycle \((\omega_\lambda)_\lambda \in \Lambda_2 \in ZR^2(\mathcal{U}_*, S)\) associated to \(\mathcal{T}\).

Assume that \((\tilde{s}_\lambda)_\lambda \in \Lambda_2\) is another Real family of continuous local sections of \(\pi\), and that \((\tilde{\omega}_\lambda)_\lambda \in \Lambda_2 \in ZR^2(\mathcal{U}_*, S)\) is its associated Real Čech 2-cocycle. Then for any \((\lambda_0, \lambda_1, \lambda_{01}) \in \Lambda_1\) and \(g \in U^1_{\lambda_0 \lambda_1 \lambda_{01}}\), there exists a unique \(c_{\lambda_0 \lambda_1 \lambda_{01}}(g) \in S\) such that

\[ \tilde{s}_{\lambda_0 \lambda_1 \lambda_{01}}(g) = c_{\lambda_0 \lambda_1 \lambda_{01}}(g) \cdot s_{\lambda_0 \lambda_1 \lambda_{01}}(g), \]

where we abusively write, for instance, \(s_{\tilde{s}_0 \lambda_1 \lambda_{01}}(g)\) for \(s_{\lambda_0 \lambda_1 \lambda_{01}}(\lambda_0, g, \lambda_1)\). Since \((\tilde{s}_\lambda_0 \lambda_1 \lambda_{01})\) and \(s_{\lambda_0 \lambda_1 \lambda_{01}}\) are Real families, we have that

\[ c_{\lambda_0 \lambda_1 \lambda_{01}}(\rho(g)) = c_{\lambda_0 \lambda_1 \lambda_{01}}(g) \]

for all \(g \in U^1_{\lambda_0 \lambda_1 \lambda_{01}}\).

It turns out that the \(c_{\lambda_0 \lambda_1 \lambda_{01}}\)'s define an element in \(CR^1(\mathcal{U}_*, S)\). Moreover, for \(\lambda \in \Lambda_2\) as previously, and for \((g_1, g_2) \in U^2_2\), we obtain from (2.51) and (2.52)

\[ s_{\lambda_0 \lambda_2 \lambda_{02}}(g_1, g_2) = c_{\lambda_0 \lambda_2 \lambda_{02}}(g_1, g_2)^{-1} \cdot c_{\lambda_0 \lambda_1 \lambda_{01}}(g_1) \cdot c_{\lambda_1 \lambda_2 \lambda_{12}}(g_2) \cdot \tilde{\omega}_\lambda(g_1, g_2) \cdot s_{\lambda_0 \lambda_1 \lambda_{01}}(g_1) \cdot s_{\lambda_1 \lambda_2 \lambda_{12}}(g_2); \]

and

\[ (\omega_\lambda \cdot \tilde{\omega}_\lambda^{-1})(g_1, g_2) = c_{\lambda_0 \lambda_2 \lambda_{02}}(g_1, g_2)^{-1} \cdot c_{\lambda_0 \lambda_1 \lambda_{01}}(g_1) \cdot c_{\lambda_1 \lambda_2 \lambda_{12}}(g_2) \]

\[ = (dc)_\lambda(g_1, g_2); \]

hence \((\omega \cdot \tilde{\omega}^{-1})_\lambda \in \Lambda_2 \in BR^2(\mathcal{U}_*, S^1)\). I.e., the class in \(HR^2(\mathcal{U}_*, S)\) of the Real 2-cocycle \((\omega_\lambda)\) does not depend on the choice of the Real family of local sections of \(\pi\).

We want now to check that the map (2.49) is well-defined. To do so, suppose that \(\mathcal{T}\) and \(\mathcal{T}'\) are equivalent in \(\text{ext}R\mathcal{U}(\mathcal{S}[\mathcal{U}_0], S)\), and that \((s_{\lambda_0 \lambda_1 \lambda_{01}})\) and \(s'_{\lambda_0 \lambda_1 \lambda_{01}}\) are Real family of local continuous sections of \(\pi\) and \(\pi'\). Let us keep the diagram (2.50). Let \((\omega_\lambda)_\lambda \in \Lambda_2\) and \((\omega'_\lambda)_\lambda \in \Lambda_2\) be the associated Real 2-cocycles in \(ZR^2(\mathcal{U}_*, S)\) of \(\mathcal{T}\) and \(\mathcal{T}'\) respectively. Then we define an
element \((b_{\lambda_0,\lambda_1,\lambda_01}) \in CR^1(\mathbb{U}_\bullet, S)\) as follows: for any \(g \in U^1_{\lambda_0,\lambda_1,\lambda_01}\), \(b_{\lambda_0,\lambda_1,\lambda_01}(g)\) is the unique element of \(S\) such that
\[
(2.53) \quad s'_{\lambda_0,\lambda_1,\lambda_01}(g) = b_{\lambda_0,\lambda_1,\lambda_01}(g) \cdot f \circ s_{\lambda_0,\lambda_1,\lambda_01}(g).
\]
This is well-defined since
\[
\pi'(s'_{\lambda_0,\lambda_1,\lambda_01}(g)) = \pi(s_{\lambda_0,\lambda_1,\lambda_01}(g)) = \pi'(f \circ s_{\lambda_0,\lambda_1,\lambda_01}(g)).
\]
Furthermore, the functions \(f \circ s_{\lambda_0,\lambda_1,\lambda_01}, (\lambda_0, \lambda_1, \lambda_01) \in \Lambda_1\), defines a globally Real family of local continuous sections of \(\pi\). Then, for all \(\lambda \in \Lambda_2\) and all \((g_1, g_2) \in U^2_{\lambda}\), we can write
\[
f \circ s_{\lambda_0,\lambda_2,\lambda_02}(g_1g_2) = \omega(\lambda)(g_1, g_2) \cdot f \circ s_{\lambda_0,\lambda_1,\lambda_01}(g_1) \cdot f \circ s_{\lambda_1,\lambda_2,\lambda_12}(g_2),
\]
up to a multiplication of \(\omega(\lambda)\) by a Real 2-coboundary. It then follows that
\[
\omega(\lambda)(g_1, g_2) \cdot \omega'(g_1, g_2)^{-1} = b_{\lambda_0,\lambda_2,\lambda_02}(g_1g_2)^{-1} \cdot b_{\lambda_0,\lambda_1,\lambda_01}(g_1) \cdot b_{\lambda_1,\lambda_2,\lambda_12}(g_2) = (db)_\lambda(g_1, g_2).
\]
Consequently, \((\omega(\lambda))_{\lambda \in \Lambda_2}\) depends only on the class of \(T\) in \(\text{extR}_U(\mathcal{S}[\mathbb{U}_0], S)\). The fact that \((d\lambda_0,\lambda_01)\) also depends only on the class of \(T\) is straightforward.

We then have proved that any element \([T]\) in \(\text{extR}_U(\mathcal{S}[\mathbb{U}_0], S)\) determines a unique cohomology class
\[
(2.54) \quad [\omega(T)] \in HR^2(\mathbb{U}_\bullet, S).
\]

Conversely, given a pair \((\omega(\lambda))_{\lambda \in \Lambda_2} \in ZR^2(\mathbb{U}_\bullet, S)\), we want to construct an ungraded Real extension of \((\mathcal{S}[\mathbb{U}_0], \rho)\) which is in \(\text{extR}^+_U(\mathcal{S}[\mathbb{U}_0], S)\). For this we proceed as in the proof of Proposition 5.6 in [21]. For \(\lambda \in \Lambda_2\), put
\[
\mu_{01} := (\lambda_0, \lambda_{01}, \lambda_1),
\mu_{02} := (\lambda_0, \lambda_{02}, \lambda_2),
\mu_{12} := (\lambda_1, \lambda_{12}, \lambda_2).
\]
Let \(\epsilon_{\mu_{01},\mu_{02},\mu_{12}} := \omega(\lambda)\). We have \(V_1 = (V^1_{\mu_{01}})_{i \in I_1}\), where \(I_1\) consists of triples \(\mu_{01} = (\lambda_0, \lambda_{01}, \lambda_1)\) and \(V^1_{\mu_{01}} := \{\lambda_0\} \times U^1_{\lambda_{01}} \times \{\lambda_1\}\). \(I_1\) is equipped with the obvious involution, so that \(V_1\) is a Real open cover of \(\mathcal{S}[\mathbb{U}_0]\). We set
\[
\widetilde{\Gamma}^\omega := \coprod_{\mu_{01} \in I_1} \{[t, g, \mu_{01}] \mid t \in S, g \in V^1_{\mu_{01}}\}/\sim,
\]
subject to the product law
\[
[t_1, g_1, \mu_{01}] \cdot [t_2, g_2, \mu_{12}] = [t_1 \cdot t_2 \cdot \epsilon_{\mu_{01},\mu_{02},\mu_{12}}(g_1, g_2), g_1g_2, \mu_{02}],
\]
where
\[
(2.55) \quad (t, g, \mu_{12}) \sim (\epsilon_{\mu_{01},\mu_{01},\mu_{01}}(r(g), r(g))^{-1} \cdot t \cdot \epsilon_{\mu_{01},\mu_{02},\mu_{12}}(r(g), g), g, \mu_{02}).
\]
The projection \(\pi : \widetilde{\Gamma}^\omega \to \mathcal{S}[\mathbb{U}_0]\) is defined by \(\pi([t, g, \mu_{01}]) := g\), and the Real structure is
\[
\overline{[t, g, \mu_{01}]} := [t, \rho(g), \mu_{01}].
\]
It is straightforward to see that these operations give $\tilde{\Gamma}^\omega$ the structure of ungraded Real $S$-twist of $\mathcal{G}[U_0]$; what is more, the maps $s_{\mu_0} : V_{\mu_0}^1 \to \tilde{\Gamma}^\omega$ defined by $s_{\mu_0}(g) := [0, g, \mu_0]$ are a Real family of continuous sections of $\pi$, so that the Real extension

$$\mathcal{J} = S \longrightarrow \tilde{\Gamma}^\omega \longrightarrow \mathcal{G}[U_0]$$

is in $\text{Ext}_{R[u]}^+(\mathcal{G}[U_0], S)$. It is also clear that $[\omega(\mathcal{T})] = [\omega]$. □

**Corollary 2.57.** We have $\text{Ext}_{R[u]}^+(\mathcal{G}, S) \cong \check{HR}^2(\mathcal{G}, S)$.

2.11. The cup-product $\check{HR}^1(\cdot, \mathbb{Z}_2) \times \check{HR}^1(\cdot, \mathbb{Z}_2) \to \check{HR}^2(\cdot, \mathbb{S}^1)$. Let $\delta, \delta' \in \check{HR}^1(\mathcal{G}, \mathbb{Z}_2)$, and let $L$ and $L'$ be representatives of their corresponding classes in $\text{Hom}_{R[u]}(\mathcal{G}, \mathbb{Z}_2)$ (see Proposition 2.49). Then by viewing $\mathbb{Z}_2 = \{\pm 1\}$ as a Real subgroup of $\mathbb{S}^1$ (identifying $-1$ with $(-1, 0)$ and $+1$ with $(1, 0)$), we define the tensor product $r^*L \otimes s^*L' \to \mathcal{G}$, and and using the same reasoning as in Example 1.45, we see that this is clearly a Real $\mathbb{Z}_2$-principal bundle; thus we have an ungraded Real $\mathbb{Z}_2$-central extension

$$\mathbb{Z}_2 \longrightarrow r^*L \otimes s^*L' \longrightarrow \mathcal{G}.$$

Therefore, we get an ungraded Real $\mathbb{S}^1$-central extension $(L \sim L', \mathcal{G})$ given by

$$L \sim L' := (r^*L \otimes s^*L') \times_{\mathbb{Z}_2} \mathbb{S}^1,$$

together with the evident Real structure and Real $\mathbb{S}^1$-action.

**Definition 2.58.** We define the cup product

$$\sim : \check{HR}^1(\mathcal{G}, \mathbb{Z}_2) \times \check{HR}^1(\mathcal{G}, \mathbb{Z}_2) \to \check{HR}^2(\mathcal{G}, \mathbb{S}^1)$$

by

$$\delta \sim \delta' := \omega(L \sim L'),$$

where $L \sim L'$ is determined by equation (2.56).

**Lemma 2.59.** The cup product $\sim$ defined above is a well-defined bilinear map; i.e.,

$$(\delta_1 + \delta_2) \sim (\delta_1' + \delta_2') = \delta_1 \sim \delta_1' + \delta_1 \sim \delta_2' + \delta_2 \sim \delta_1' + \delta_2 \sim \delta_2'.$$

**Proof.** If $\delta_i$ is realized by the generalized Real homomorphism

$$L_i : \mathcal{G} \to \mathbb{Z}_2,$$

then $\delta_1 + \delta_2$ is realized by $L_1 + L_2$. The result follows from the easy to check bilinearity of the tensor product $r^*L \otimes s^*L'$ with respect to the sum in $\text{Hom}_{R[u]}(\mathcal{G}, \mathbb{Z}_2)$. □
2.12. Cohomological picture of the group $\hat{\text{Ext}}R(\mathcal{S}, S^1)$. Let \\
$\mathcal{T} = (\hat{\mathcal{S}}, \delta) \in \text{ext}R(\mathcal{S}[\mathcal{U}_0], S^1),$ \\
where as usual $\mathcal{U}_0$ is a Real open cover of $X$. Let $\mathcal{U}_* \text{ be the pre-simplexial}$ \\
Real open cover of $(\mathcal{S}, \rho)$ defined as in (2.42).

Define a continuous map $\delta_{j_{01}} : U^1(\mathcal{S}, \delta_{j_{01}}) \to \mathbb{Z}_2$ over all $U^1(\mathcal{S}, \delta_{j_{01}}) \subseteq \mathcal{U}_1$ by \\
$\delta_{j_{01}}(g) := \delta(j_0, g, j_1).$ Then, over all $U^2(\mathcal{S}, \delta_{j_{01}}),$ we have that \\
$\delta_{j_{01}}(g_1, g_2) = \delta((j_0, g_1, j_1) \cdot (j_1, g_2, j_2)) = \delta_{j_{01}}(g_1) \cdot \delta_{j_{01}}(g_2).$

Moreover, since $\delta$ is a Real morphism, we have that $\delta_{j_{01}}(\rho(g)) = \delta_{j_{01}}(g);$ hence $\mathcal{T}$ determines a Real Čech 1-cocycle \\
(2.57) $\delta(\mathcal{T}) := (\delta_{j_{01}}(g_{j_{01}})) \in ZR^1(\mathcal{U}_*, \mathbb{Z}_2),$

Then, (2.57) gives a Real Čech 1-cocycle $(\delta_{\lambda_0, \lambda_1, \lambda_{01}}) \in ZR^1(\mathcal{U}_*, \mathbb{Z}_2)$ defined by \\
$\delta_{\lambda_0, \lambda_1, \lambda_{01}}(g) := \delta(\lambda_0, g, \lambda_1)$ for any $g \in U^1_{\lambda_0, \lambda_1, \lambda_{01}};$ this does make sense, for \\
we know from Section 2.9 that Real Čech 1-cocycles do not depend on $\lambda_{01}.$

If $\mathcal{T}'$ is another $\mathcal{S}^1$-central extension over $\mathcal{S},$ we may suppose it is represented by a $\mathcal{S}^1$-twisted $(\hat{\mathcal{S}}', \delta')$ of $\mathcal{S} \mathcal{U}_0.$ Then by definition of the \\
grading of $\mathcal{T} \mathcal{T}'$, we have $\delta(\mathcal{T} \mathcal{T}') = \delta(\mathcal{T}) + \delta(\mathcal{T}').$

**Theorem 2.60** (Cf. [6, Proposition 2.13]). Let $(\mathcal{S}, \rho)$ be a locally compact \\
Hausdorff Real groupoid. There is a set-theoretic split-exact sequence \\
(2.58) $0 \to \check{H}R^2(\mathcal{S}_*, \check{S}^1) \to \hat{\text{Ext}}\check{R}(\mathcal{S}, \check{S}^1) \to \check{H}R^1(\mathcal{S}_*, \check{Z}_2) \to 0$

so that we have a canonical group isomorphism \\
(2.59) $dd : \hat{\text{Ext}}\check{R}(\mathcal{S}, \check{S}^1) \cong \check{H}R^1(\mathcal{S}_*, \check{Z}_2) \times \check{H}R^2(\mathcal{S}_*, \check{S}^1),$

where the semi-direct product $\check{H}R^1(\mathcal{S}_*, \check{Z}_2) \times \check{H}R^2(\mathcal{S}_*, \check{S}^1)$ is defined by the \\
operation \\
$(\delta, \omega) + (\delta', \omega') := (\delta + \delta', (\delta - \delta') \cdot \omega \cdot \omega').$

The image of a Real graded extension $\mathcal{E}$ by $dd$ is called the Dixmier–Douady \\
class of $\mathcal{E}$.

**Proof.** The first arrow is the canonical inclusion \\
$\hat{\text{Ext}}\check{R}(\mathcal{S}, \check{S}^1) \subseteq \hat{\text{Ext}}\check{R}(\mathcal{S}, \check{S}^1),$

and hence is injective. The exactness of the sequence (2.58) is obvious, by \\
definition of $\delta$ and $\hat{\text{Ext}}\check{R}(\mathcal{S}, \check{S}^1).$

The map $\delta$ is well-defined; indeed, if $\mathcal{T} \sim \mathcal{T}'$ in $\hat{\text{Ext}}\check{R}(\mathcal{S}[\mathcal{U}_0], \check{S}^1)$, they differ from a twist coming from an element of $\text{PicR}(\mathcal{S}[\mathcal{U}_0]),$ and hence by \\
construction of $\delta$, one has $\delta(\mathcal{T}) = \delta(\mathcal{T}').$ Moreover, $\delta$ is surjective, for if $L \in \text{Hom}_{\mathcal{R} \mathcal{S}}(\mathcal{S}, \mathcal{Z}_2)$ represents the Real 1-cocycle $(\varepsilon_{j_{01}}) \in ZR^1(\mathcal{U}_*, \mathcal{Z}_2),$ \\
then $L \sim L$ is graded as follows: \\
$L \sim L := (S^1 \to (r^*L \otimes \check{s}^*L) \times_{\mathcal{Z}_2} S^1 \to \mathcal{S}[\mathcal{U}_0], \delta'),$
where
\[ \delta'(j_0, \gamma, j_1) := \varepsilon_{j_0j_1}(\gamma). \]
We see that \( \delta(L \sim L) = \varepsilon. \) Finally, note that the operation law comes from the definition of the sum in \( \hat{\text{Ext}}^R(\mathcal{G}, \mathbb{S}^1). \)

\section{2.13. The proper case.}
In this subsection, we are interested in some particular Abelian Real sheaves on \( (\mathcal{G}_\bullet, \rho_\bullet), \) where \((\mathcal{G}, \rho)\) is a proper groupoid. More precisely, we aim to generalize a result by Crainic (see [4, Proposition 1]) stating that for a proper Lie groupoid \( \mathcal{G}, \) and “representation” \( E \) of \( \mathcal{G} \) ([4, 1.2]), the differentiable cohomology \( H^n_d(\mathcal{G}, E) = 0 \) for all \( n \geq 1. \) Let us first introduce some few notions and properties.

\begin{definition}[Real Haar measure]
Let \((\mathcal{G}, \rho)\) be a locally compact Real groupoid, and let \( \{\mu_x\}_{x \in X} \) be a (left) Haar system for \( \mathcal{G} \) (see [19, §2]). Define a new family \( \{\tilde{\mu}_x\}_{x \in X} \) of measures \( \mu^x, \) with support \( \mathcal{G}^x \) for all \( x \in X, \) defined by
\[ \tilde{\mu}_x^x(C) := \mu^x(\rho(C)), \]
for all measurable subset \( C \subset \mathcal{G}^x. \)
We say that \( \{\mu_x\}_{x \in X} \) is Real if
\[ \mu_x^x = \mu^x, \forall x \in X. \]

\begin{lemma}
Any Haar system for \( \mathcal{G} \) gives rise to a Real one.
\end{lemma}

\begin{proof}
Assume \( \{\mu^x\} \) is a Haar system for \( \mathcal{G}. \) For every \( x \in X, \) we set
\[ \tilde{\mu}_x := \frac{1}{2}(\mu^x + \mu^x_x). \]
It is clear that \( \{\tilde{\mu}_x\}_{x \in X} \) is a Haar system for \( \mathcal{G}; \) measurable subsets for \( \tilde{\mu}_x \) being exactly those for \( \mu^x. \) Moreover, one has
\[ \tilde{\mu}_x^x = \frac{1}{2} (\mu^x \circ \rho + \mu^x \circ \rho) = \frac{1}{2} (\mu^x + \mu^x) = \tilde{\mu}_x, \forall x \in X. \]
\end{proof}

\begin{remark}
From the lemma above, we will always assume Haar systems for \( \mathcal{G} \) to be Real.
\end{remark}

In what follows, the Real group \( \mathbb{K} \) is either the additive group \( \mathbb{R} \) equipped with the Real structure \( t \mapsto \bar{t} := -t, \) or the additive group \( \mathbb{C} \) equipped with the complex conjugation \( z \mapsto \bar{z} \) as Real structure.

\begin{definition}
Let \( (\mathcal{G}, \rho) \) be a locally compact Real groupoid. A Real representation of \( (\mathcal{G}, \rho) \) is a locally trivial Real \( \mathbb{K} \)-vector bundle
\[ \pi : (E, \nu) \rightarrow (X, \rho) \]
edowed with a (left) continuous Real \( \mathcal{G} \)-action; that is a Real open cover \( (U_j) \) of \( (X, \rho) \) and isomorphisms \( \phi_j : U_j \times \mathbb{K}^r \rightarrow E_{U_j} \) such that
\[ \nu(\phi_j(x, (a_1, \ldots, a_r))) = \phi_j(\rho(x), (\bar{a}_1, \ldots, \bar{a}_r)), \forall x \in U_j, (a_1, \ldots, a_r) \in \mathbb{K}^r, \] and
• ∀x ∈ X, the induced isomorphism νₙ : Eₓ → Eᵱₓ is K-antilinear:
  νₙ(ξ ⋅ a) = νₙ(ξ) ⋅ ¯a, ∀ξ ∈ Eₓ, a ∈ K;

• ∀g ∈ G, the isomorphism Eₛ(g) → Eᵱᵣ(g), induced by the G-action, is linear.

Note that such a Real representation (E, ν) can be viewed as a Real G-module in the following way: E is the groupoid E → X with
  rₓE := π(ξ) for every ξ ∈ E, for any x ∈ X, Eₓ = Eₓ = Eₓ is isomorphic
to the group K, then the product in E is defined by the sum on the fibres. The Real sheaf on (G•, ρ•) associated to the Real G-module (E, ν) will be
denoted (E•, ν•).

Remark 2.65. More generally, we define a Real representation of of type
Rₚ,ₚ as a locally trivial real vector bundle E → X of rank p + q, together
with a Real structure ν : E → E, and a Real G-action on E with respect to
the projection map, such that locally, the Real space (E, ν) identifies with
Rₚ,ₚ; that is there is a Real open cover (Uᵢ) of X and commutative diagrams

\[
\begin{array}{ccc}
Uᵢ × ℝₑq & \xrightarrow{φᵢ} & E|Uᵢ \\
\downarrow{ρ×bar} & & \downarrow{ν} \\
Uᵢ × ℝₑq & \xrightarrow{φᵢ} & E|Uᵢ 
\end{array}
\]

where bar : ℝₑq → ℝₑq is the Real structure defined in the first section.

Definition 2.66 ([25, Definition 2.20]). A locally compact Real groupoid
(𝐺, ρ) is said to be proper if either of the following equivalent conditions is satisfied:

(i) The Real map (s, r) : 𝐺 → X × X is proper.

(ii) For every K ⊂ X compact, 𝐺𝐾 is compact.

Proper Real groupoids can be characterized by the following (we refer to
Propositions 6.10 and 6.11 in [22] for a proof).

Proposition 2.67. Let (𝐺, ρ) be a locally compact Real groupoid with a Haar
system {µₓ}ₓ∈X. Then (𝐺, ρ) is proper if and only it admits a cutoff Real
function; that is, a function x : X → ℝ₊ such that:

(i) ∀x ∈ X, c(ρ(x)) = c(x).
(ii) ∀x ∈ X, ∫ₓ, c(s(g))dµₓ(g) = 1.
(iii) The map r : supp(c ◦ s) → X is proper; i.e., for every K ⊂ X
compact, supp(c) ∩ s(K) is compact.

Theorem 2.68. Suppose (𝐺, ρ) is a locally compact proper Real groupoid
with a Haar system. Then, for any Real representation (E, ν) of (𝐺, ρ), we have

\[\check{H}R^n(𝐺•, E•) = 0, \ \forall n ≥ 1.\]
To prove this result, we shall recall fundamentals of vector-valued integration exposed, for instance, in [26, Appendix B.1], and then adapt them to the case when we deal with Real structures. Let $X$ be a locally compact Hausdorff space, and let $B$ be a separable Banach space. Let $\mu$ be a Radon measure on $X$. Then measurable functions $f : X \to B$ are defined as usual, and such function is integrable if

$$\|f\|_1 := \int_X \|f(x)\|d\mu(x) < \infty.$$ 

The collection of all $B$-valued integrable functions on $X$ is denoted by $L^1(X, B)$, and the set of equivalence classes of functions in $L^1(X, B)$ is a Banach space denoted by $L^1(X, B)$ ([26, Proposition B.31]). Furthermore, $C_c(X, B)$ is dense in $L^1(X, B)$. The $B$-valued integration of elements of $L^1(X, B)$ is defined as a linear map $I : C_c(X, B) \to B$ given by

$$I(f) := \int_X f(x)d\mu(x), \text{ and } \|I(f)\| \leq \|f\|_1.$$ 

Moreover, this integral is characterized by the following:

**Proposition 2.69** (Cf. Proposition B.34 [26]). Let $\mu$ be a Radon measure on $X$, and let $B$ be a Banach space. Then, the integral is characterized by:

(a) For all $f \in C_c(X, B)$ and $\varphi \in B^*$,

$$\varphi\left(\int_X f(x)d\mu(x)\right) = \int_X \varphi(f(x))d\mu(x).$$

(b) If $L : B \to B'$ is any bounded linear map between two Banach spaces, then

$$L\left(\int_X f(x)d\mu(x)\right) = \int_X L(f(x))d\mu(x).$$

Now suppose $(X, \rho)$ is a locally compact Hausdorff Real space, $\mu$ is a Real Radon measure; i.e., $\mu(\rho(C)) = \rho(C)$ for every measurable set $C \subset X$. Let $(B, \varsigma)$ be a separable Real Banach space. Then from the above, we deduce the

**Lemma 2.70.** Let $C_c(X, B)$ be equipped with the Real structure denoted by $\tilde{\rho} : C_c(X, B) \to C_c(X, B)$, and given by $\rho(f)(x) := \varsigma(f(\rho(x)))$. Then, under the above assumption, the integral $\int : C_c(X, B) \to B$ is Real, in that it commutes with the Real structures $\varsigma$ and $\tilde{\rho}$; i.e.,

$$\int_X \varsigma(f(\rho(x)))d\mu(x) = \varsigma\left(\int_X f(x)d\mu(x)\right), \forall f \in C_c(X, B).$$

**Proof.** For any $\varphi \in B^*$, define $\tilde{\varphi} \in B^*$ by $\tilde{\varphi}(b) := \overline{\varphi(b)}$. Then, from Proposition 2.69(a) and the definition of $\tilde{\varphi}$, one has

$$\varphi\left(\varsigma\left(\int_X f(x)d\mu(x)\right)\right) = \int_X \tilde{\varphi}(\varsigma(f(x)))d\mu(x) = \int_X \varphi(\varsigma(f(x)))d\mu(x).$$
Thus,
\[
\varphi \left( \zeta \left( \int_X f(x) d\mu(x) \right) \right) = \int_X \varphi(\zeta(f(x))) d\mu(x).
\]
Again from (b) of Proposition 2.69 and from the fact that \( \mu \) is Real, we then get
\[
\varphi \left( \zeta \left( \int_X f(x) d\mu(x) \right) \right) = \varphi \left( \int_X \zeta(f(\rho(x))) d\mu(x) \right), \forall \varphi \in B^*,
\]
and the result holds. \( \square \)

Let us investigate the case of a Real groupoid \((\mathcal{G}, \rho)\) together with a Real representation \((E, \nu)\). Let \( \mu = \{ \mu^x \}_{x \in X} \) be a Real Haar system for \((\mathcal{G}, \rho)\). For any \( x \in X \), we can apply (2.63) to \( E_x \) and get the integral \( \int_{G^x} C_c(G^x, E_x) \rightarrow E_x \). Further, it is very easy to check that
\[
(2.65) \quad \nu_x \left( \int_{g^x} f(\gamma) d\mu^x(\gamma) \right) = \int_{g^{\rho(\gamma)}} \nu_x(f(\rho(\gamma))) d\mu^{\rho(x)}(\gamma), \forall f \in C_c(G^x, E_x).
\]

**Proof of Theorem 2.68.** Fix a Real Haar system \( \{ \mu^x \}_{x \in X} \) for \((\mathcal{G}, \rho)\) and a cutoff Real function \( c : X \rightarrow \mathbb{R}_+ \). Let \( U_s \) be a Real open cover of \((\mathcal{G}, \rho_s)\). Let \( \lambda := (\lambda_0, \lambda_1, \ldots, \lambda_{01} \cdots n) \in \Lambda_n \) and \( U^n_{\lambda} \in \mathfrak{U}_n \). Denote by \( \Lambda_{n+1}|_{\lambda} \) the subset of \( \Lambda_{n+1} \) consisting of those \( \tilde{\lambda} \in \Lambda_{n+1} \) such that \( \tilde{\lambda}(S) = \lambda_S \) for all \( \emptyset \neq S \subseteq [n] \). Then, if for any \( x \in U^n_{\lambda} \), we denote
\[
(U^n_{\lambda} \ast G^x) \cap \text{supp}(c \circ s) := \{(g_1, \ldots, g_n, \gamma) \in U^n_{\lambda} \times (G^x \cap \text{supp}(c \circ s)) \mid s(g_n) = r(\gamma) = x\},
\]
we have that
\[
(2.66) \quad (U^n_{\lambda} \ast G^x) \cap \text{supp}(c \circ s) \subset \bigcup_{\tilde{\lambda} \in \Lambda_{n+1}|_{\lambda}} U^{n+1}_{\tilde{\lambda}}.
\]
Notice that for \( \tilde{\lambda} \) running over \( \Lambda_{n+1}|_{\lambda} \), only its images \( \tilde{\lambda}_S \in \Lambda_{#S-1} \), for \( S \subseteq [n+1] \) containing \( n+1 \), are led to vary. On the other hand, since \( G^x \cap \text{supp}(c \circ s) \) is compact in \( G \) (by (iii) of Proposition 2.67), the union (2.66) is finite. In particular, for every \( S \in S(n+1) := \{ S \subseteq [n+1] \mid n+1 \in S \neq \emptyset \} \), where elements of \( S(n+1) \) are ranged in cardinality and in lexicographic order, there is \( \tilde{\lambda}_S^{LS} \in \Lambda_{#S-1}, l_S = 0, \ldots, m_S \), such that
\[
(2.67) \quad (U^n_{\lambda} \ast G^x) \cap \text{supp}(c \circ s) \subset \bigcup_{l=(l_S)_{S \in S(n+1)}} U^{n+1}_{\lambda_l},
\]
where for any \( l = (l_S)_{S \in S(n+1)} \in \mathbb{N}^{2n+1} \) written as
\[
l = (l_{n+1}, l_{0,n+1}, l_{1,n+1}, \ldots, l_{n,n+1}, \ldots, l_{1,\ldots,n+1}, l_{0,1,\ldots,n+1}),
\]
the element $\lambda^l \in \Lambda_{n+1}\lambda$ is given by the following

\[
\begin{cases}
\lambda^l(S) := \lambda_S, & \text{for any } S \subseteq [n]; \\
\lambda^l(S) := \lambda^l_S, & \text{for any } S \in S(n+1).
\end{cases}
\]

Now for each $S \in S(n+1)$, $\varepsilon_{S+1}^n =: \varepsilon_S : [\#S - 1] \rightarrow [n + 1]$ denotes the unique morphism in $\text{Hom}_{\Delta'}([\#S - 1], [n + 1])$ whose range is exactly $S$. It is then clear that

\[
\varepsilon_S((U^\lambda_\Lambda \ast \mathcal{S}^x) \cap \text{supp}(c \circ s)) \subset \bigcup_{l_S} U^{\#S-1}_{\lambda^l_S}, \forall S \in S(n+1).
\]

Next, choose for every $S \in S(n+1)$, a partition of unity

\[
\varphi_{\lambda^l_S} : \varepsilon_S((U^\lambda_\Lambda \ast \mathcal{S}^x) \cap \text{supp}(c \circ s)) \rightarrow \mathbb{R}_+
\]

subordinate to the open covering $\left( U^{\#S-1}_{\lambda^l_S} \right)_{l_S=0}^{m_S}$.

For all $n \geq 1$, we define the map

\[
h^n : CR_{ss}^{n+1}(\mathcal{U}_\bullet, E^\bullet) \rightarrow CR_{ss}^n(\mathcal{U}_\bullet, E^\bullet)
\]

by

\[
(h^n f)_\lambda(g_1, \ldots, g_n) :=
\]

\[
(-1)^{n+1} \int_{g_\lambda} \sum_{l=(l_S) \in S(n+1)} f_\lambda(g_1, \ldots, g_n, \gamma)
\]

\[
\prod_{S \in S(n+1)} \prod_{l_S} \varepsilon_S((U^\lambda_\Lambda \ast \mathcal{S}^x) \cap \text{supp}(c \circ s \circ \rho)) \cdot c(s(\gamma)) d\mu^{s(g_n)}(\gamma).
\]

Observe that

\[
(U^\lambda_\Lambda \ast \mathcal{S}^x) \cap \text{supp}(c \circ s \circ \rho) \subset \bigcup_{l=(l_S) \in S(n+1)} U^{n+1}_{\lambda^l},
\]

where the $\lambda^l$'s are defined in the obvious way. Hence, we get a partition of unity of $\varepsilon_S((U^\lambda_\Lambda \ast \mathcal{S}^x) \cap \text{supp}(c \circ s \circ \rho))$ subordinate to the open covering $\left( U^{\#S-1}_{\lambda^l_S} \right)_{l_S=0}^{m_S}$ by setting $\varphi_{\lambda^l_S} : \varepsilon_S((U^\lambda_\Lambda \ast \mathcal{S}^x) \cap \text{supp}(c \circ s \circ \rho)) := \varphi_{\lambda^l_S}(\varepsilon_S(g_1, \ldots, g_n))$.

Next, using (2.65), it is straightforward that

\[
(h^n f)_\lambda(\rho(g_1), \ldots, \rho(g_n)) = \nu_{U^\lambda_\Lambda} \circ (h^n f)_\lambda(g_1, \ldots, g_n),
\]

which means that $((h^n f)_\lambda)_{\lambda \in \Lambda_n} \in CR_{ss}^n(\mathcal{U}_\bullet, E^\bullet)$.

Assume now that $(f_\lambda)_{\lambda \in \Lambda_n} \in CR_{ss}^n(\mathcal{U}_\bullet, E^\bullet)$. Then, for every $U^\lambda_\Lambda \in \mathcal{U}_n$ and $(g_1, \ldots, g_n) \in U^\lambda_\Lambda$, one has
\[(2.71) \quad (h^n d^n f) \lambda (g_1, \ldots, g_n) \]
\[= (-1)^{n+1} \sum_{S \in S(n+1)} \int_{g^n \cap (I_S)} (d^n f) \lambda (g_1, \ldots, g_n, \gamma) \]
\[\cdot \prod_{S \in S(n+1)} \prod_{s \in S} \varphi_{\lambda_S}^{g_S}(\tilde{\varepsilon}_S^{n+1}(g_1, \ldots, g_n, \gamma)) \cdot c(s(\gamma)) d\mu^{s(g_n)}(\gamma) \]
\[= f_\lambda (g_1, \ldots, g_n) - A_\lambda (g_1, \ldots, g_n), \]

where
\[A_\lambda (g_1, \ldots, g_n) := (-1)^n \sum_{k=0}^{n} (-1)^k \int_{g^n \cap (I_S)} (d^n f) \lambda (g_1, \ldots, g_n, \gamma) \]
\[\cdot \prod_{S \in S(n+1)} \prod_{s \in S} \varphi_{\lambda_S}^{g_S}(\tilde{\varepsilon}_S^{n+1}(g_1, \ldots, g_n, \gamma)) \cdot c(s(\gamma)) d\mu^{s(g_n)}(\gamma). \]

We want to show that
\[(2.72) \quad A_\lambda (g_1, \ldots, g_n) = (d^{m-1} h^{n-1} f) \lambda (g_1, \ldots, g_n). \]

One has
\[(2.73) \quad (d^{m-1} h^{n-1} f) \lambda (g_1, \ldots, g_n) \]
\[= (-1)^n \sum_{k=0}^{n-1} \sum_{r_k := (r_k, T) \in S(n)} \int_{g^{r_k}(g_n)} f_{\lambda_T}^{r_k}(\tilde{\varepsilon}_T^{r_k}(g_1, \ldots, g_n, \gamma)) \cdot c(s(\gamma)) d\mu^{s(g_n)}(\gamma) \]
\[\cdot \prod_{T \in S(n)} \prod_{r_n} \varphi_{\lambda_T}^{g_T}(\tilde{\varepsilon}_T^{n}(g_1, \ldots, g_n, \gamma)) \cdot c(s(\gamma)) d\mu^{s(g_n)}(\gamma) \]
\[+ \sum_{r_n := (r_n, T) \in S(n)} \int_{g^{r_n}(g_n)} f_{\lambda_T}^{r_n}(g_1, \ldots, g_n, \gamma) \]
\[\cdot \prod_{T \in S(n)} \prod_{r_n} \varphi_{\lambda_T}^{g_T}(\tilde{\varepsilon}_T^{n}(g_1, \ldots, g_n, \gamma)) \cdot c(s(\gamma)) d\mu^{s(g_n)}(\gamma) \]
\[= B_\lambda (g_1, \ldots, g_n) + C_\lambda (g_1, \ldots, g_n). \]

Notice that by the left-invariance of \(\mu^x \) for \(x \in X\), the second integral \(C_\lambda\) in the right hand side of \(2.73\) can be written as
\[C_\lambda (g_1, \ldots, g_n) \]
\[= \int_{g^{r_n}(g_n)} \int_{(r_n, T) \in S(n)} f_{\lambda_T}^{r_n}(g_1, \ldots, g_n, \gamma) \]
\[\cdot \prod_{T \in S(n)} \prod_{r_n} \varphi_{\lambda_T}^{g_T}(\tilde{\varepsilon}_T^{n}(g_1, \ldots, g_n, \gamma)) \cdot c(s(\gamma)) d\mu^{s(g_n-1)}(\gamma) \]
\[
\begin{align*}
&= \int_{s(g_n)} \sum_{(r_k, T) \in S(n)} f_{\varepsilon_k^n}(\lambda) r_k(\varepsilon_n^{n+1}(g_1, \ldots, g_n, \gamma)) \\
&\quad \cdot \prod_{T \in S(n), r_k, T} \prod_{T \in S(n), r_k, T} \varphi_{\varepsilon_k^n}(\lambda) r_k, T(\varepsilon_n^{n+1}(g_1, \ldots, g_n, \gamma))) \cdot c(s(\gamma)) d\mu^s(g_{n-1})(\gamma).
\end{align*}
\]

On the other hand, for all \( k = 0, \ldots, n - 1 \), one has

\[
(\varepsilon_k^n(g_1, \ldots, g_n, \gamma)) = \varepsilon_k^{n+1}(g_1, \ldots, g_n, \gamma);
\]

hence

\[
B_{\lambda}(g_1, \ldots, g_n)
\]

\[
= (-1)^n \sum_{k=0}^{n-1} (-1)^k \int_{s(g_n)} \sum_{(r_k, T) \in S(n)} f_{\varepsilon_k^n}(\lambda) r_k(\varepsilon_n^{n+1}(g_1, \ldots, g_n, \gamma)) \\
\quad \cdot \prod_{T \in S(n), r_k, T} \prod_{T \in S(n), r_k, T} \varphi_{\varepsilon_k^n}(\lambda) r_k, T(\varepsilon_n^{n+1}(g_1, \ldots, g_n, \gamma))) \cdot c(s(\gamma)) d\mu^s(g_{n-1})(\gamma).
\]

Thus, (2.73) becomes

(2.74) \quad (d^{n-1}h^{n-1}f)_{\lambda}(g_1, \ldots, g_n)

\[
= (-1)^n \sum_{k=0}^{n-1} (-1)^k \int_{s(g_n)} \sum_{(r_k, T) \in S(n)} f_{\varepsilon_k^n}(\lambda) r_k(\varepsilon_n^{n+1}(g_1, \ldots, g_n, \gamma)) \\
\quad \cdot \prod_{T \in S(n), r_k, T} \prod_{T \in S(n), r_k, T} \varphi_{\varepsilon_k^n}(\lambda) r_k, T(\varepsilon_n^{n+1}(g_1, \ldots, g_n, \gamma))) \cdot c(s(\gamma)) d\mu^s(g_{n-1})(\gamma).
\]

Now, for any \( k = 0, \ldots, n, r_k = (r_k, T)_{T \in S(n)} \), let \( \gamma \in S(g_n) \) such that

\[
\varepsilon_k^{n+1}(g_1, \ldots, g_n, \gamma) \in U_{\varepsilon_k^n}(\lambda)^r_k.
\]

Then, there exists \( l = (l_s)_{s \in S(n+1)} \) such that \((g_1, \ldots, g_n, \gamma) \in U^{n+1}_\lambda\), so that

\[
\varepsilon_k^{n+1}(g_1, \ldots, g_n, \gamma) \in U_{\varepsilon_k^n}(\lambda)^r_k \bigcup U^{n+1}_\lambda.
\]

One can then suppose that for any \( k \in [n] \) and any family \( r_k = (r_k, T)_{T \in S(n)} \), there exists a family \( l = (l_s)_{s \in S(n+1)} \) such that \( \varepsilon_k^n(\lambda)^r_k = \varepsilon_k^{n+1}(\lambda)^r_k \). Moreover, in virtue of the identities (2.2), it is straightforward that for each \( k \in [n] \) and any \( T \in S(n) \), there exists a unique \( S \in S(n+1) \) such that \( \varepsilon_S^{n+1} = \varepsilon_k^{n+1} \circ \varepsilon_T^n \), so that \( \varepsilon_S^{n+1} = \varepsilon_T^n \circ \varepsilon_k^{n+1} \). Therefore, we obtain from (2.74) that
\[(d^{n-1}h_{n-1}^n f)_{\lambda}(g_1, \ldots, g_n)\]
\[= (-1)^n \sum_{k=0}^{n} (-1)^k \int_{S(n)} \sum_{(l_S) \in S(n+1)} f_{\tilde{\xi}_{k+1}^n(\lambda)}(\tilde{\xi}_{k}^n(g_1, \ldots, g_n, \gamma)) \]
\[\times \prod_{S \in S(n+1)} \prod_{l_S} \varphi_{\lambda_S}^n(\tilde{\xi}_{S}^n(g_1, \ldots, g_n, \gamma)) c(s(\gamma)) d\mu^{s(g_n)}(\gamma)\]
\[= A_{\lambda}(g_1, \ldots, g_n).\]

Combining with (2.71), we thus have shown that
\[(2.75)\]
\[h^n \circ d^n + d^{n-1} \circ h^{n-1} = \text{Id}_{\text{CR}^n_{ss}((\mathcal{U}_*, E^*))}, \forall n \geq 1;\]
i.e., \(h^*\) defines a contraction of \(\text{CR}^n_{ss}((\mathcal{U}_*, E^*))\) for any Real open cover \(\mathcal{U}_*\) of \((\mathcal{G}_*, \rho_* )\) and this ends our proof. \(\square\)

**Remark 2.71.** It is straightforward, using the same arguments, that Theorem 2.68 remains true for a Real representation of type \(\mathbb{R}^{p,q} \) (see Remark 2.65).

**Corollary 2.72.** Let \(\mathcal{G} \) be a proper groupoid. Let \(E \longrightarrow X\) be a representation of \(\mathcal{G}\) in the sense of Crainic [4]; that is, a real \(\mathcal{G}\)-equivariant vector bundle of rank \(p\). Then \(H^n(\mathcal{G}_*, E^*) = 0\), \(\forall n \geq 1\).

**Proof.** Let \(\mathcal{G}\) be endowed with the trivial Real structure. Form the Real representation \((F, \nu)\) of type \(\mathbb{R}^{p,p}\) of \((\mathcal{G}, \text{Id})\) by \(F := E \oplus E\) endowed with the diagonal \(\mathcal{G}\)-action and the Real structure \(\nu(e_1, e_2) := (e_1, -e_2)\). Then by Theorem 2.68, we have \(\tilde{H}R^n(\mathcal{G}_*, F^*) = 0\) for all \(n \geq 1\). But since the Real structure is trivial, we have \(\tilde{H}R^n(\mathcal{G}_*, F^*) = \tilde{H}R^n(\mathcal{G}_*, rF^*)\), thanks to the discussion following Proposition 2.44. Moreover, we obviously have \(rF^* = E^*\). \(\square\)

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**References**


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LMAM, Université de Lorraine - Metz, CNRS UMR 7122, Ile du Saulcy, F-57045 Metz Cedex 1, France
Current address: School of Mathematics, University of Southampton, Highfield, Southampton, SO17 1BJ, UK
e.mohamedmoutuou@soton.ac.uk
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