Horospherical limit points of $S$-arithmetic groups

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Abstract. Suppose $\Gamma$ is an $S$-arithmetic subgroup of a connected, semisimple algebraic group $G$ over a global field $Q$ (of any characteristic). It is well-known that $\Gamma$ acts by isometries on a certain CAT(0) metric space $X_S = \prod_{v \in S} X_v$, where each $X_v$ is either a Euclidean building or a Riemannian symmetric space. For a point $\xi$ on the visual boundary of $X_S$, we show there exists a horoball based at $\xi$ that is disjoint from some $\Gamma$-orbit in $X_S$ if and only if $\xi$ lies on the boundary of a certain type of flat in $X_S$ that we call “$Q$-good.” This generalizes a theorem of G. Avramidi and D. W. Morris that characterizes the horospherical limit points for the action of an arithmetic group on its associated symmetric space $X$.

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1. Introduction

Definition 1.1 ([6, Defn. B]). Suppose the group $\Gamma$ acts by isometries on the CAT(0) metric space $X$, and fix $x \in X$. A point $\xi$ on the visual boundary of $X$ is a horospherical limit point for $\Gamma$ if every horoball based at $\xi$ intersects the orbit $x \cdot \Gamma$. Notice that this definition is independent of the choice of $x$. Also note that if $\Lambda$ is a finite-index subgroup of $\Gamma$, then $\xi$ is a horospherical limit point for $\Lambda$ if and only if it is a horospherical limit point for $\Gamma$.

In the situation where $\Gamma$ is an arithmetic group, with its natural action on its associated symmetric space $X$, the horospherical limit points have a simple geometric characterization:
Theorem 1.2 (Avramidi–Morris [1, Thm. 1.3]). Let:
• $G$ be a connected, semisimple algebraic group over $\mathbb{Q}$,
• $K$ be a maximal compact subgroup of the Lie group $G(\mathbb{R})$,
• $X = K \backslash G(\mathbb{R})$ be the corresponding symmetric space of noncompact type (with the natural metric induced by the Killing form of $G(\mathbb{R})$), and
• $\Gamma$ be an arithmetic subgroup of $G$.

Then a point $\xi \in \partial X$ is not a horospherical limit point for $\Gamma$ if and only if $\xi$ is on the boundary of some flat $F$ in $X$, such that $F$ is the orbit of a $\mathbb{Q}$-split torus in $G(\mathbb{R})$.

This note proves a natural generalization that allows $\Gamma$ to be $S$-arithmetic (of any characteristic), rather than arithmetic. The precise statement assumes familiarity with the theory of Bruhat–Tits buildings [12], and requires some additional notation.

Notation 1.3.
(1) Let:
• $Q$ be a global field (of any characteristic),
• $G$ be a connected, semisimple algebraic group over $Q$,
• $S$ be a finite set of places of $Q$ (containing all the archimedean places if the characteristic of $Q$ is 0),
• $G_v = G(Q_v)$ for each $v \in S$, where $Q_v$ is the completion of $Q$ at $v$,
• $K_v$ be a maximal compact subgroup of $G_v$, for each $v \in S$, and
• $Z_S$ be the ring of $S$-integers in $Q$.
(2) Adding the subscript $S$ to any symbol other than $Z$ denotes the Cartesian product over all elements of $S$. Thus, for example, we have $G_S = \prod_{v \in S} G_v = \prod_{v \in S} G(Q_v)$.
(3) For each $v \in S$, let

$$X_v = \begin{cases} 
\text{the symmetric space } K_v \backslash G(Q_v) & \text{if } v \text{ is archimedean,} \\
\text{the Bruhat–Tits building of } G(Q_v) & \text{if } v \text{ is nonarchimedean.}
\end{cases}$$

Thus, $G_v = G(Q_v)$ acts properly and cocompactly by isometries on the CAT(0) metric space $X_v$. So $G_S$ acts properly and cocompactly by isometries on the CAT(0) metric space $X_S = \prod_{v \in S} X_v$.

Definition 1.4. We say a family $\{Y_t\}_{t \in \mathbb{R}}$ of subsets of $X_S$ is uniformly coarsely dense in $X_S/G(Z_S)$ if there exists $C > 0$, such that, for every $t \in \mathbb{R}$, each $G(Z_S)$-orbit in $X_S$ has a point that is at distance $< C$ from some point in $Y_t$.

See Definition 3.2 for the definition of a $Q$-good flat in $X_S$.

Theorem 1.5 (cf. [1, Cor. 4.5]). For a point $\xi$ on the visual boundary of $X_S = \prod_{v \in S} X_v$, the following are equivalent:
(1) \( \xi \) is a horospherical limit point for \( G(Z_S) \).
(2) \( \xi \) is not on the boundary of any \( Q \)-good flat.
(3) There does not exist a parabolic \( Q \)-subgroup \( P \) of \( G \), such that \( P_S \) fixes \( \xi \), and \( P(Z_S) \) fixes some (or, equivalently, every) horosphere based at \( \xi \).
(4) The horospheres based at \( \xi \) are uniformly coarsely dense in \( X_S/G(Z_S) \).
(5) The horoballs based at \( \xi \) are uniformly coarsely dense in \( X_S/G(Z_S) \).
(6) \( \pi(B) = X_S/G(Z_S) \) for every horoball \( B \) based at \( \xi \), where \( \pi : X_S \to X_S/G(Z_S) \) is the natural covering map.

**Remark 1.6.** The implications (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) are proved in the following sections, by fairly straightforward adaptations of arguments in [1]. This suffices to establish the theorem, since:

- (1) \( \Leftrightarrow \) (6) is a restatement of Definition 1.1.
- (4) \( \Rightarrow \) (5) is obvious, because horoballs are bigger than horospheres.
- (5) \( \Rightarrow \) (1) is well-known (see, for example, [1, Lem. 2.3(\( \Leftarrow \))]).

The minimal parabolic \( Q \)-subgroups of \( G \) are all conjugate under \( G(Q) \) [4, Thm. 4.13(b)], and the proof of Proposition 3.4 shows that the nonhorospherical limit points fixed by a given parabolic \( Q \)-subgroup are all contained in the boundary of a single \( Q \)-good flat, so Theorem 1.5 implies the following alternative characterization of the horospherical limit points:

**Corollary 1.7** (cf. [1, Cor. 1.4]). If \( B \) is the boundary of any maximal \( Q \)-good flat in \( X_S \), then the set of horospherical limit points for \( G(Z_S) \) is the complement of \( \bigcup_{g \in G(Q)} Bg \).

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**2. Proof of** (3) \( \Rightarrow \) (4)

(3) \( \Rightarrow \) (4) of Theorem 1.5 is the contrapositive of the following result.

**Proposition 2.1** (cf. [1, Thm. 4.3]). If the horospheres based at \( \xi \) are not uniformly coarsely dense in \( X_S/G(Z_S) \), then there is a parabolic \( Q \)-subgroup \( P \) of \( G \), such that:

- (1) \( P_S \) fixes \( \xi \).
- (2) \( P(Z_S) \) fixes some (or, equivalently, every) horosphere based at \( \xi \).

**Proof.** We modify the proof of [1, Thm. 4.3] to deal with minor issues, such as the fact that \( G_S \) is not (quite) transitive on \( X_S \). To avoid technical complications, assume \( G \) is simply connected. We begin by introducing yet more notation:
(Γ) Let $\Gamma = G(Z_S)$.

(x) Let $x \in X_S$. If $v \in S$ is a nonarchimedean place, then we choose $x$ so that its projection to $X_v$ is a vertex.

(γ) Let $\gamma: \mathbb{R} \to X_S$ be a geodesic with $\gamma(0) = x$ and $\gamma(+\infty) = \xi$. Let $\gamma^+: [0, \infty) \to X$ be the forward geodesic ray of $\gamma$. For each $v \in S$, let $\gamma_v$ be the projection of $\gamma$ to $X_v$, so $\gamma_v$ is a geodesic in $X_v$.

(F$S$) For each $v \in S$, choose a maximal flat (or “apartment”) $F_v$ in $X_v$ that contains $\gamma_v$. Then $F_S$ is a maximal flat in $X_S$ that contains $\gamma$.

(A$S$) For each $v \in S$, there is a maximal $Q_v$-split torus $A_v$ of $G(Q_v)$, such that $A_v$ acts properly and cocompactly on the Euclidean space $F_v$ by translations. Then $A_S$ acts properly and cocompactly on $F_S$ (by translations).

(C$S$) For each $v \in S$, choose a compact subset $C_v$ of $F_v$, such that $C_v A_v = F_v$. Then $C_v A_S = F_S$.

(A$\gamma$) Let $A_\gamma = \{ a \in A_S \mid C_S a \cap \gamma \neq \emptyset \}$ and $A_\gamma^+ = \{ a \in A_S \mid C_S a \cap \gamma^+ \neq \emptyset \}$.

(F$_\perp$, A$_\perp$) Let $F_\perp$ be the (codimension-one) hyperplane in $F_S$ that is orthogonal to the geodesic $\gamma$ and contains $x$. Let $A_\perp = \{ a \in A_S \mid C_S a \cap F_\perp \neq \emptyset \}$.

(P$^\xi_v$, N$_v$) For each $v \in S$, let $P_v^\xi = \{ g \in G(Q_v) \mid \{ aga^{-1} \mid a \in A_\gamma^+ \} \text{ is bounded} \}$, so $P_v^\xi$ is a parabolic $Q_v$-subgroup of $G(Q_v)$ that fixes $\xi$. The Iwasawa decomposition [12, §3.3.2] allows us to choose a maximal horospherical subgroup $N_v$ of $G(Q_v)$ that is contained in $P_v^\xi$ and is normalized by $A_v$, such that $F_v N_v = X_v$.

(P$^\xi_v$, M$^*_v$, T$^\ast_v$, M$_v^\ast$) By applying the $S$-arithmetic generalization of Ratner’s Theorem that was proved independently by Margulis-Tomanov [7] and Ratner [11] (or, if char $Q \neq 0$, by applying a theorem of Mohammadi [8, Cor. 4.2]), we obtain an $S$-arithmetic analogue of [1, Cor. 2.13]. Namely, for some parabolic $Q$-subgroup $P$ of $G$, if we let $P_v = P(Q_v)$ for each $v \in S$, and let $P_v = M_v T_v U_v$ be the Langlands decomposition over $Q_v$ (so $T_v$ is the maximal $Q_v$-split torus in the center of the reductive group $M_v T_v$, and $U_v$ is the unipotent radical), then we have $N_S \subseteq M_v^\ast U_S$ and $M_v^\ast U_S \gamma \subseteq N_S \Gamma$,

where $M_v^\ast$ is the product of all the isotropic almost-simple factors of $M_v$.

Since $N_v \subseteq P_v$ for every $v$ (and $P_S$ is parabolic), we have $U_S \subseteq N_S$ and $A_S \subseteq P_S$ (cf. proof of [1, Lem. 2.10]). Therefore, since all maximal $Q_v$-split tori of $P_v$ are conjugate [2, Thm. 20.9(ii), p. 228], and $M_v^\ast T_v$ contains
a maximal $Q_v$-split torus, there is no harm in assuming $A_S \subseteq M_S^* T_S$, by replacing $M_S^* T_S$ with a conjugate. Let $A_S^M = A_S \cap M_S = A_S \cap M_S^*$.

Note that $N_v$ is in the kernel of every continuous homomorphism from $F_v^\times$ to $\mathbb{R}$. Since $F_v^\times$ acts continuously on the set of horospheres based at $\xi$, and these horospheres are parametrized by $\mathbb{R}$, this implies that $N_v$ fixes every horosphere based at $\xi$. Then, since $F_S N_S = X_S$, we see that, for each $a \in A_\gamma$, the set $F_\perp a N_S$ is the horosphere based at $\xi$ through the point $xa$. By the definition of $A_\perp$, this implies that the horosphere is at bounded Hausdorff distance from

$$\mathcal{H}_a = xaA_\perp N_S.$$ 

(Also note that every horosphere is at bounded Hausdorff distance from some $\mathcal{H}_a$, since $A_S$ acts cocompactly on $F_S$.) We have

$$aA_\perp N_S \Gamma \supseteq aA_\perp \cdot N_S \Gamma \supseteq aA_\perp \cdot M_S^* U_S \Gamma. \tag{2.2}$$

We claim that $F_\perp A_S^M$ is not coarsely dense in $F_S$. Indeed, suppose, for the sake of a contradiction, that the set is coarsely dense. Then $A_\perp A_S^M$ is coarsely dense in $A_S$, which means there is a compact subset $K_1$ of $A_S$, such that $A_S = K_1 A_\perp A_S^M$. Also, the Iwasawa decomposition $[12, \S 3.3.2]$ of each $G(Q_v)$ implies there is a compact subset $K_S$ of $G_S$, such that $K_S A_S N_S = G_S$. Then, for every $a \in A_\gamma$, we have

$$K_S K_1 \cdot aA_\perp M_S^* U_S = K_S a(K_1 A_\perp M_S^*) U_S \supseteq K_S a A_S M_S^* U_S \supseteq K_S A_S N_S = G_S.$$

Since the compact set $K_S K_1$ is independent of $a$, this (together with (2.2)) implies that the sets $\mathcal{H}_a$ are uniformly coarsely dense in $X/\Gamma$. This contradicts the fact that the horospheres based at $\xi$ are not uniformly coarsely dense.

Since $F_\perp$ is a hyperplane of codimension one in $F_S$ (and $A_S^M$ is a group that acts by translations), the claim proved in the preceding paragraph implies $F_\perp = F_\perp A_S^M \supseteq xaA_S^M$. This means that $\gamma$ is orthogonal to the convex hull of $xA_S^M$.

On the other hand, we know that $M_S$ centralizes $T_S$. Therefore, $M_S$ fixes the endpoint $\xi_T$ of any geodesic ray $\gamma_T$ in the convex hull of $xT_S$. So $M_S$ acts (continuously) on the set of horospheres based at $\xi_T$. However, $M_S$ is the almost-direct product of compact groups and semisimple groups over local fields, so it has no no nontrivial homomorphism to $\mathbb{R}$. (For the semisimple groups, this follows from the truth of the Kneser–Tits Conjecture [10, Thm. 7.6].) Since the horospheres are parametrized by $\mathbb{R}$, we conclude that $M_S$ fixes every horosphere based at $\xi_T$. Hence $A_S^M$ also fixes these horospheres. So $xA_S^M$ is contained in the horosphere through $x$, which means the convex hull of $xA_S^M$ must be perpendicular to the convex hull of $xT_S$. Since $A_S^M T_S$ has finite index in $A_S$, the conclusion of the preceding paragraph now implies that $\gamma$ is contained in the convex hull of $xT_S$, so $C_{G_S}(T_S)$ fixes $\xi$. 

We also have
\[ P_S = M_S T_S U_S = C_{G_S} (T_S) U_S \subseteq C_{G_S} (T_S) N_S. \]
Since \( C_{G_S} (T_S) \) and \( N_S \) each fix the point \( \xi \), we conclude that \( P_S \) fixes \( \xi \). This completes the proof of (1).

From here, the proof of (2) is almost identical to the proof of Thm. 4.3(2) in [1]. □

3. Proof of \((2) \Rightarrow (3)\)

\((2) \Rightarrow (3)\) of Theorem 1.5 is the contrapositive of Proposition 3.4 below.

**Notation 3.1.** Suppose \( T \) is a torus that is defined over \( Q \). Let:

1. \( X^*_Q (T) \) be the set of \( Q \)-characters of \( T \);
2. \( T_S^{(1)} = \{ g \in T_S \mid \prod_{v \in S} \| \chi (g_v) \|_v = 1, \ \forall \chi \in X_Q (T) \} \).

**Definition 3.2.** Suppose \( F \) is a flat in \( X_S \) (not necessarily maximal). We say \( F \) is \( Q \)-good if there exists a \( Q \)-torus \( T \), such that:

- \( T \) contains a maximal \( Q \)-split torus of \( G \).
- \( T \) contains a maximal \( Q_v \)-split torus \( A_v \) of \( G_v \) for every \( v \in S \).
- \( F \) is contained in the maximal flat \( F_S \) that is fixed by \( A_S \).
- \( F \) is orthogonal to the convex hull of an orbit of \( T_S^{(1)} \) in \( F_S \).

**Remark 3.3.** \( Q \)-good flats are a natural generalization of \( Q \)-split flats. Indeed, the two notions coincide in the setting of arithmetic groups. Namely, suppose:

- \( Q \) is an algebraic number field.
- \( S \) is the set of all archimedean places of \( Q \).
- \( T \) is a maximal \( Q \)-split torus in \( G \).
- \( H = \text{Res}_{Q/Q} G \) is the \( Q \)-group obtained from \( G \) by restriction of scalars.

Then \( T_S \) can be viewed as the real points of a \( Q \)-torus in \( H(\mathbb{R}) \), and \( T_S^{(1)} \) is the group of real points of the \( Q \)-anisotropic part of \( T_S \). Thus, in this setting, the \( Q \)-good flats in the symmetric space of \( G_S \) are naturally identified with the \( Q \)-split flats in the symmetric space of \( H(\mathbb{R}) \).

**Proposition 3.4 (cf. [1, Prop. 4.4]).** If there is a parabolic \( Q \)-subgroup \( P \) of \( G \), such that \( P_S \) fixes \( \xi \), and \( P(Z_S) \) fixes every horosphere based at \( \xi \), then \( \xi \) is on the boundary of a \( Q \)-good flat in \( X_S \).

**Proof.** Choose a maximal \( Q \)-split torus \( R \) of \( P \). The centralizer of \( R \) in \( G \) is an almost direct product \( RM \) for some reductive \( Q \)-subgroup \( M \) of \( P \).

Choose a \( Q \)-torus \( L \) of \( M \), such that \( L(Q_v) \) contains a maximal \( Q_v \)-split torus \( B_v \) of \( M(Q_v) \) for each \( v \in S \). (This is possible when \( \text{char} \ Q = 0 \) by [10, Cor. 3 of §7.1, p. 405], and the same proof works in positive characteristic, because a theorem of A. Grothendieck tells us that the variety of maximal tori is rational [5, Exp. XIV, Thm. 6.1, p. 334], [3, Thm. 7.9].) Let \( T = RL \)
and \( A_v = R(Q_v)B_v \), so that \( T \) is a \( Q \)-torus that contains the maximal \( Q \)-split torus \( R \) as well as the maximal \( Q_v \)-split tori \( A_v \) for all \( v \in S \).

Let \( F_S \) be the maximal flat corresponding to \( A_S \), and choose some \( x \in F_S \). Since \( P_S \) fixes \( \xi \), there is a geodesic \( \gamma = \{ \gamma_t \} \) in \( F_S \), such that \( \lim_{t \to \infty} \gamma_t = \xi \) (and \( \gamma_0 = x \)).

Now \( T(Z_S) \) is a cocompact lattice in \( T_S^{(1)} \) (because the “Tamagawa number” of \( T \) is finite: see [10, Thm. 5.6, p. 264] if \( \text{char } Q = 0 \); or see [9, Thm. IV.1.3] for the general case), and, by assumption, \( T(Z_S) \) fixes the horosphere through \( x \). This implies that all of \( T_S^{(1)} \) fixes this horosphere, so \( xT_S^{(1)} \) is contained in the horosphere. Therefore, the convex hull of \( xT_S^{(1)} \) is perpendicular to the geodesic \( \gamma \), so \( \gamma \) is a \( Q \)-good flat. \( \square \)

4. Proof of \((1) \Rightarrow (2)\)

\((1) \Rightarrow (2)\) of Theorem 1.5 is the contrapositive of the following result.

**Proposition 4.1** (cf. [1, Prop. 3.1] or [6, Thm. A]). If \( \xi \) is on the boundary of a \( Q \)-good flat, then \( \xi \) is not a horospherical limit point for \( G(Z_S) \).

**Proof.** Let:

- \( F \) be a \( Q \)-good flat, such that \( \xi \) is on the boundary of \( F \).
- \( \gamma \) be a geodesic in \( F \), such that \( \lim_{t \to \infty} \gamma(t) = \xi \).
- \( T, A_S, \) and \( F_S \) be as in Definition 3.2.
- \( x = \gamma(0) \in F_S \).
- \( F_S \) be considered as a real vector space with Euclidean inner product, by specifying that the point \( x \) is the zero vector.
- \( C_x \) be a compact set, such that \( C_xA_S = F_S \) (and \( x \in C_x \)).
- \( \gamma^+ \) be the orthogonal complement of the 1-dimensional subspace \( \gamma \) in the vector space \( F_S \).
- \( \gamma^+ \setminus \gamma \) \( \{a \in A_S \, | \, C_x a \cap \gamma^+ \neq \emptyset \} \).
- \( \gamma^+_t \in A_S \), such that \( \gamma(t) \in C_x \gamma^+_t \), for each \( t \in \mathbb{R} \).
- \( R \) be a maximal \( Q \)-split torus of \( G \) that is contained in \( T \).
- \( \Phi \) be the system of roots of \( G \) with respect to \( R \).
- \( \alpha^S : T_S \to \mathbb{R}^+ \) be defined by \( \alpha^S(g) = \prod_{v \in S} ||\alpha(g_v)||_v \) for \( \alpha \in \Phi \) (where \( ||.||_v \) is extended to be defined on all of \( T(Q_v) \) by making it trivial on the \( Q \)-anisotropic part).
- \( \hat{\alpha}^S : F_S \to \mathbb{R} \) be the linear map satisfying \( \hat{\alpha}^S(xa) = \log \alpha^S(a) \) for all \( a \in A_S \).
- \( \alpha^F \in F_S \), such that \( \langle \alpha^F \, | \, y \rangle = \hat{\alpha}^S(y) \) for all \( y \in F_S \).
- \( \Phi^++ \) \( \{ \alpha \in \Phi \, | \, \hat{\alpha}^S(\gamma(t)) > 0 \text{ for } t > 0 \} \).
- \( \Delta \) be a base of \( \Phi \), such that \( \Phi^+ \) contains \( \Phi^{++} \).
- \( \Delta^{++} = \Delta \cap \Phi^{++} \).
- \( P_\alpha = R_\alpha M_\alpha N_\alpha \) be the parabolic \( Q \)-subgroup corresponding to \( \alpha \), for \( \alpha \in \Delta \), where:
By definition, we have $T_{\text{convex hull of } x}$ for $x \in X$. There is no harm in renormalizing the metric on $X$. Let $\alpha$ be the inverse of $N$. Since $S$ is nonarchimedean, this means $\gamma(t)$ is in the span of $\{\alpha^F\}_{\alpha \in \Delta}$. Also, for $\alpha \in \Delta$, we have

$$\langle \alpha^F | \gamma(t) \rangle = \hat{\alpha}^S(\gamma(t)) \geq 0.$$ 

There is no harm in renormalizing the metric on $X_S$ by a positive scalar on each irreducible factor (cf. [1, Rem. 5.4]). This allows us to assume $\langle \alpha^F | \beta^F \rangle \leq 0$ whenever $\alpha \neq \beta$ (see Lemma 4.2 below). Therefore, for any $b \in \gamma_1$, there is some $\alpha \in \Delta$, such that $\hat{\alpha}^S(x\gamma_A(t)b)$ is large (see Lemma 4.3 below). This means $\alpha^S(\gamma_A(t)b)$ is large.

Since conjugation by the inverse of $\gamma_A(t)b$ contracts the Haar measure on $(N_\alpha)_S$ by a factor of $\alpha^S(\gamma_A(t)b)^k$ for some $k \in \mathbb{Z}^+$, and the action of $N_S$ on $(N_\alpha)_S$ is volume-preserving, this implies that, for any $g \in \gamma_A(t)b N_S$, conjugation by the inverse of $g$ contracts the Haar measure on $(N_\alpha)_S$ by a large factor. Since $N_\alpha(Z_S)$ is a cocompact lattice in $(N_\alpha)_S$ (because the “Tamagawa number” of $N_\alpha$ is finite: see [10, Thm. 5.6, p. 264] if char $Q = 0$; or see [9, Thm. IV.1.13] for the general case), this implies there is some nontrivial $h \in N_\alpha(Z_S)$, such that $\|ghg^{-1} - e\|$ is small. We conclude that $\xi$ is not a horospherical limit point for $G(Z_S)$ (cf. [1, Lem. 2.5(2)]).

**Lemma 4.2.** Assume the notation of the proof of Proposition 4.1. The metric on $X_S$ can be renormalized so that we have $\langle \alpha^F | \beta^F \rangle \leq 0$ for all $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$.

**Proof.** When $v$ is archimedean, the Killing form provides a metric on $X_v$. We now construct an analogous metric when $v$ is nonarchimedean. To do this, let $\Phi_v$ be the root system of $G$ with respect to the maximal $Q_v$-split torus $A_v$, let $t = \bigoplus_{\alpha \in \Phi_v} \mathfrak{g}_\alpha$ be the corresponding weight-space decomposition of the Lie algebra of $G_v$, choose a uniformizer $\pi_v$ of $Q_v$, let $\mathcal{X}_v(A_v)$ be the group of co-characters of $A_v$, and define a $\mathbb{Z}$-bilinear form

$$\langle \cdot, \cdot \rangle_v : \mathcal{X}_v(A_v) \times \mathcal{X}_v(A_v) \to \mathbb{R}$$

by

$$\langle \varphi_1, \varphi_2 \rangle_v = \sum_{\alpha \in \Phi_v} v\left(\alpha\left(\varphi_1(\pi_v)\right)\right) v\left(\alpha\left(\varphi_2(\pi_v)\right)\right) \left(\dim \mathfrak{g}_\alpha\right).$$

This extends to a positive-definite inner product on $\mathcal{X}_v(A_v) \otimes \mathbb{R}$ (and the extension is also denoted by $\langle \cdot, \cdot \rangle_v$). It is clear that this inner product is invariant under the Weyl group, so it determines a metric on $X_v$ [12, §2.3].
By renormalizing, we may assume that the given metric on $X_v$ coincides with this one.

Let $E$ be the $Q$-anisotropic part of $T$. Then it is not difficult to see that $\mathcal{X}(R) \otimes R$ is the orthogonal complement of $\mathcal{X}(E(Q_v)) \otimes R$, with respect to the inner product $\langle | \rangle_v$ (cf. [1, Lem. 2.8]). Since every $Q$-root annihilates $E(Q_v)$, this implies that the $F_v$-component $\alpha^F_v$ of $\alpha^F$ belongs to the convex hull of $x R(Q_v)$, for every $\alpha \in \Phi$.

From [4, Cor. 5.5], we know that the Weyl group over $Q$ is the restriction to $R$ of a subgroup of the Weyl group over $Q_v$. So the restriction of $\langle | \rangle_v$ to $X^*(R) \otimes R$ is invariant under the $Q$-Weyl group. Assume, for simplicity, that $G$ is $Q$-simple, so the invariant inner product on $X^*(R) \otimes R$ is unique (up to a positive scalar). (The general case is obtained by considering the simple factors individually.) This means that, after passing to the dual space $X^*(R) \otimes R$, the inner product $\langle | \rangle_v$ must be a positive scalar multiple $c_v$ of the usual inner product (for which the reflections of the root system $\Phi$ are isometries), so $\langle \alpha^F_v | \beta^F_v \rangle_v = c_v \langle \alpha | \beta \rangle$ for all $\alpha, \beta \in \Delta$. Since it is a basic property of bases in a root system that $\langle \alpha | \beta \rangle \leq 0$ whenever $\alpha \neq \beta$, we therefore have

$$\langle \alpha^F | \beta^F \rangle = \sum_{v \in S} \langle \alpha^F_v | \beta^F_v \rangle_v = \sum_{v \in S} c_v \langle \alpha | \beta \rangle = \sum_{v \in S} (> 0) (\leq 0) \leq 0. \quad \square$$

**Lemma 4.3** ([1, Lem. 2.6]). *Suppose:*

1. $v, v_1, \ldots, v_n \in R^k$, with $v \neq 0$.
2. $v$ is in the span of $\{v_1, \ldots, v_n\}$.
3. $\langle v | v_i \rangle \geq 0$ for all $i$.
4. $\langle v_i | v_j \rangle \leq 0$ for $i \neq j$.
5. $T \in R^+$.

*Then, for all sufficiently large $t \in R^+$ and all $w \perp v$, there is some $i$, such that $\langle tv + w | v_i \rangle > T$.***

**References**


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