Nonreciprocal units in a number field with an application to Oeljeklaus–Toma manifolds

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Abstract. In this paper we show that if a number field $K$ contains a nonreciprocal unit $u$ of degree $s + 2t$ with $s$ positive conjugates and $2t$ complex conjugates of equal moduli, where $t \geq 2$, then $s = (2t+2m)q - 2t$ for some integers $m \geq 0$ and $q \geq 2$. On the other hand, for any $s$ and $t \geq 2$ related as above we construct a number field $K$ with $s$ real and $2t$ complex embeddings that contains a nonreciprocal unit $u$ of degree $s + 2t$ with $s$ positive conjugates and $2t$ complex conjugates of equal moduli. From this, for any pair of integers $s \geq 1$, $t \geq 2$ satisfying $s \neq (2t+2m)q - 2t$ we deduce that the rank of the subgroup of units $U$ whose $2t$ complex conjugates have equal moduli is smaller than $s$ and, therefore, for any choice of an admissible subgroup $A$ of $K$ the corresponding Oeljeklaus–Toma manifold $X(K, A)$ admits no locally conformal Kähler metric.

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1. Introduction

Let throughout $K$ be a number field of degree $d = [K : \mathbb{Q}]$ with $s \geq 0$ real embeddings $\sigma_1, \ldots, \sigma_s$ into $\mathbb{C}$ and $2t$ complex embeddings $\sigma_{s+1}, \sigma_{s+2}, \ldots, \sigma_{s+t}$ into $\mathbb{C}$, so that $d = s + 2t$. Here, for each $i = 1, \ldots, t$ the embedding $\sigma_{s+i} : K \to \mathbb{C}$ is defined as $\sigma_{s+i}(a) = \overline{\sigma_{s+i}(a)}$ for every $a \in K$, where $\overline{z}$ stands for the complex conjugate of $z \in \mathbb{C}$. Let $O_K^*$ be the group of units in the ring of integers of $K$. Put

$$U_K := \{ u \in O_K^* : \sigma_i(u) > 0 \text{ for every } i = 1, \ldots, s \}$$

for a subgroup of $O_K^*$ consisting of units whose real conjugates are all positive.

Consider the logarithmic representation of units $l : O_K^* \to \mathbb{R}^{s+t}$ given by

$$l(u) := (\log |\sigma_1(u)|, \ldots, \log |\sigma_s(u)|, 2 \log |\sigma_{s+1}(u)|, \ldots, 2 \log |\sigma_{s+t}(u)|).$$

By the Dirichlet’s unit theorem, $l(U_K)$ is a full discrete lattice in the subspace

$$S := \{(x_1, \ldots, x_{s+t}) \in \mathbb{R}^{s+t} : \sum_{i=1}^{s+t} x_i = 0\}$$

of $\mathbb{R}^{s+t}$. Equivalently (see, e.g., [13] and [20]), one can choose $s + t - 1$ multiplicatively independent units in $U_K$, say $u_1, \ldots, u_{s+t-1}$, such that every other unit in $U_K$ can be written as $wu_1^{k_1} \ldots u_{s+t-1}^{k_{s+t-1}}$ with a root of unity $w \in K$ and some $k_1, \ldots, k_{s+t-1} \in \mathbb{Z}$. From now on, assume that $s \geq 1$. Then the projection $P : S \to \mathbb{R}^s$ given by the first $s$ coordinates is surjective. Thus, there are subgroups $A$ of rank $s$ in $U_K$ such that $P(l(A))$ is a full discrete lattice in $\mathbb{R}^s$. Throughout, such a subgroup $A$ will be called admissible for $K$. An admissible subgroup $A$ for $K$ is generated by $s$ units $u_1, \ldots, u_s \in U_K$ such that the matrix

$$M(u_1, \ldots, u_s) := (\log |\sigma_j(u_i)|)_{1 \leq i, j \leq s}$$

has rank $s$, so that these units are multiplicatively independent.

The results of this paper are motivated by some applications to the so-called locally conformal Kähler complex manifolds $X$ (according to Vaisman [29], such a manifold is defined as a Hermitian manifold whose metric is conformal to a Kähler metric in some neighborhood of every point) and the corresponding study of locally conformal Kähler metrics (LCK metrics) on $X$ (see, e.g., [3], [4], [17], [19], [28], [29], [30]). In [16], Oeljeklaus and Toma introduced some compact complex manifold $X(K, A)$ associated to a number field $K$ and to an admissible subgroup $A$ for $K$. These manifolds were named as Oeljeklaus–Toma manifolds and have many interesting properties (see the recent papers of Battisti and Oeljeklaus [1], Kasuya [14], Ornea and Verbitsky [18], Verbitsky [32], Vuletescu [33], etc.). In particular, it is
known that if an Oeljeklaus–Toma manifold \( X(K, A) \) admits an LCK metric then for all \( u \in A \) we have
\[
|\sigma_{s+1}(u)| = \cdots = |\sigma_{s+t}(u)|
\]
(see the proof of Proposition 2.9 in [16]). Since the numbers \( \sigma_{s+i}(u) \) and \( \sigma_{s+i}(u) \) are complex conjugate, the previous condition can be written in the form
\[
|\sigma_{s+1}(u)| = |\sigma_{s+1}(u)| = \cdots = |\sigma_{s+t}(u)| = |\sigma_{s+t}(u)|.
\]

In the appendix of this paper (written by Laurent Battisti), it is shown (see Theorem 8) that the Oeljeklaus–Toma manifold \( X(K, A) \) admits an LCK metric if and only if for all \( u \in A \) the condition (1) holds (which is stronger than just the ‘only if’ condition that was proved in the previous result in [16]).

This raises the following natural question: are there \( s \) multiplicatively independent units \( u_1, \ldots, u_s \) in \( U_K \) such that (1) holds for each \( u = u_1, \ldots, u_s \)? Such units would generate an admissible subgroup \( A \) for \( K \) and a corresponding manifold \( X(K, A) \) with LCK metric. The answer is ‘yes’ for \( s \geq 1 \) and \( t = 1 \) (so far this is the only known case with a positive answer) and ‘no’ for \( s = 1 \) and \( t \geq 2 \) (see Proposition 2.9 in [16]). It is not clear whether or not there are some cases with \( s, t \geq 2 \) when the answer is positive. Vuletescu [33] has shown recently that the answer is ‘no’ for \( 1 < s < t \). Below, we will show that the answer is ‘no’ for any \( s \) that is not of the form (2) below. In particular, for \( t \geq 2 \) this implies a negative answer for \( 1 < s < 2t \) and also for \( s \) odd. Unfortunately, the second statement of Theorem 1 shows that for all other \( s \) the field \( K \) may contain a nonreciprocal unit. This leaves the problem open for some special pairs \( s, t \geq 2 \) satisfying (2), since our construction gives only one nonreciprocal unit in \( U_K \) instead of \( s \) multiplicatively independent units.

2. Main results

Recall that an algebraic number \( \alpha \) is called \textit{reciprocal} if \( \alpha^{-1} \) is its conjugate over \( \mathbb{Q} \) and \textit{nonreciprocal} otherwise. The main result of this paper is the following:

\textbf{Theorem 1.} If a number field \( K \) of degree \( d = s + 2t \) over \( \mathbb{Q} \) with \( s \) real and \( 2t \) complex embeddings, where \( t \geq 2 \), contains a nonreciprocal unit \( u \in U_K \) of degree \( d \) whose \( 2t \) nonreal algebraic conjugates satisfy (1) then for some integers \( m \geq 0 \) and \( q \geq 2 \) we have
\[
s = (2t + 2m)q - 2t.
\]

On the other hand, if \( s \) and \( t \geq 2 \) satisfy (2) with some integers \( m \geq 0 \) and \( q \geq 2 \) then there is a number field \( K \) with \( s \) real and \( 2t \) complex embeddings that contains a nonreciprocal unit \( u \in U_K \) of degree \( d = s + 2t \) satisfying (1).
In general, the situation when a number field $K$ contains a nonreciprocal unit $u \in \mathcal{U}_K$ as described in Theorem 1 happens very rarely. If, for instance, the Galois group $\text{Gal}(F/\mathbb{Q})$, where $F$ is the normal closure of $K$ over $\mathbb{Q}$, is 'large' (say the group $\text{Gal}(F/\mathbb{Q})$ acts on $d$ conjugates of $\alpha \in K$ as a full symmetric group $S_d$ which is the 'generic' situation, by an old result of van der Waerden ([31]), then the equality (5) below cannot hold (see, e.g., [27]). Hence, such fields $K$ do not contain units with the required properties.

From Theorem 1 we shall derive the following:

**Theorem 2.** Let $K$ be a number field of degree $d = s + 2t$ over $\mathbb{Q}$ with $s$ real and $2t$ complex embeddings, where $s \geq 1$ and $t \geq 2$ are not of the form (2). Then the rank of the subgroup $U$ of $\mathcal{U}_K$ of units satisfying (1) is smaller than $s$ and, therefore, for any choice of an admissible subgroup $A$ for $K$ the Oeljeklaus–Toma manifold $X(K, A)$ has no LCK metric.

This implies the main result of [33], where the same conclusion as that of Theorem 2 has been proved under the assumption $1 < s < t$.

In the next section we shall give some auxiliary results. The proof of Theorem 1 is then given in Sections 4 and 5. In Section 5 one can also find an explicit example corresponding to the case $s = 4$, $t = 2$, $m = 0$ and $q = 2$ of Theorem 1. In Section 6 we shall give the proof of Theorem 2. Finally, in an appendix Laurent Battisti gives the proof of his Theorem 8 and using an alternative (geometrical) approach derives Theorem 2 from Theorem 1 as well.

### 3. Auxiliary results

An algebraic integer $\alpha > 1$ is called a *Perron number* if all of its conjugates over $\mathbb{Q}$ are less than $\alpha$ in absolute value. In particular, a Perron number is a *Pisot number* if its conjugates over $\mathbb{Q}$ (if any) are less than 1 in absolute value. We shall use *totally positive Pisot units* (Pisot numbers that are units whose algebraic conjugates are all positive) in Lemma 5 and subsequently in the proof of Theorem 1.

A version of the next lemma appears in [26]. Its proof is based on the argument of applying an automorphism of the Galois group that maps an algebraic number to its maximal (or minimal) conjugate which leads to a contradiction. This simple argument also plays a crucial role in the papers [7], [9], [27]. Below, we shall give a proof of the next lemma, since a similar argument appears several times in this paper as well.

**Lemma 3.** Let $\alpha$ or $\alpha^{-1}$ be a Perron number of degree $d \geq 3$, and let $\alpha_1, \alpha_2, \alpha_3$ be any three distinct conjugates of $\alpha$. Then $\alpha_1^2 \neq \alpha_2 \alpha_3$.

**Proof.** Assume that $\alpha_1^2 = \alpha_2 \alpha_3$. Let $F$ be the normal closure of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. Take an automorphism of the Galois group of $F/\mathbb{Q}$ which maps $\alpha_1$ to $\alpha$. It maps $\alpha_2, \alpha_3$ to some conjugates $\alpha_i \neq \alpha_j$ of $\alpha$, different from $\alpha$, and so it maps the equality $\alpha_1^2 = \alpha_2 \alpha_3$ into the equality $\alpha_2^2 = \alpha_i \alpha_j$. However,
the modulus of its left hand side is greater than the modulus of its right hand side if \( \alpha \) is Perron number (resp. smaller if \( \alpha^{-1} \) is a Perron number), a contradiction.

\begin{proof}
There is nothing to prove if \( s = 0 \). Assume that \( s > 0 \). If one of the real conjugates of \( u \), say, \( \alpha_1 > 0 \) has the same modulus as the complex (nonreal) conjugate \( \alpha_2 \) then \( \alpha_1^2 = \alpha_2 \overline{\alpha}_2 \). Here, \( \alpha_1, \alpha_2, \overline{\alpha}_2 \) are distinct. By Lemma 3, none of the conjugates of \( u \) (and of \( u^{-1} \)) is a Perron number. This happens only if there are no other positive conjugates of \( u \) except for \( \alpha_1 \). Thus, \( s = 1 \) and all the conjugates of \( u \) lie on the circle \( |z| = \alpha_1 \). Since \( u \) is a unit, the radius of the circle must be 1, so that \( u = \alpha_1 = 1 \). Hence, \( \deg u = 1 \), contrary to \( \deg u = s + 2t' \geq 2 \).
\end{proof}

We remark that an alternative proof of Corollary 4 can be given by applying the results of Boyd [2] and Ferguson [12].

A standard construction of Pisot numbers in a real field gives Pisot numbers but not Pisot units [23]; see also [11] for a construction of a dense set of Pisot numbers in a given field with very small conjugates and [6] for a construction of Pisot units. In the proof of Theorem 1 we shall need the following result (which is also of independent interest):

\begin{lemma}
For each number field \( L \), each constant \( c > 1 \) and each integer \( q \geq 2 \) there is a totally positive Pisot unit \( \beta > c \) of degree \( q \) whose minimal polynomial over \( \mathbb{Q} \) is irreducible in the ring \( L[x] \).
\end{lemma}

\begin{proof}
Consider the polynomial
\[
H(x) := (-x)^q + k_{q-1}(-x)^{q-1} + \cdots + k_1(-x) + 1,
\]
where \( k_1 < k_2 < \cdots < k_{q-1} \) is a rapidly increasing sequence of positive integers; for example, \( k_{j+1} > k_j^2 \) for \( j = 1, \ldots, q-2 \) and \( k_1 \) is large enough. (This construction is similar in spirit to that of Lemma 3 in [6].) Then \( H(0) = 1 > 0 \) and \( H(1/k_1) < 0 \). Also, it is easy to see that the sign of \( H(2k_j) \) is the same as that of \( (-1)^{j+1} \). Indeed, inserting \( x = 2k_j \) into \( H(x) \) we see that among the two largest terms \( k_{q-1} \ldots k_{j+1}(-2k_j)^{j+1} \) and \( k_{q-1} \ldots k_{j+1}k_j(-2k_j)^j \) the first one is greater in absolute value. Similarly, the sign of \( H(k_j/2) \) is the same as that of \( (-1)^j \). So in each of the \( q \) intervals \( (0, 1/k_1) \) and \( (k_j/2, 2k_j) \), where \( j = 1, \ldots, q-1 \), there is a root of the polynomial \( H \). Consequently, the polynomial \( G(x) = x^q H(1/x) \) reciprocal to \( H \) has \( q-1 \) roots in the interval \( (0,1) \) and one root \( \beta \) in \( (k_1, \infty) \). As \( \deg G = q \), this polynomial must be irreducible in \( \mathbb{Z}[x] \), since the product of any number of its roots without \( \beta \) is of modulus smaller than 1. Therefore, \( \beta > k_1 \) is a totally positive Pisot unit of degree \( q \).
Note that the polynomial (3) is linear in $k_{q-1}$, so the polynomial (3), as a polynomial in two variables $H(x, k_{q-1})$, is irreducible in the ring $L[x, a_{d-1}]$. Indeed, otherwise the polynomials $(-x)^q + 1$ and

$$H_1(x) := \frac{H(x) - (-x)^q - 1}{k_{q-1}} = (-x)^q + k_{q-2}(-x)^{q-2} + \cdots + k_{q-2} \cdots k_1(-x)$$

must have a common factor in $L[x]$. Hence, $(-x)^q + 1$ and $H_1(x)$ must have a common root. However, this is not the case, since the roots of $(-x)^q + 1$ are roots of unity whereas the modulus of lowest term in $H_1$ is greater than the sum of the moduli of the other terms for any $x$ of modulus 1. Thus, by Hilbert’s irreducibility theorem (see Theorem 46 on p. 298 in [24]), for some positive integer $k_{q-1} > k_{q-1}$ the polynomial $H(x, k_{q-1}^*)$ is irreducible in the ring $L[x]$, and so the polynomial $G(x, k_{q-1}^*) = x^q H(1/x, k_{q-1}^*)$ is irreducible in $L[x]$ too. \hfill \Box

For any $u_1, \ldots, u_s \in U_K$ we write

$$S(u_1, \ldots, u_s) := \{u_1^{k_1} \cdots u_s^{k_s} : k_1, \ldots, k_s \in \mathbb{N} \cup \{0\}\}$$

for the multiplicative semigroup generated by $u_1, \ldots, u_s$.

**Lemma 6.** Let $u_1, \ldots, u_s$ be some $s \geq 1$ multiplicatively independent units in $U_K$ satisfying (1). Then either $\mathbb{Q}(u_1, \ldots, u_s)$ is a proper subfield of $K$ or $U_K$ contains $s$ multiplicatively independent units $v_1, \ldots, v_s \in S(u_1, \ldots, u_s)$, each of degree $d = [K : \mathbb{Q}]$.

**Proof.** Assume that $\mathbb{Q}(u_1, \ldots, u_s) = K$ (otherwise there is nothing to prove). Suppose $S(u_1, \ldots, u_s)$ does not contain $s$ multiplicatively independent units of degree $d$ each. Choose multiplicatively independent units $v_1, \ldots, v_s \in S(u_1, \ldots, u_s)$ satisfying $\mathbb{Q}(v_1, \ldots, v_s) = K$ for which the sum $D := \deg v_1 + \cdots + \deg v_s$ is the largest possible. If $D = sd$ we are done. In case $D < sd$, we will show that $D$ can be increased, and so arrive to a contradiction.

Without restriction of generality we may assume that $h := \deg v_1 < d$. Then $s > 1$ and for some $v_j$ with $j \geq 2$, say for $v_2$, we have $v_2 \notin \mathbb{Q}(v_1)$, since otherwise $\mathbb{Q}(v_1, \ldots, v_s) = \mathbb{Q}(v_1)$ is a proper subfield of $K$. Now, replace the set $v_1, v_2, \ldots, v_s$ by the set $v_1 v_2^p, v_2, \ldots, v_s$, where $p$ is an integer that will be chosen later. The latter set is also multiplicatively independent, all of its elements belong to $S(u_1, \ldots, u_s)$ and also

$$\mathbb{Q}(v_1 v_2^p, v_2, \ldots, v_s) = \mathbb{Q}(v_1, v_2, \ldots, v_s) = K.$$

In order to complete the proof it remains to show that

$$\deg v_1 v_2^p > h = \deg v_1$$

for some large positive integer $p$.

It is clear that $v_1, v_2 > 0$, since $v_1, v_2 \in S(u_1, \ldots, u_s)$ and $u_1, \ldots, u_s > 0$. If all the conjugates of $v_2$ are of equal moduli then, as $v_2$ is a unit, they all lie on the circle $|z| = 1$. Hence, $v_2 = 1$, which is a contradiction to
\(v_2 \notin \mathbb{Q}(v_1)\). It follows that not all conjugates of \(v_2\) have the same modulus. Since \(v_2 \in S(u_1, \ldots, u_s)\), and the units \(u_1, \ldots, u_s\) satisfy the condition (1), the unit \(v_2\) satisfies (1) as well. Consequently, either the largest positive conjugate of \(v_2\) (it can be \(v_2\) itself) is a Perron number or a reciprocal of the smallest positive conjugate of \(v_2\) is a Perron number.

Select the smallest \(\ell \in \mathbb{N}\) for which \(v_2^\ell \in \mathbb{Q}(v_1)\), if such an \(\ell\) exists. Take \(p\) of the form \(\ell k + 1\) with large \(k \in \mathbb{N}\) if \(\ell \in \mathbb{N}\) as above exists and take any large \(p\) otherwise. For such \(p\) we have \(v_2^2 \notin \mathbb{Q}(v_1)\). Thus, \(v_2^2\) has a conjugate over the field \(\mathbb{Q}(v_1)\) distinct from \(v_2^2\). Assume that \(w_2^p\) is such a conjugate, where \(w_2 \neq v_2\) are conjugate over \(\mathbb{Q}\). It follows that the numbers \(v_1 v_2^2 \neq v_1 w_2^p\) are conjugate over \(\mathbb{Q}\). Now, consider some \(h\) automorphisms of the Galois group of \(\mathbb{Q}(v_1, v_2)/\mathbb{Q}\) that map \(v_1\) into its \(h\) conjugates over \(\mathbb{Q}\). These map \(v_2\) and \(w_2\) to some of their conjugates over \(\mathbb{Q}\) and the pair \(v_1 v_2^2, v_1 w_2^p\) into some \(h\) pairs of two distinct conjugates of \(v_1 v_2^2\). Therefore, either \(\deg v_1 v_2^2 \geq 2h\) (which finishes the proof of (4)) or the list of \(2h\) conjugates contains some equal elements. This means that for some two distinct conjugates of \(v_1\), say for \(v_1\) itself and \(w_1 \neq v_1\), we have \(v_1 v_2^p = w_1 (w_2^*)^p\), where \(w_2^*\) is a conjugate of \(v_2\) over \(\mathbb{Q}\). Then \(w_2^* \neq v_2\).

Now, take an automorphism \(\sigma\) of the Galois group of \(\mathbb{Q}(v_1, v_2)/\mathbb{Q}\) that maps \(w_2^*\) into \(v\). (Recall that \(v\) is a conjugate of \(v_2\) which is a Perron number, or \(v^{-1}\) is a Perron number.) This maps the equality \(v_1 v_2^p = w_1 (w_2^*)^p\) into \(\sigma(v_1) \sigma(v_2)^p = \sigma(w_1) v^p\), where \(\sigma(v_1) \neq \sigma(w_1)\) and \(\sigma(v_2) \neq v\). However, this is impossible, since \(|\sigma(v_2)| \neq v\) and so the modulus of the right hand side, \(\sigma(w_1) v^p\), is greater (resp. smaller) than that of the left hand side, \(\sigma(v_1) \sigma(v_2)^p\), if \(v\) is a Perron number (resp. \(v^{-1}\) is a Perron number) and \(p\) is large enough.

Finally, in the proof of Theorem 2 we shall use the next lemma (which is Lemma 1.6 in [16]):

**Lemma 7.** Let \(K'\) be a proper subfield of \(K\) and a proper extension of \(\mathbb{Q}\), i.e., \(\mathbb{Q} \subset K' \subset K\), and let \(A \subset \mathcal{U}_{K'}\) be an admissible subgroup for \(K\). Suppose that \(s'\) and \(2t'\) are the numbers of distinct real and complex embeddings of \(K'\), respectively. Then \(t'\) is positive, \(s = s'\) and \(A\) is admissible for \(K'\).

### 4. The restriction on the number of real embeddings in Theorem 1

The aim of this section is to prove (2). Take a nonreciprocal unit in \(\mathcal{U}_K\) of degree \(d\) with conjugates

\[\alpha_1, \ldots, \alpha_s, \alpha_{s+1}, \overline{\alpha_{s+1}}, \ldots, \alpha_{s+t}, \overline{\alpha_{s+t}},\]

satisfying (1). Then

\[\alpha_{s+1} \overline{\alpha_{s+1}} = \cdots = \alpha_{s+t} \overline{\alpha_{s+t}} = \beta\]
for some $\beta > 0$. Here, $\beta \neq 1$, since otherwise the unit $\alpha_1$ is reciprocal. Thus, $s > 0$. We claim that the set $\{\alpha_1, \ldots, \alpha_s, \alpha_1^{-1}, \ldots, \alpha_s^{-1}\}$ contains a Perron number. As the conjugates $\alpha_1, \ldots, \alpha_s$ are positive, at most one of them can lie on the circle $|z| = \sqrt{\beta}$. If $s = 1$ then we are done, unless all the conjugates of $\alpha_1$ lie on the circle $|z| = \sqrt{\beta}$. However, then the norm of $\alpha_1$ is $\beta^{d/2}$. In view of $\alpha_1 \in U_K$ we obtain $\beta = 1$, a contradiction. In the alternative case, $s \geq 2$, we take $\alpha$ to be the largest number in the set $\{\alpha_1, \ldots, \alpha_s\}$ if it is greater than $\sqrt{\beta}$, and the smallest one if all the conjugates of $\alpha_1$ lie in $|z| \leq \sqrt{\beta}$. Then $\alpha$ (resp. $\alpha^{-1}$) is a Perron number.

Obviously, $\beta$ cannot be written as a product of two complex conjugates of $\alpha$ other than given in (5), and it is not a product of a real conjugate and a complex (nonreal) conjugate. Assume that among the real conjugates of $\alpha$ there are $m \geq 0$ pairs of conjugates that multiply to $\beta$, where $m = 0$ if there are no such pairs. Then, without restriction of generality (5) can be extended to

$$
(6) \quad \beta = \alpha_{s-2m+1} \alpha_{s-2m+2} = \cdots = \alpha_{s-1} \alpha_s = \alpha_{s+1} \bar{\alpha}_{s+1} = \cdots = \alpha_{s+t} \bar{\alpha}_{s+t}.
$$

Note that $s > 2m$, since otherwise the norm of $\alpha$ is equal to $\beta^{d/2} \neq 1$ and $\alpha$ is not a unit.

Assume that the degree of $\beta$ over $\mathbb{Q}$ is $q$, and the conjugates of $\beta$ are $\beta_1 = \beta, \beta_2, \ldots, \beta_q$. If $q = 1$ then mapping $\alpha_{s-2m+1}$ to $\alpha_1$, we find that $\alpha_1 \alpha' = \beta$ for some conjugate $\alpha' \neq \alpha_1$ of $\alpha$, since $\beta \mapsto \beta$. But the pair $\alpha_1, \alpha'$ does not appear in (6), a contradiction. Hence, $q \geq 2$.

Take any automorphism $\sigma = \sigma_j$ of the Galois group $\text{Gal}(F/\mathbb{Q})$, where $F$ is the normal closure of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$, that maps $\beta$ to $\beta_j$. Then (6) (which corresponds to $\sigma = \text{id}$) maps to

$$
(7) \quad \beta_j = \sigma(\alpha_{s-2m+1}) \sigma(\alpha_{s-2m+2}) = \cdots = \sigma(\alpha_{s-1}) \sigma(\alpha_s) = \sigma(\alpha_{s+1}) \sigma(\bar{\alpha}_{s+1}) = \cdots = \sigma(\alpha_{s+t}) \sigma(\bar{\alpha}_{s+t}).
$$

Here, $\sigma$ acts as a permutation of the set $C := \{\alpha_1, \alpha_2, \ldots, \alpha_{s+t}, \bar{\alpha}_{s+t}\}$. Consider $q$ multiple equalities (7), where $j = 1, \ldots, q$. Evidently, each of the $q$ sets

$$
C_\sigma := \{\sigma(\alpha_{s-2m+1}), \sigma(\alpha_{s-2m+2}), \ldots, \sigma(\alpha_{s-1}), \sigma(\alpha_s), \sigma(\alpha_{s+1}), \sigma(\bar{\alpha}_{s+1}), \ldots, \sigma(\alpha_{s+t}), \sigma(\bar{\alpha}_{s+t})\}
$$

contains $2t + 2m$ distinct elements. We will show that they are disjoint, so that $\cup_{\sigma} C_\sigma = C$.

Suppose first that some set $C_\sigma$, where $\sigma \neq \text{id}$, contains a complex (nonreal) number. Then $\alpha_i \alpha_j = \alpha_k \alpha_l = \beta_j \neq \beta$, where the indices $i, j, k, l$ are distinct and

$$
\mathcal{I} := \{\alpha_i, \alpha_j, \alpha_k, \alpha_l\} \cap \{\alpha_{s+1}, \bar{\alpha}_{s+1}, \ldots, \alpha_{s+t}, \bar{\alpha}_{s+t}\} \neq \emptyset.
$$

If $|\mathcal{I}| = 1$, then one side of the equality

$$
(8) \quad \alpha_i \alpha_j = \alpha_k \alpha_l
$$
is real and the other side is nonreal, a contradiction. Suppose next that \(|I| = 2\). If both complex numbers are on one side of (8), say on its right hand side, then \(|\alpha_i\alpha_j| = |\alpha_k| \cdot |\alpha_l| = \sqrt{3} \cdot \sqrt{3} = \beta\), so \(\alpha_i\alpha_j = \beta\), contrary to \(\alpha_i\alpha_j = \beta_J \neq \beta\). If the two complex numbers are on different sides of (8), say \(\alpha_j\) and \(\alpha_i\), then \(|\alpha_i/\alpha_k| = |\alpha_l/\alpha_j| = \sqrt{3}/\sqrt{3} = 1\). Thus, \(\alpha_i = \pm \alpha_k\), which is impossible in view of \(i \neq k\) and \(\alpha_i, \alpha_k > 0\). Next, if \(|I| = 3\) then, assuming that the remaining real conjugate in (8) is \(\alpha_i\), we obtain \(|\alpha_i| = |\alpha_k||\alpha_l|/|\alpha_j| = \sqrt{3}\). Thus, \(\alpha_i = \sqrt{3}\). Then \(\beta = \alpha_i^2 = \alpha_{s+1}\overline{\alpha_{s+1}}\), which contradicts Lemma 3. Finally, if \(|I| = 4\), then all 4 conjugates of \(\alpha\) in (8) are complex, \(\alpha_j \neq \overline{\alpha_i}\) and \(\alpha_i \neq \overline{\alpha_k}\). We have already proved that the product of such \(\alpha_i\) and \(\alpha_j\) cannot be the product of two real conjugates or a real and a complex conjugate. Hence, the set \(C_\sigma\) corresponding to \(\beta_J\) (which is equal to \(\alpha_i\alpha_j\)) consists entirely of complex (nonreal) numbers. As \(|C_\sigma| = 2t + 2m \geq 2t\), all the complex conjugates of \(\alpha\) must belong to \(C_\sigma\). Thus, \(\overline{\alpha_j} \in C_\sigma\), and so \(\alpha_i\alpha_j = \overline{\alpha_j} \alpha_\ell\) with some complex (nonreal) conjugate \(\alpha_\ell\). Multiplying both sides by \(\alpha_i\overline{\alpha_j}/\beta\), we deduce that

\[
\alpha_i^2 = \alpha_i^2 \alpha_j \overline{\alpha_j}/\beta = \overline{\alpha_i} \alpha_\ell \alpha_j/\beta = \alpha_\ell \overline{\alpha_j}.
\]

Now, if \(\overline{\alpha_j} = \alpha_\ell\) then \(\alpha_i = \alpha_\ell\), which is not the case. If otherwise \(\overline{\alpha_j} \neq \alpha_\ell\) then the conjugates \(\alpha_i, \alpha_\ell\) and \(\overline{\alpha_j}\) are distinct. Then equality \(\alpha_i^2 = \alpha_\ell \overline{\alpha_j}\) is impossible, by Lemma 3. Hence, the set \(I\) is empty.

We have thus proved that all the numbers in (8) are distinct positive numbers. Hence, for each \(\sigma \neq \id\) the set \(C_\sigma\) consists of \(2t + 2m\) positive conjugates of \(\alpha\). Assume that some positive conjugate, say \(\alpha_1\), appears in \(b > 1\) sets \(C_\sigma\). Then an automorphism of \(\Gal(F/Q)\) that maps \(\alpha_1\) to \(\alpha_{s+1}\) acts as a permutation of the set \(C\) and as a permutation of the set \(\{\beta_1, \ldots, \beta_q\}\). In this way we will obtain \(q\) equalities of the type (7), where the complex conjugate \(\alpha_{s+1}\) appears \(b\) times. As \(b > 1\), this contradicts the fact that \(\beta_J = \alpha_i\alpha_j\) with complex (nonreal) \(\alpha_i, \alpha_j\) only happens once when \(J = 1\) and \(\alpha_j = \overline{\alpha_i}\). By the same argument, a conjugate of \(\alpha\) cannot appear \(b = 0\) times in the sets \(C_\sigma\). Consequently, every conjugate of \(\alpha\) appears exactly once in the union of \(q\) sets \(C_\sigma\). Hence,

\[
s + 2t = d = |C| = |C_\id|q = (2t + 2m)q.
\]

As \(q \geq 2\), this finishes the proof of (2).

5. The construction of a nontrivial unit in Theorem 1

Let \(I(n)\) be the infimum among all positive numbers \(I_n\) with the following property: any closed real interval of length at least \(I_n\) contains a full set of conjugates of an algebraic integer of degree \(n\). By a result of Robinson [21], every interval of length greater than 4 contains infinitely many full sets of conjugates of algebraic integers (see also [22]). Later, Ennola [10] proved that such an interval contains full sets of conjugates of algebraic integers of degree \(n\) for all \(n\) sufficiently large. Hence, for each positive \(\varepsilon\) we have \(I(n) < \ldots\)
4 + $\varepsilon$ for every $n > n(\varepsilon)$. From [5] we know that $I(2) = (1 + \sqrt{5})/2 + \sqrt{2}$ and it is evident that $I(1) = 1$. It seems very likely that every interval of length, say 5, or even smaller (although greater than 4, by an old result of Schur [25]) contains a full set of conjugates of an algebraic integer of degree $n$ for every $n \in \mathbb{N}$. However, since no result of such type is given explicitly in the literature, we simply put

$$I := \max \{5, \sup_{n \in \mathbb{N}} I(n)\}.$$  

Therefore, every interval of length $I$ contains a full set of conjugates of an algebraic integer of degree $n$ for every $n \in \mathbb{N}$.

We claim that for any integers $t \geq 1$ and $m \geq 0$ there is an algebraic integer $\gamma$ of degree $t + m$ with $t$ conjugates in the interval $[1, I + 1]$ and $m$ conjugates in the interval $(2I, \infty)$, say

$$1 \leq \gamma = \gamma_1 < \cdots < \gamma_t \leq I + 1 < 2I < \gamma_{t+1} < \cdots < \gamma_{t+m}. \tag{9}$$

Indeed, by the definition of $I$, such an algebraic integer $\gamma$ exists for $m = 0$. To show the existence of such $\gamma$ for $m \geq 1$ we can use a theorem of Motzkin [15]. Let us take, for instance, arbitrary $t$ points $\lambda_1 < \cdots < \lambda_t$ in the interval $(1, I + 1)$ and arbitrary $m - 1$ points $\lambda_{t+1} < \cdots < \lambda_{t+m-1}$ in the interval $(2I, 2I + 1)$. Then, by the main result of [8], for each $\varepsilon > 0$ there is a constant $c(\varepsilon, \lambda_1, \ldots, \lambda_{t+m-1})$ and a totally positive algebraic integer $\gamma_{t+m} > \max\{c(\varepsilon, \lambda_1, \ldots, \lambda_{t+m-1}), 2I + 1\}$

of degree $t + m$ such that the numbers $\gamma_1, \ldots, \gamma_{t+m-1}$ conjugate to $\gamma_{t+m}$ over $\mathbb{Q}$ lie in the $\varepsilon$-neighborhoods of the points $\lambda_1, \ldots, \lambda_{t+m-1}$, respectively. By taking a sufficiently small $\varepsilon > 0$, we see that this algebraic integer $\gamma_{t+m}$ of degree $t + m$ with conjugates $\gamma_1 = \gamma, \ldots, \gamma_{t+m}$ satisfies (9).

By Lemma 5, we can take a totally positive Pisot unit $\beta$ of degree $q \geq 2$ such that, firstly,

$$\beta = \beta_1 > \frac{I(I + 1)}{I - 1} > 1 > \beta_2 > \cdots > \beta_q \tag{10}$$

and, secondly, the minimal polynomial of $\beta$ over $\mathbb{Q}$ is irreducible in the ring $\mathbb{Q}(\gamma_1, \ldots, \gamma_{t+m})[x]$.

Set

$$k := [(1 + I^{-1}) \beta] \geq (1 + I^{-1}) \beta > (1 + I^{-1}) \frac{I(I + 1)}{I - 1} > I + 1 \geq 6 \tag{11}$$

Consider the polynomial

$$H(x, \beta_j) := \prod_{i=1}^{t+m} (x^2 - \gamma_i(k - \beta_j)x + \beta_j^2).$$

If $j > 1$ then the discriminant of each quadratic factor in $H(x, \beta_j)$ is positive. Indeed, using (9), (10) and (11), we obtain

$$(\gamma_i(k - \beta_j))^2 - 4\beta_j^2 \geq (k - \beta_j)^2 - 4\beta_j^2 > (6 - 1)^2 - 4 > 0.$$
Since \( \gamma_i(k - \beta_j) > 0 \), the factor \( x^2 - \gamma_i(k - \beta_j)x + \beta_j^2 \) has two positive roots. Hence, the polynomial \( H(x, \beta_j) \) has \( 2t + 2m \) positive roots.

We claim that for \( j = 1 \) the polynomial

\[
H(x, \beta_1) = \prod_{i=1}^{t+m} (x^2 - \gamma_i(k - \beta)x + \beta^2)
\]

has \( 2m \) positive roots and \( 2t \) complex roots lying on the circle \(|z| = \beta\).

We claim that for \( j = 1 \) the polynomial

\[
\prod_{i=1}^{t+m} (x^2 - \gamma_i(k - \beta)x + \beta^2)
\]

has \( 2m \) positive roots and \( 2t \) complex roots lying on the circle \(|z| = \beta\).

Indeed, this time the discriminant

\[
\Delta_i := (\gamma_i(k - \beta))^2 - 4\beta^2
\]

is positive for \( i = t + 1, \ldots, t + m \). To see this, we use \( k - \beta \geq \beta/I \) and \( \gamma_i > 2I \) which gives

\[
(\gamma_i(k - \beta))^2 - 4\beta^2 \geq \gamma_i^2(\beta/I)^2 - 4\beta^2 > 4I^2(\beta/I)^2 - 4\beta^2 = 0.
\]

Thus, the quadratic polynomial \( x^2 - \gamma_i(k - \beta)x + \beta^2 \) has two positive roots for every \( i = t + 1, \ldots, t + m \).

Similarly, we may check that \( \Delta_i \) is negative for \( i = 1, \ldots, t \). Indeed, by (9), (10) and (11),

\[
(\gamma_i(k - \beta))^2 - 4\beta^2 \leq (I + 1)^2((1 + I^{-1})\beta - \beta)^2 - 4\beta^2
\]

\[
< (I + 1)^2(\beta/I + 1)^2 - 4\beta^2
\]

\[
< (I + 1)^2(2\beta/(I + 1))^2 - 4\beta^2 = 4\beta^2 - 4\beta^2 = 0,
\]

where the inequality \( \beta/I + 1 < 2\beta/(I + 1) \) follows from (10). Consequently, for each \( i = 1, \ldots, t \) the roots of \( x^2 - \gamma_i(k - \beta)x + \beta^2 \) are

\[
\gamma_i(k - \beta) \pm \frac{\sqrt{(\gamma_i(k - \beta))^2 - 4\beta^2}}{2}.
\]

These are complex conjugate numbers lying on the circle \(|z| = \beta\). Thus, \( H(x, \beta_1) \) has \( 2m \) positive roots and \( 2t \) complex roots all lying on the circle \(|z| = \beta\).

Summarizing, we conclude that the polynomial

\[
P(x) := \prod_{j=1}^{q} H(x, \beta_j) = \prod_{j=1}^{q} \prod_{i=1}^{t+m} (x^2 - \gamma_i(k - \beta_j)x + \beta_j^2) \in \mathbb{Z}[x]
\]

has \( (2t + 2m)(q - 1) + 2m = (2t + 2m)q - 2t \) positive roots and \( 2t \) complex roots.

We next show that the polynomial \( P(x) \) of (12), of degree \( d = s + 2t \), where \( s = (2t + 2m)q - 2t \), is irreducible in \( \mathbb{Z}[x] \). Let \( \alpha \) be one of its complex roots, say

\[
\alpha = \gamma(k - \beta) + i\frac{\sqrt{4\beta^2 - (\gamma(k - \beta))^2}}{2},
\]
where \( i = \sqrt{-1}, \beta = \beta_1 \) and \( \gamma = \gamma_1 \). Assume that \( \ell := \deg \alpha < 2(t + m)q \) and consider the set of conjugates of \( \alpha \), say \( \alpha_1 = \alpha, \alpha_2, \ldots, \alpha_\ell \). Evidently,
\[
\overline{\alpha} = \frac{\gamma(k - \beta) - i\sqrt{4\beta^2 - (\gamma(k - \beta))^2}}{2}
\]
is a conjugate of \( \alpha \) over \( \mathbb{Q} \), so both \( \alpha \) and \( \overline{\alpha} \) belong to the set \( \{\alpha_1, \ldots, \alpha_\ell\} \).

Let \( F \) be the Galois closure of \( \mathbb{Q}(\beta, \gamma) \) over \( \mathbb{Q} \) and \( G := \text{Gal}(F/\mathbb{Q}) \). Note that the numbers \( \beta_j, j = 1, \ldots, q \), are conjugate over \( \mathbb{Q}(\gamma) \), since, by the choice of \( \beta \), the minimal polynomial of \( \beta \) is irreducible in the ring \( \mathbb{Q}(\gamma)[x] \).

Thus, given any \( j \) in the range \( 1 \leq j \leq q \), there is an automorphism \( \sigma \in G \) that fixes \( \gamma \) and maps \( \beta \mapsto \beta_j \). This automorphism maps the factor \( x^2 - \gamma(k - \beta)x + \beta^2 \) to the factor \( x^2 - \gamma(k - \beta_j)x + \beta_j^2 \), so it takes the pair of roots of the first quadratic polynomial, \( \alpha = \alpha_1(\gamma, \beta), \overline{\alpha} = \alpha_2(\gamma, \beta) \), to the pair of roots \( \alpha_1(\gamma, \beta_j), \alpha_2(\gamma, \beta_j) \) of the second quadratic polynomial. In particular, this implies that these four roots are conjugate over \( \mathbb{Q} \) for each \( j = 1, \ldots, q \).

Next, map \( \gamma \) to \( \gamma_i \), where \( i \) is one of the indices \( 1, \ldots, t + m \). This automorphism of \( G \) takes \( \beta \) to some \( \beta_j \) and \( x^2 - \gamma(k - \beta)x + \beta^2 \) to \( x^2 - \gamma_i(k - \beta_j)x + \beta_j^2 \). Hence, their roots \( \alpha_1(\gamma, \beta), \alpha_2(\gamma, \beta) \) and \( \alpha_1(\gamma_i, \beta_j), \alpha_2(\gamma_i, \beta_j) \) are conjugate over \( \mathbb{Q} \). Furthermore, by the same argument as above and the fact that the minimal polynomial of \( \beta_j \) is irreducible in \( \mathbb{Q}(\gamma_i)[x] \), the roots of \( x^2 - \gamma_i(k - \beta_j)x + \beta_j^2 \) (namely \( \alpha_1(\gamma_i, \beta_j), \alpha_2(\gamma_i, \beta_j) \)) and the roots of \( x^2 - \gamma_i(k - \beta_r)x + \beta_r^2 \) (say \( \alpha_1(\gamma_i, \beta_r), \alpha_2(\gamma_i, \beta_r) \)) are conjugate over \( \mathbb{Q} \) for any indices \( J, r \) in the range \( 1 \leq J, r \leq q \). Thus, we conclude that all \( (2t + 2m)q \) roots of the polynomial \( P \) defined in (12) are conjugate over \( \mathbb{Q} \).

Hence, \( \ell = \deg \alpha \) can be smaller than \( \deg P = 2(t + m)q \) only if \( \alpha \) is a multiple root of \( P \). However, if \( \alpha \) is equal to another complex root \( \alpha' \) of \( P \) corresponding, say to \( \gamma_i \neq \gamma \) and \( \beta \) (which is the only possibility to get a complex root), then
\[
\alpha + \overline{\alpha} = \gamma(k - \beta) = \gamma_i(k - \beta) = \alpha' + \overline{\alpha'}.
\]
This yields \( \gamma = \gamma_i \), a contradiction. The proof of Theorem 1 is now completed.

We conclude this section with an example which shows that the unit
\[
\alpha := 15 + 5\sqrt{2} + 6\sqrt{3} + 2\sqrt{6} + \sqrt{310} + 222\sqrt{2} + 276\sqrt{3} + 120\sqrt{6} = 74.724635 \ldots
\]
is a nonreciprocal unit of degree 8 with 4 real conjugates and two pairs of complex conjugates of equal moduli. This corresponds to the case \( K = \mathbb{Q}(\alpha) \) and \( s = 4, t = 2, m = 0, q = 2 \) in equality (2) of Theorem 1.

Take a quadratic algebraic integer \( \gamma = 3 - \sqrt{2} \) with conjugate \( \gamma' = 3 + \sqrt{2} \) and a quadratic Pisot unit \( \beta := 7 + 4\sqrt{3} \) with conjugate \( \beta' = 7 - 4\sqrt{3} \). Then the conditions (9) and (10) are satisfied with \( I = 5 \). Evidently, the minimal
polynomial of $\beta$ is irreducible in the ring $\mathbb{Q}(\gamma, \gamma')$. By (11), we obtain $k = 17$. Hence, $H(x, \beta)$ is the product of the polynomials

$$x^2 - (3 - \sqrt{2})(10 - 4\sqrt{3})x + 97 + 56\sqrt{3}$$

and

$$x^2 - (3 + \sqrt{2})(10 - 4\sqrt{3})x + 97 + 56\sqrt{3}.$$  

Thus, $H(x, \beta)$ is equal to

$$x^4 - (60 - 24\sqrt{3})x^3 + (1230 - 448\sqrt{3})x^2 - (1788 + 1032\sqrt{3})x + 18817 + 10864\sqrt{3}.$$  

Similarly, $H(x, \beta')$ equals

$$x^4 - (60 + 24\sqrt{3})x^3 + (1230 + 448\sqrt{3})x^2 - (1788 - 1032\sqrt{3})x + 18817 - 10864\sqrt{3}.$$  

Now, calculating the product $H(x, \beta)H(x, \beta')$ we find the polynomial (12)

$$P(x) = x^8 - 120x^7 + 4332x^6 - 86664x^5 + 1311590x^4 - 10994952x^3 + 75494124x^2 - 19704x + 1,$$

which is irreducible in $\mathbb{Z}[x]$. It has four positive roots

$$0.000068\ldots, 0.000192\ldots, 0.00068\ldots, 26.844323\ldots, 74.724635\ldots$$

where the last one is the root $\alpha$ defined in (13), which is the larger of the roots of the quadratic factor $x^2 - (3 + \sqrt{2})(10 + 4\sqrt{3})x + (7 - 4\sqrt{3})^2$ and is the largest positive root of $P$, and two pairs of complex conjugate roots

$$6.779783\ldots \pm i12.166732\ldots, 2.435606\ldots \pm i13.713594\ldots$$

on the circle $|z| = \beta = 7 + 4\sqrt{3} = 13.928203\ldots$.

6. Proof of Theorem 2

Consider the subgroup $U$ of $U_K$ of units satisfying (1). If $U$ has rank at least $s$ then it contains $s$ multiplicatively independent units $u_1, \ldots, u_s$. In particular, $u_1 \not\in \mathbb{Q}$. Suppose first that $K' := \mathbb{Q}(u_1, \ldots, u_s)$ is a proper subfield of $K$. Then $K'$ is a proper extension of $\mathbb{Q}$, since $u_1 \not\in \mathbb{Q}$. Applying Lemma 7 we find that $K'$ has $s$ real and $2t' > 0$ complex embeddings. By Corollary 4, the conjugates of $u_1$ have at least $s + 1$ distinct moduli. Note that the restrictions of the embeddings $\sigma_1, \ldots, \sigma_s$ of $K$ to $K'$ are the real embeddings of $K'$. Hence, the list

$$\sigma_1(u_1), \ldots, \sigma_s(u_1), \sigma_{s+1}(u_1), \sigma_{s+1}(u_1), \ldots, \sigma_{s+t}(u_1), \sigma_{s+t}(u_1)$$

contains at least $s + 1$ numbers with distinct moduli. Since the last $2t$ numbers in this list have the same modulus, the first $s$ must have distinct moduli. Now, as $\sigma_1(u_1)$ appears in the list exactly $k = (s + 2t)/(s + 2t') > 1$ times and $k \in \mathbb{N}$, it must appear at least once among the last $2t$ numbers of the list. However, then the number of distinct moduli in the list is at most $s$, a contradiction.
It remains to consider the alternative case when \( \mathbb{Q}(u_1, \ldots, u_s) = K \). Then, by Lemma 6, the semigroup \( S(u_1, \ldots, u_s) \) contains \( s \) multiplicatively independent units \( v_1, \ldots, v_s \) of degree \( d \) each. Since \( v_1, \ldots, v_s \in S(u_1, \ldots, u_s) \) and the units \( u_1, \ldots, u_s \) satisfy the condition (1), the units \( v_1, \ldots, v_s \) must satisfy (1) as well. In particular, this implies that the matrix

\[
M = M(v_1, \ldots, v_s) := (\log |\sigma_j(v_i)|)_{1 \leq i, j \leq s}
\]

has rank \( s \). However, by Theorem 1, the units \( v_1, \ldots, v_s \) of degree \( d \) must be reciprocal. Hence, for each \( i = 1, \ldots, s \) the product over real embeddings \( \prod_{j=1}^s \sigma_j(v_i) \) is equal to 1. Thus, the columns of the matrix \( M \) are linearly dependent, which implies that the rank of \( M \) is smaller than \( s \). (This is also true for \( s = 1 \) when \( M \) is the \( 1 \times 1 \) matrix with entry 0.) Therefore, the rank of \( U \) is smaller than \( s \). This completes the proof of Theorem 2, by the result of Oeljeklaus and Toma [16] stated in Section 1. (See also a stronger result given in Theorem 8 of the Appendix.)

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**Appendix (by Laurent Battisti)**

This appendix has two objectives. First, we prove a criterion for detecting OT-manifolds admitting a locally conformally Kähler metric and in the second part we give an alternative proof of Theorem 2 by using a geometric property of OT-manifolds (namely, their non-Kählerianity). In what follows, we keep the notation defined in the introduction of the main article.

A complex manifold \( X \) is said to admit an LCK metric if there exists a closed positive \((1,1)\)-form \( \tilde{\omega} \) on the universal cover \( \tilde{X} \) of \( X \) and a representation \( \rho : \pi_1(X) \rightarrow \mathbb{R}_{>0} \) such that for all \( g \in \pi_1(X) \), one has \( g^* \tilde{\omega} = \rho(g) \tilde{\omega} \). This notion was introduced by Vaisman in [28]. See also the introduction of the main article for further references on the subject.

In the case of an OT-manifold \( X(K,A) \), its fundamental group is (up to isomorphism) the semi-direct product \( A \rtimes \mathcal{O}_K \) and its universal cover is \( \mathbb{H}^s \times \mathbb{C}^t \). In [16] (proof of Proposition 2.9) and in [33] the authors prove that if an OT-manifold \( X(K,A) \) admits an LCK metric then one has \( \rho(g) = |\sigma_{s+1}(u)|^2 = \ldots = |\sigma_{s+t}(u)|^2 \) for all \( g = (u,a) \in A \rtimes \mathcal{O}_K \). It turns out that this relation between the absolute values of the complex embeddings of the elements of \( A \) is in fact a characterization:

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Theorem 8. An OT-manifold $X(K, A)$ admits an LCK metric if and only if the following holds:

\[(14) \quad \text{for all } u \in A, \quad |\sigma_{s+1}(u)| = \cdots = |\sigma_{s+t}(u)|.\]

Proof. We only need to check that this condition is sufficient. Let $X(K, A)$ be an OT-manifold with $A$ satisfying condition (14) and define the following real function on $\mathbb{H}^s \times \mathbb{C}^t$:

$$
\varphi(z) := \left( \prod_{j=1}^{s} \frac{i}{z_j - \bar{z}_j} \right)^{\frac{1}{t}} + \sum_{k=1}^{t} |z_{s+k}|^2.
$$

This definition of $\varphi$ is very natural: when $t = 1$, this function is the same as the function $F$ defined in [16], example p. 169.

It is enough to prove that it is a Kähler potential on $\mathbb{H}^s \times \mathbb{C}^t$. For this, we will see that the matrix $(\partial z_p \partial \overline{z}_q \varphi_1)$ is positive definite, where we set $\varphi_1(z) = \left( \prod_{j=1}^{s} \frac{i}{z_j - \bar{z}_j} \right)^{\frac{1}{t}}$. For all $q \in \{1, ..., s\}$, one has:

$$
\partial z_q \varphi_1(z) = \frac{1}{t} \frac{1}{z_q - \bar{z}_q} \varphi_1,
$$

and for all $p \in \{1, ..., s\}$, one has:

$$
\partial z_p \partial \overline{z}_q \varphi_1(z) = \begin{cases} 
\frac{1}{t^2} \frac{-1}{(z_p - \bar{z}_p)(z_q - \bar{z}_q)} \varphi_1 & \text{if } p \neq q \\
\frac{1}{t^2} \frac{1}{(1+t)} \frac{-1}{(z_p - \bar{z}_p)^2} \varphi_1 & \text{if } p = q.
\end{cases}
$$

Hence, $(\partial z_p \partial \overline{z}_q \varphi_1) = \frac{1}{t^2} \varphi_1 B$ where the matrix $B$ is

$$
B = \begin{pmatrix}
\frac{(1+t)}{4y_1^2} & \frac{1}{4y_1 y_2} & \cdots & \frac{1}{4y_1 y_s} \\
\frac{1}{4y_1 y_2} & \frac{(1+t)}{4y_2^2} & \cdots & \frac{1}{4y_2 y_s} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{4y_s y_1} & \frac{1}{4y_s y_2} & \cdots & \frac{(1+t)}{4y_s^2}
\end{pmatrix},
$$

and $z_j = x_j + iy_j$ for all $j \in \{1, ..., s + t\}$. As in [19], we notice that $B$ is the sum of a diagonal positive definite matrix and a positive semidefinite one. Hence, $B$ is positive definite.

Now, let $\omega_0 := i \partial \bar{\partial} \varphi$ and for all $g = (u, a) \in A \times O_K$ set $\rho(g) := |\sigma_{s+1}(u)|^2$. First, notice that because $u$ is a unit we have

$$
(\sigma_1(u) \ldots \sigma_s(u))(|\sigma_{s+1}(u)|^2 \ldots |\sigma_{s+t}(u)|^2) = 1.
$$
Then, write
\[
\partial\bar{\partial}(\varphi \circ g)(z) = \frac{1}{(\sigma_1(u)\cdots\sigma_s(u))^t} \partial\bar{\partial}\varphi_1(z)
\]
\[
+ \partial\bar{\partial} \sum_{k=1}^t |\sigma_{s+k}(u)z_{s+k} + \sigma_{s+k}(a)|^2
\]
\[
= \rho(g)\partial\bar{\partial}\varphi_1(z) + \rho(g)\partial\bar{\partial} \sum_{k=1}^t |z_{s+k}|^2
\]
\[
= \rho(g)\partial\bar{\partial}\varphi(z).
\]
We now obtain the following equalities:
\[
g^*\omega_0 = g^*(i\partial\bar{\partial}\varphi) = i\partial\bar{\partial}(\varphi \circ g) = i\rho(g)\partial\bar{\partial}(\varphi) = \rho(g)\omega_0.
\]
This concludes the proof. \qed

Recall that in [16], the authors showed that no OT-manifold admits a Kähler structure (this is Proposition 2.5, loc. cit.). Using this fact, we now see how to prove Theorem 2.

Assume that $K$ is a number field of degree $d = s + 2t$ with $t \geq 2$ and with $s$ not being of the form (2). We now suppose that the rank of the subgroup $U$ of $U_K$ of units satisfying equation (1) is at least (therefore, equal to) $s$ and we want to show that this leads to a contradiction.

First, notice that $l(U)$ has a trivial intersection with the kernel of the projecting map $\mathcal{P}: \mathcal{S} \to \mathbb{R}^s$, where $l$ and $\mathcal{P}$ are defined in the introduction of the main article. Thus, $U$ is an admissible subgroup of $U_K$. Now, consider the OT-manifold $X(K,U)$; it admits an LCK metric by Theorem 8. As a consequence of Theorem 1, all the elements of $U$ are reciprocal. In particular, $|\sigma_{s+j}(u)| = 1$ for all $u \in U$ and for all $j \in \{1,...,t\}$.

Let $\omega$ be a Kähler form on $\mathbb{H}^s \times \mathbb{C}^t$ giving rise to an LCK metric on $X(K,U)$. For all $g = (u,a) \in U \times O_K$, one has $g^*\omega = |\sigma_{s+1}(u)|^2\omega$ (see the paragraph before Theorem 8), which simplifies as $g^*\omega = \omega$. The form $\omega$ being invariant under the action of $A \ltimes O_K$, it descends to a Kähler form on $X(K,U)$. This implies that $X(K,U)$ is a Kähler manifold, which is the desired contradiction.

References


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