Conway–Gordon type theorem for the complete four-partite graph $K_{3,3,1,1}$

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Abstract. We give a Conway–Gordon type formula for invariants of knots and links in a spatial complete four-partite graph $K_{3,3,1,1}$ in terms of the square of the linking number and the second coefficient of the Conway polynomial. As an application, we show that every rectilinear spatial $K_{3,3,1,1}$ contains a nontrivial Hamiltonian knot.

1. Introduction

Throughout this paper we work in the piecewise linear category. Let $G$ be a finite graph. An embedding $f$ of $G$ into the Euclidean 3-space $\mathbb{R}^3$ is called a spatial embedding of $G$ and $f(G)$ is called a spatial graph. We denote the set of all spatial embeddings of $G$ by $\text{SE}(G)$. We call a subgraph $\gamma$ of $G$ which is homeomorphic to the circle a cycle of $G$ and denote the set of all cycles of $G$ by $\Gamma(G)$. We also call a cycle of $G$ a $k$-cycle if it contains exactly $k$ edges and denote the set of all $k$-cycles of $G$ by $\Gamma_k(G)$. In particular, a $k$-cycle is said to be Hamiltonian if $k$ equals the number of all vertices of $G$. For a positive integer $n$, $\Gamma^{(n)}(G)$ denotes the set of all cycles of $G (= \Gamma(G))$ if $n = 1$ and the set of all unions of $n$ mutually disjoint cycles of $G$ if $n \geq 2$. For an element $\gamma$ in $\Gamma^{(n)}(G)$ and an element $f$ in $\text{SE}(G)$, $f(\gamma)$ is none other than a knot in $f(G)$ if $n = 1$ and an $n$-component link in $f(G)$ if $n \geq 2$. In particular, we call $f(\gamma)$ a Hamiltonian knot in $f(G)$ if $\gamma$ is a Hamiltonian cycle.
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For an edge $e$ of a graph $G$, we denote the subgraph $G \setminus \text{int } e$ by $G - e$. Let $e = uv$ be an edge of $G$ which is not a loop, where $u$ and $v$ are distinct end vertices of $e$. Then we call the graph which is obtained from $G - e$ by identifying $u$ and $v$ the edge contraction of $G$ along $e$ and denote it by $G/e$. A graph $H$ is called a minor of a graph $G$ if there exists a subgraph $G'$ of $G$ and the edges $e_1, e_2, \ldots, e_m$ of $G'$ each of which is not a loop such that $H$ is obtained from $G'$ by a sequence of edge contractions along $e_1, e_2, \ldots, e_m$. A minor $H$ of $G$ is called a proper minor if $H$ does not equal $G$. Let $P$ be a property of graphs which is closed under minor reductions; that is, for any graph $G$ which does not have $P$, all minors of $G$ also do not have $P$. A graph $G$ is said to be minor-minimal with respect to $P$ if $G$ has $P$ but all proper minors of $G$ do not have $P$. Then it is known that there exist finitely many minor-minimal graphs with respect to $P$ [RS].

Let $K_m$ be the complete graph on $m$ vertices, namely the simple graph consisting of $m$ vertices in which every pair of distinct vertices is connected by exactly one edge. Then the following are very famous in spatial graph theory, which are called the Conway–Gordon theorems.

**Theorem 1.1** (Conway–Gordon [CG]).

(1) For any element $f$ in $\text{SE}(K_6)$,

\[
\sum_{\gamma \in \Gamma^{(2)}(K_6)} \text{lk}(f(\gamma)) \equiv 1 \pmod{2},
\]

where $\text{lk}$ denotes the linking number.

(2) For any element $f$ in $\text{SE}(K_7)$,

\[
\sum_{\gamma \in \Gamma^{(2)}(K_7)} a_2(f(\gamma)) \equiv 1 \pmod{2},
\]

where $a_2$ denotes the second coefficient of the Conway polynomial.

A graph is said to be intrinsically linked if for any element $f$ in $\text{SE}(G)$, there exists an element $\gamma$ in $\Gamma^{(2)}(G)$ such that $f(\gamma)$ is a nonsplittable 2-component link, and to be intrinsically knotted if for any element $f$ in $\text{SE}(G)$, there exists an element $\gamma$ in $\Gamma(G)$ such that $f(\gamma)$ is a nontrivial knot. Theorem 1.1 implies that $K_6$ (resp. $K_7$) is intrinsically linked (resp. knotted).

Moreover, the intrinsic linkedness (resp. knottedness) is closed under minor reductions [NeTh] (resp. [FL]), and $K_6$ (resp. $K_7$) is minor-minimal with respect to the intrinsically linkedness [S] (resp. knottedness [MRS]).

A $\triangle Y$-exchange is an operation to obtain a new graph $G_Y$ from a graph $G_\triangle$ by removing all edges of a 3-cycle $\triangle$ of $G_\triangle$ with the edges $\overline{uv}, \overline{vw}$ and $\overline{wu}$, and adding a new vertex $x$ and connecting it to each of the vertices $u, v$ and $w$ as illustrated in Figure 1.1 (we often denote $\overline{ux} \cup \overline{vx} \cup \overline{wx}$ by $Y$). A $Y\triangle$-exchange is the reverse of this operation. We call the set of all graphs obtained from a graph $G$ by a finite sequence of $\triangle Y$ and $Y\triangle$-exchanges the $G$-family and denote it by $\mathcal{F}(G)$. In particular, we denote
the set of all graphs obtained from \( G \) by a finite sequence of \( \triangle Y \)-exchanges by \( F_{\Delta}(G) \). For example, it is well known that the \( K_6 \)-family consists of exactly seven graphs as illustrated in Figure 1.2, where an arrow between two graphs indicates the application of a single \( \triangle Y \)-exchange. Note that \( F_{\Delta}(K_6) = F(K_6) \setminus \{P_7\} \). Since \( P_{10} \) is isomorphic to the Petersen graph, the \( K_6 \)-family is also called the Petersen family. It is also well known that the \( K_7 \)-family consists of exactly twenty graphs, and there exist exactly six graphs in the \( K_7 \)-family each of which does not belong to \( F_{\Delta}(K_7) \). Then the intrinsic linkedness and the intrinsic knottedness behave well under \( \triangle Y \)-exchanges as follows.

**Proposition 1.2** (Sachs [S]).

1. If \( G_\Delta \) is intrinsically linked, then \( G_Y \) is also intrinsically linked.
2. If \( G_\Delta \) is intrinsically knotted, then \( G_Y \) is also intrinsically knotted.

![Figure 1.1](image1.png)

**Figure 1.1.**

![Figure 1.2](image2.png)

**Figure 1.2.**

Proposition 1.2 implies that any element in \( F_{\Delta}(K_6) \) (resp. \( F_{\Delta}(K_7) \)) is intrinsically linked (resp. knotted). In particular, Robertson–Seymour–Thomas showed that the set of all minor-minimal intrinsically linked graphs equals the \( K_6 \)-family, so the converse of Proposition 1.2(1) is also true [RST].
On the other hand, it is known that any element in $F_\Delta(K_7)$ is minor-minimal with respect to the intrinsic knottedness [KS], but any element in $F(K_7) \setminus F_\Delta(K_7)$ is not intrinsically knotted [FN], [HNTY], [GMN], so the converse of Proposition 1.2(2) is not true. Moreover, there exists a minor-minimal intrinsically knotted graph which does not belong to $F_\Delta(K_7)$ as follows. Let $K_{n_1,n_2,\ldots,n_m}$ be the complete $m$-partite graph, namely the simple graph whose vertex set can be decomposed into $m$ mutually disjoint nonempty sets $V_1, V_2, \ldots, V_m$ where the number of elements in $V_i$ equals $n_i$ such that no two vertices in $V_i$ are connected by an edge and every pair of vertices in the distinct sets $V_i$ and $V_j$ is connected by exactly one edge, see Figure 1.3 which illustrates $K_{3,3}$, $K_{3,3,1}$ and $K_{3,3,1,1}$. Note that $K_{3,3,1}$ is isomorphic to $P_7$ in the $K_6$-family, namely $K_{3,3,1}$ is a minor-minimal intrinsically linked graph. On the other hand, Motwani–Raghunathan–Saran claimed in [MRS] that it may be proven that $K_{3,3,1,1}$ is intrinsically knotted by using the same technique of Theorem 1.1, namely, by showing that for any element in $SE(K_{3,3,1,1})$, the sum of $a_2$ over all of the Hamiltonian knots is always congruent to one modulo two. But Kohara–Suzuki showed in [KS] that the claim did not hold; that is, the sum of $a_2$ over all of the Hamiltonian knots is dependent to each element in $SE(K_{3,3,1,1})$. Actually, they demonstrated the specific two elements $f_1$ and $f_2$ in $SE(K_{3,3,1,1})$ as illustrated in Figure 1.4. Here $f_1(K_{3,3,1,1})$ contains exactly one nontrivial knot $f_1(\gamma_0)$ (= a trefoil knot, $a_2 = 1$) which is drawn by bold lines, where $\gamma_0$ is an element in $\Gamma_8(K_{3,3,1,1})$, and $f_2(K_{3,3,1,1})$ contains exactly two nontrivial knots $f_2(\gamma_1)$ and $f_2(\gamma_2)$ (= two trefoil knots) which are drawn by bold lines, where $\gamma_1$ and $\gamma_2$ are elements in $\Gamma_8(K_{3,3,1,1})$. Thus the situation of the case of $K_{3,3,1,1}$ is different from the case of $K_7$. By using another technique different from Conway–Gordon’s, Foisy proved the following.

**Theorem 1.3** (Foisy [F02]). For any element $f$ in $SE(K_{3,3,1,1})$, there exists an element $\gamma$ in $\bigcup_{k=4}^7 \Gamma_k(K_{3,3,1,1})$ such that $a_2(f(\gamma)) \equiv 1 \pmod{2}$.

**Figure 1.3.**

Theorem 1.3 implies $K_{3,3,1,1}$ is intrinsically knotted. Moreover, Proposition 1.2(2) and Theorem 1.3 imply that any element $G$ in $F_\Delta(K_{3,3,1,1})$ is
also intrinsically knotted. It is known that there exist exactly twenty six elements in $F_{\Delta}(K_{3,3,1,1})$. Since Kohara–Suzuki pointed out that each of the proper minors of $G$ is not intrinsically knotted [KS], it follows that any element in $F_{\Delta}(K_{3,3,1,1})$ is minor-minimal with respect to the intrinsic knottedness. Note that a $\Delta Y$-exchange does not change the number of edges of a graph. Since $K_7$ and $K_{3,3,1,1}$ have different numbers of edges, the families $F_{\Delta}(K_7)$ and $F_{\Delta}(K_{3,3,1,1})$ are disjoint.

Our first purpose in this article is to refine Theorem 1.3 by giving a kind of Conway–Gordon type formula for $K_{3,3,1,1}$ not over $\mathbb{Z}_2$, but integers $\mathbb{Z}$. In the following, $\Gamma_{k,l}^{(2)}(G)$ denotes the set of all unions of two disjoint cycles of a graph $G$ consisting of a $k$-cycle and an $l$-cycle, and $x$ and $y$ denotes the two vertices of $K_{3,3,1,1}$ with valency seven. Then we have the following.

**Theorem 1.4.**

1. For any element $f$ in $SE(K_{3,3,1,1})$,

$$
(1.3) \quad 4 \sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_7(K_{3,3,1,1}) \setminus \{x,y\} \subseteq \gamma} a_2(f(\gamma))
$$

$$
- 4 \sum_{\gamma \in \Gamma_6^{'}} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_5(K_{3,3,1,1}) \setminus \{x,y\} \subseteq \gamma} a_2(f(\gamma))
$$

$$
= \sum_{\lambda \in \Gamma_{3,5}^{(2)}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 + 2 \sum_{\lambda \in \Gamma_{4,4}^{(2)}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 - 18,
$$

where $\Gamma_6'$ is a specific proper subset of $\Gamma_6(K_{3,3,1,1})$ which does not depend on $f$ (see (2.31)).

2. For any element $f$ in $SE(K_{3,3,1,1})$,

$$
(1.4) \quad \sum_{\lambda \in \Gamma_{3,5}^{(2)}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 + 2 \sum_{\lambda \in \Gamma_{4,4}^{(2)}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 \geq 22.
$$
We prove Theorem 1.4 in the next section. By combining the two parts of Theorem 1.4, we immediately obtain the following.

**Corollary 1.5.** For any element \( f \) in \( SE(K_{3,3,1,1}) \),

\[(1.5) \sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) - \sum_{\gamma \in \Gamma_7(K_{3,3,1,1}) \setminus \{x,y\} \not\subset \gamma} a_2(f(\gamma)) - \sum_{\gamma \in \Gamma_6(K_{3,3,1,1}) \setminus \{x,y\} \not\subset \gamma} a_2(f(\gamma)) - \sum_{\gamma \in \Gamma_5(K_{3,3,1,1}) \setminus \{x,y\} \not\subset \gamma} a_2(f(\gamma)) \geq 1.
\]

Corollary 1.5 gives an alternative proof of the fact that \( K_{3,3,1,1} \) is intrinsically knotted. Moreover, Corollary 1.5 refines Theorem 1.3 by identifying the cycles that might be nontrivial knots in \( f(K_{3,3,1,1}) \).

**Remark 1.6.** We see the left side of (1.5) is not always congruent to one modulo two by considering two elements \( f_1 \) and \( f_2 \) in \( SE(K_{3,3,1,1}) \) as illustrated in Figure 1.4. Thus Corollary 1.5 shows that the argument over \( \mathbb{Z} \) has a nice advantage. In particular, \( f_1 \) gives the best possibility for (1.5), and therefore for (1.4) by Theorem 1.4(1). Actually \( f_1(K_{3,3,1,1}) \) contains exactly fourteen nontrivial links all of which are Hopf links, where the six of them are the images of elements in \( \Gamma_{3,3,1,1}^{(2)} \) by \( f_1 \) and the eight of them are the images of elements in \( \Gamma_{4,4,1,1}^{(2)}(K_{3,3,1,1}) \) by \( f_1 \).

As we said before, any element \( G \) in \( F_\Delta(K_7) \cup F_\Delta(K_{3,3,1,1}) \) is a minor-minimal intrinsically knotted graph. If \( G \) belongs to \( F_\Delta(K_7) \), then it is known that Conway–Gordon type formula over \( \mathbb{Z}_2 \) as in Theorem 1.1 also holds for \( G \) as follows.

**Theorem 1.7** (Nikkuni–Taniyama [NT]). Let \( G \) be an element in \( F_\Delta(K_7) \). Then, there exists a map \( \omega \) from \( \Gamma(G) \) to \( \mathbb{Z}_2 \) such that for any element \( f \) in \( SE(G) \),

\[
\sum_{\gamma \in \Gamma(G)} \omega(\gamma)a_2(f(\gamma)) \equiv 1 \pmod{2}.
\]

Namely, for any element \( G \) in \( F_\Delta(K_7) \), there exists a subset \( \Gamma \) of \( \Gamma(G) \) which depends on only \( G \) such that for any element \( f \) in \( SE(G) \), the sum of \( a_2 \) over all of the images of the elements in \( \Gamma \) by \( f \) is odd. On the other hand, if \( G \) belongs to \( F_\Delta(K_{3,3,1,1}) \), we have a Conway–Gordon type formula over \( \mathbb{Z} \) for \( G \) as in Corollary 1.5 as follows. We prove it in Section 3.

**Theorem 1.8.** Let \( G \) be an element in \( F_\Delta(K_{3,3,1,1}) \). Then, there exists a map \( \omega \) from \( \Gamma(G) \) to \( \mathbb{Z} \) such that for any element \( f \) in \( SE(G) \),

\[
\sum_{\gamma \in \Gamma(G)} \omega(\gamma)a_2(f(\gamma)) \geq 1.
\]
Our second purpose in this article is to give an application of Theorem 1.4 to the theory of rectilinear spatial graphs. A spatial embedding $f$ of a graph $G$ is said to be rectilinear if for any edge $e$ of $G$, $f(e)$ is a straight line segment in $\mathbb{R}^3$. We denote the set of all rectilinear spatial embeddings of $G$ by $\text{RSE}(G)$. We can see that any simple graph has a rectilinear spatial embedding by taking all of the vertices on the spatial curve $(t, t^2, t^3)$ in $\mathbb{R}^3$ and connecting every pair of two adjacent vertices by a straight line segment. Rectilinear spatial graphs appear in polymer chemistry as a mathematical model for chemical compounds, see [A] for example. Then by an application of Theorem 1.4, we have the following.

**Theorem 1.9.** For any element $f$ in $\text{RSE}(K_{3,3,1,1})$,
\[
\sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) \geq 1.
\]

We prove Theorem 1.9 in section 4. As a corollary of Theorem 1.9, we immediately have the following.

**Corollary 1.10.** For any element $f$ in $\text{RSE}(K_{3,3,1,1})$, there exists a Hamiltonian cycle $\gamma$ of $K_{3,3,1,1}$ such that $f(\gamma)$ is a nontrivial knot with $a_2(f(\gamma)) > 0$.

Corollary 1.10 is an affirmative answer to the question of Foisy–Ludwig [FL, QUESTION 5.8] which asks whether the image of every rectilinear spatial embedding of $K_{3,3,1,1}$ always contains a nontrivial Hamiltonian knot.

**Remark 1.11.**

1. In [FL, QUESTION 5.8], Foisy–Ludwig also asked that whether the image of every spatial embedding of $K_{3,3,1,1}$ (which may not be rectilinear) always contains a nontrivial Hamiltonian knot. As far as the authors know, it is still open.

2. In addition to the elements in $\mathcal{F}_\Delta(K_7) \cup \mathcal{F}_\Delta(K_{3,3,1,1})$, many minor-minimal intrinsically knotted graph are known [F04], [GMN]. In particular, it has been announced by Goldberg–Mattman–Naimi that all of the thirty two elements in $\mathcal{F}(K_{3,3,1,1}) \setminus \mathcal{F}_\Delta(K_{3,3,1,1})$ are minor-minimal intrinsically knotted graphs [GMN]. Note that their method is based on Foisy’s idea in the proof of Theorem 1.3 with the help of a computer.

2. **Conway–Gordon type formula for $K_{3,3,1,1}$**

To prove Theorem 1.4, we recall a Conway–Gordon type formula over $\mathbb{Z}$ for a graph in the $K_6$-family which is as below.

**Theorem 2.1.** Let $G$ be an element in $\mathcal{F}(K_6)$. Then there exist a map $\omega$ from $\Gamma(G)$ to $\mathbb{Z}$ such that for any element $f$ in $\text{SE}(G)$,
\[
2 \sum_{\gamma \in \Gamma(G)} \omega(\gamma)a_2(f(\gamma)) = \sum_{\gamma \in \Gamma^{(2)}(G)} \text{lk}(f(\gamma))^2 - 1.
\]
We remark here that Theorem 2.1 was shown by Nikkuni (for the case $G = K_6$) [N], O’Donnol ($G = P_7$) [O] and Nikkuni–Taniyama (for the others) [NT]. In particular, we use the following explicit formulae for $Q_8$ and $P_7$ in the proof of Theorem 1.4. For the other cases, see Hashimoto–Nikkuni [HN].

**Theorem 2.2.**

1. (Hashimoto–Nikkuni [HN]). For any element $f$ in $\text{SE}(Q_8)$,

$$2 \sum_{\gamma \in \Gamma(P_7)} a_2(f(\gamma)) + 2 \sum_{\gamma \in \Gamma_6(Q_8), v, v' \notin \gamma} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_6(Q_8), \gamma \cap \{v, v'\} \neq \emptyset} a_2(f(\gamma)) = \sum_{\gamma \in \Gamma_4^{(2)}(Q_8)} \text{lk}(f(\gamma))^2 - 1,$$

where $v$ and $v'$ are exactly two vertices of $Q_8$ with valency three.

2. (O’Donnol [O]). For any element $f$ in $\text{SE}(P_7)$,

$$2 \sum_{\gamma \in \Gamma(P_7)} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_6(P_7), u \notin \gamma} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_5(P_7)} a_2(f(\gamma)) = \sum_{\gamma \in \Gamma_4^{(2)}(P_7)} \text{lk}(f(\gamma))^2 - 1,$$

where $u$ is the vertex of $P_7$ with valency six.

By taking the modulo two reduction of (2.1), we immediately have the following fact containing Theorem 1.1(1).

**Corollary 2.3** (Sachs [S], Taniyama–Yasuhara [TY]). Let $G$ be an element in $\mathcal{F}(K_6)$. Then, for any element $f$ in $\text{SE}(G)$,

$$\sum_{\gamma \in \Gamma^{(2)}(G)} \text{lk}(f(\gamma)) \equiv 1 \pmod{2}.$$

Now we give labels for the vertices of $K_{3,3,1,1}$ as illustrated in the left figure in Figure 2.1. We also call the vertices 1, 3, 5 and 2, 4, 6 the black vertices and the white vertices, respectively. We regard $K_{3,3}$ as the subgraph of $K_{3,3,1,1}$ induced by all of the white and black vertices. Let $G_x$ and $G_y$ be two subgraphs of $K_{3,3,1,1}$ as illustrated in Figure 2.1(1) and (2), respectively. Since each of $G_x$ and $G_y$ is isomorphic to $P_7$, by applying Theorem 2.2(2) to $f|_{G_x}$ and $f|_{G_y}$ for an element $f$ in $\text{SE}(K_{3,3,1,1})$, it follows that

$$2 \sum_{\gamma \in \Gamma(P_7)} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_5(K_{3,3})} a_2(f(\gamma)) = \sum_{\gamma \in \Gamma_4^{(2)}(G_x)} \text{lk}(f(\gamma))^2 - 1,$$
Let $\gamma$ be an element in $\Gamma(K_{3,3,1,1})$ which is a 8-cycle or a 6-cycle containing $x$ and $y$. We will assign a type to $\gamma$ as follows:

- $\gamma$ is of **Type A** if the neighbor vertices of $x$ in $\gamma$ consist of both a black vertex and a white vertex (if and only if the neighbor vertices of $y$ in $\gamma$ consist of both a black vertex and a white vertex).
- $\gamma$ is of **Type B** if the neighbor vertices of $x$ in $\gamma$ consist of only black (resp. white) vertices and the neighbor vertices of $y$ in $\gamma$ consist of only white (resp. black) vertices.
- $\gamma$ is of **Type C** if $\gamma$ contains the edge $xy$.
- $\gamma$ is of **Type D** if $\gamma \in \Gamma_6(K_{3,3,1,1})$ and the neighbor vertices of $x$ and $y$ in $\gamma$ consist of only black or only white vertices.

Note that any element in $\Gamma_8(K_{3,3,1,1})$ is of Type A, B or C, and any element in $\Gamma_6(K_{3,3,1,1})$ containing $x$ and $y$ is of Type A, B, C or D.

On the other hand, let $\lambda$ be an element in $\Gamma^{(2)}_{4,4}(K_{3,3,1,1})$. We assign types to $\lambda$ as follows:

- $\lambda$ is of **Type A** if $\lambda$ does not contain the edge $xy$ and both $x$ and $y$ are contained in either connected component of $\lambda$.
- $\lambda$ is of **Type B** if $x$ and $y$ are contained in different connected components of $\lambda$.
- $\lambda$ is of **Type C** if $\lambda$ contains the edge $xy$.

Note that any element in $\Gamma^{(2)}_{4,4}(K_{3,3,1,1})$ is of Type A, B or C.

Then the following three lemmas hold.
Lemma 2.4. For any element \( f \) in \( \text{SE}(K_{3,3,1,1}) \),

\[
(2.4) \quad \sum_{\lambda \in \Gamma^{(2)}_{3,3}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 + 2 \sum_{\begin{subarray}{c}
\lambda \in \Gamma^{(2)}_{3,3}(K_{3,3,1,1}) \\
\text{Type A}
\end{subarray}} \text{lk}(f(\lambda))^2
\]

\[
= 4 \sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) - 4 \left\{ \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_7(G_y)} a_2(f(\gamma)) \right\}
\]

\[
+ 8 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_6(K_{3,3,1,1})} a_2(f(\gamma))
\]

\[
- 4 \left\{ \sum_{\gamma \in \Gamma_5(G_x)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_5(G_y)} a_2(f(\gamma)) \right\} + 10.
\]

Proof. For \( i = 1, 3, 5 \) and \( j = 2, 4, 6 \), let us consider subgraphs \( F_x^{(ij)} = (G_x - ij) \cup jy \cup jy \) and \( F_y^{(ij)} = (G_y - ij) \cup ix \cup ix \) of \( K_{3,3,1,1} \) as illustrated in Figure 2.2(1) and (2), respectively. Since each of \( F_x^{(ij)} \) and \( F_y^{(ij)} \) is isomorphic to \( P_7 \), by applying Theorem 2.2(2) to \( f|_{F_x^{(ij)}} \), it follows that

\[
(2.5) \quad \sum_{\begin{subarray}{c}
\lambda = \gamma \cup \gamma' \in \Gamma^{(2)}_{3,3}(F_x^{(ij)}) \\
\gamma \in \Gamma_3(F_x^{(ij)}), \ \gamma' \in \Gamma_3(F_x^{(ij)}) \\
x \in \gamma, \ y \in \gamma'
\end{subarray}} \text{lk}(f(\lambda))^2 + \sum_{\begin{subarray}{c}
\lambda = \gamma \cup \gamma' \in \Gamma^{(2)}_{3,3}(F_y^{(ij)}) \\
x \in \gamma, \ y \in \gamma'
\end{subarray}} \text{lk}(f(\lambda))^2
\]

\[
+ \sum_{\begin{subarray}{c}
\lambda = \gamma \cup \gamma' \in \Gamma^{(2)}_{3,3}(G_x) \\
\gamma \in \Gamma_3(G_x), \ \gamma' \in \Gamma_3(G_x) \\
j \notin \lambda, \ x \in \gamma
\end{subarray}} \text{lk}(f(\lambda))^2
\]

\[
= 2 \left\{ \sum_{\gamma \in \Gamma_8(F_x^{(ij)})} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) \right\}
\]

\[
- 4 \left\{ \sum_{\gamma \in \Gamma_7(F_x^{(ij)})} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) \right\}
\]

\[
- 2 \left\{ \sum_{\gamma \in \Gamma_6(F_x^{(ij)})} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_5(G_x)} a_2(f(\gamma)) \right\} + 1.
\]
Let us take the sum of both sides of (2.5) over \( i = 1, 3, 5 \) and \( j = 2, 4, 6 \). For an element \( \gamma \) in \( \Gamma_8(K_{3,3,1,1}) \) of Type A, there uniquely exists \( F_x^{(ij)} \) containing \( \gamma \). This implies that

\[
(2.6) \quad \sum_{i,j} \left( \sum_{\gamma \in \Gamma_8(F_x^{(ij)})} a_2(f(\gamma)) \right) = \sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)).
\]

For an element \( \gamma \) of \( \Gamma_7(G_x) \), there exist exactly four edges of \( K_{3,3} \) which are not contained in \( \gamma \). Thus \( \gamma \) is common for exactly four \( F_x^{(ij)} \)'s. This implies that

\[
(2.7) \quad \sum_{i,j} \left( \sum_{\gamma \in \Gamma_7(G_x), \gamma \not\in \gamma} a_2(f(\gamma)) \right) = 4 \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)).
\]

For an element \( \gamma \) in \( \Gamma_7(G_y) \), there uniquely exists \( F_x^{(ij)} \) containing \( \gamma \). This implies that

\[
(2.8) \quad \sum_{i,j} \left( \sum_{\gamma \in \Gamma_7(F_x^{(ij)}), \gamma \not\in \gamma} a_2(f(\gamma)) \right) = \sum_{\gamma \in \Gamma_7(G_y)} a_2(f(\gamma)).
\]

For an element \( \gamma \) in \( \Gamma_6(K_{3,3}) \), there exist exactly three edges of \( K_{3,3} \) which are not contained in \( \gamma \). Thus \( \gamma \) is common for exactly three \( F_x^{(ij)} \)'s. This
implies that

\[
(2.9) \quad \sum_{i,j} \left( \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) \right) = 3 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)).
\]

For an element \( \gamma \) in \( \Gamma_6(K_{3,3,1,1}) \) containing \( x \) and \( y \), if \( \gamma \) is of Type A, then there uniquely exists \( F_x^{(ij)} \) containing \( \gamma \). This implies that

\[
(2.10) \quad \sum_{i,j} \left( \sum_{\gamma \in \Gamma_6(F_x^{(ij)})} a_2(f(\gamma)) \right) = \sum_{\gamma \in \Gamma_6(K_{3,3,1,1})} a_2(f(\gamma)).
\]

For an element \( \gamma \) in \( \Gamma_5(G_x) \), there exist exactly six edges of \( K_{3,3} \) which are not contained in \( \gamma \). Thus \( \gamma \) is common for exactly six \( F_x^{(ij)} \)'s. This implies that

\[
(2.11) \quad \sum_{i,j} \left( \sum_{\gamma \in \Gamma_5(G_x)} a_2(f(\gamma)) \right) = 6 \sum_{\gamma \in \Gamma_5(G_x)} a_2(f(\gamma)).
\]

For an element \( \lambda = \gamma \cup \gamma' \) in \( \Gamma_3,5^{(2)}(K_{3,3,1,1}) \) where \( \gamma \) is a 3-cycle and \( \gamma' \) is a 5-cycle, if \( \gamma \) contains \( x \) and \( \gamma' \) contains \( y \), then there uniquely exists \( F_x^{(ij)} \) containing \( \lambda \). This implies that

\[
(2.12) \quad \sum_{i,j} \left( \sum_{\lambda = \gamma \cup \gamma' \in \Gamma_3,5^{(2)}(F_x^{(ij)})} \text{lk}(f(\lambda))^2 \right) = \sum_{\lambda = \gamma \cup \gamma' \in \Gamma_3,5^{(2)}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2.
\]
For an element $\lambda$ in $\Gamma'(4,4')$ of Type A, there uniquely exists $F_x^{(ij)}$ containing $\lambda$. This implies that

$$\sum_{i,j} \left( \sum_{\lambda = \gamma \cup \gamma' \in \Gamma'(2,2') \lambda \in \Gamma'(2,2') \gamma, \gamma' \in \Gamma'(2,2') \gamma \in \Gamma'(2,2') \gamma' \in \Gamma'(2,2')} \text{lk}(f(\lambda))^2 \right) = \sum_{\lambda \in \Gamma(2,2') \lambda \in \Gamma(2,2')} \text{lk}(f(\lambda))^2.$$

For an element $\lambda$ in $\Gamma'(3,4)$, there exist exactly four edges of $K_{3,3}$ which are not contained in $\lambda$. Thus $\lambda$ is common for exactly four $F_x^{(ij)}$s. This implies that

$$\sum_{i,j} \left( \sum_{\lambda = \gamma \cup \gamma' \in \Gamma'(3,4)} \text{lk}(f(\lambda))^2 \right) = 4 \sum_{\lambda \in \Gamma'(3,4)} \text{lk}(f(\lambda))^2.$$

Thus by (2.5), (2.6), (2.7), (2.8), (2.9), (2.10), (2.11), (2.12), (2.13) and (2.14), we have

$$\sum_{\lambda = \gamma \cup \gamma' \in \Gamma'(2,2') \lambda \in \Gamma'(2,2') \gamma, \gamma' \in \Gamma'(2,2')} \text{lk}(f(\lambda))^2 + \sum_{\lambda \in \Gamma'(2,2')} \text{lk}(f(\lambda))^2$$

$$= 2 \sum_{\gamma \in \Gamma'(2,2')} a_2(\gamma) + 8 \sum_{\gamma \in \Gamma'(2,2')} a_2(\gamma)$$

$$- 4 \sum_{\gamma \in \Gamma'(2,2')} a_2(\gamma) - 12 \sum_{\gamma \in \Gamma'(2,2')} a_2(\gamma)$$

$$- 2 \sum_{\gamma \in \Gamma'(2,2')} a_2(\gamma) 12 \sum_{\gamma \in \Gamma'(2,2')} a_2(\gamma) + 9.$$
Lemma 2.5. For any element \( f \) in \( \text{SE}(K_{3,3,1,1}) \),

\[
(2.18) \quad \sum_{\lambda \in \Gamma_{3,4}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 = 2 \sum_{\gamma \in \Gamma_3(K_{3,3,1,1})} a_2(f(\gamma)) + 4 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) - 2 \left\{ \sum_{\gamma \in \Gamma_6(G_x)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_5(G_y)} a_2(f(\gamma)) \right\} - 2 \sum_{\gamma \in \Gamma_6(K_{3,3,1,1})} a_2(f(\gamma)) + 2.
\]

By applying Theorem 2.2(2) to \( f \mid_{F_y}^{(ij)} \) and combining the same argument as in the case of \( F_x^{(ij)} \) with (2.3), we also have

\[
(2.17) \quad \sum_{\lambda \in \Gamma_{3,4}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 = 2 \sum_{\gamma \in \Gamma_3(K_{3,3,1,1})} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) + 4 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_6(K_{3,3,1,1})} a_2(f(\gamma)) + 5.
\]
Proof. Let us consider subgraphs $Q^{(1)}_8 = K_{3,3} \cup x \cup K_{3,3} \cup y \cup 2 \cup 4 \cup y\bar{6}$ and $Q^{(2)}_8 = K_{3,3} \cup x \cup 2 \cup 4 \cup y \cup 6 \cup y\bar{7} \cup y\bar{3} \cup y\bar{5}$ of $K_{3,3,1,1}$ as illustrated in Figure 2.3(1) and (2), respectively. Since each of $Q^{(1)}_8$ and $Q^{(2)}_8$ is homeomorphic to $Q_8$, by applying Theorem 2.2(1) to $f|_{Q^{(1)}_8}$ and $f|_{Q^{(2)}_8}$, it follows that

\begin{align}
\sum_{\lambda \in \Gamma^{(2)}_{4,4}(Q_8^{(i)})} \text{lk}(f(\lambda))^2 &= 2 \sum_{\gamma \in \Gamma_8(Q_8^{(i)})} a_2(f(\gamma)) + 2 \sum_{\gamma \in \Gamma_{6}(K_{3,3})} a_2(f(\gamma)) \\
&\quad - 2 \sum_{\gamma \in \Gamma_{6}(Q_8^{(i)})} a_2(f(\gamma)) - 2 \sum_{x \in \gamma, \ y \notin \gamma} a_2(f(\gamma)) \\
&\quad - 2 \sum_{\gamma \in \Gamma_{6}(Q_8^{(i)})} a_2(f(\gamma)) + 1
\end{align}

for $i = 1, 2$. By adding (2.19) for $i = 1, 2$, we have the result. \qed

![Figure 2.3. (1) $Q_8^{(1)}$, (2) $Q_8^{(2)}$](image)

**Lemma 2.6.** For any element $f$ in $SE(K_{3,3,1,1})$,

\begin{align}
\sum_{\lambda \in \Gamma^{(2)}_{4,4}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 \\
&= 2 \sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) - 8 \sum_{\gamma \in \Gamma_{6}(K_{3,3})} a_2(f(\gamma)) \\
&\quad - 2 \sum_{\gamma \in \Gamma_{6}(K_{3,3,1,1})} a_2(f(\gamma)) + 2.
\end{align}

**Proof.** For $k = 1, 2, \ldots, 6$, let us consider subgraphs

$F_x^{(k)} = (G_x - k) \cup xy \cup k\bar{y}$,
\[ F_y^{(k)} = (G_y - \overline{y}) \cup \overline{kx} \cup \overline{y}x, \]
of \(K_{3,3,1,1}\) as illustrated in Figure 2.4(1) and (2), respectively. Since each of \(F_x^{(k)}\) and \(F_y^{(k)}\) is also homeomorphic to \(P_7\), by applying Theorem 2.2(2) to \(f|_{F_x^{(k)}}\), it follows that

\[
\sum_{\lambda = \gamma \cup \gamma' \in \Gamma_8^{(2)}(F_x^{(k)})} \sum_{x,y \in \gamma, \text{ Type C}} \text{lk}(f(\lambda))^2 + \sum_{\lambda \in \Gamma_7^{(2)}(G_x) \atop xk \not\subseteq \lambda} \sum_{\gamma \in \Gamma_6(F_x^{(k)})} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_5(G_x) \atop xk \not\subseteq \gamma} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) - 2 \left\{ \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_5(G_x) \atop xk \not\subseteq \gamma} a_2(f(\gamma)) \right\} + 1.
\]

\[ \text{Figure 2.4. } (1) F_x^{(k)}, (2) F_y^{(k)} \text{ } (k = 1, 2, 3, 4, 5, 6) \]

Let us take the sum of both sides of (2.21) over \(k = 1, 2, \ldots, 6\). For an element \(\gamma\) in \(\Gamma_8(K_{3,3,1,1})\), if \(\gamma\) is of Type C, then there uniquely exists \(F_x^{(k)}\) containing \(\gamma\). This implies that

\[
\sum_k k \left( \sum_{\gamma \in \Gamma_6(F_x^{(k)})} a_2(f(\gamma)) \right) = \sum_{\gamma \in \Gamma_8(K_{3,3,1,1}) \atop \text{Type C}} a_2(f(\gamma)).
\]
For an element $\gamma$ of $\Gamma_7(G_x)$, there exist exactly four edges which are incident to $x$ such that they are not contained in $\gamma$. Thus $\gamma$ is common for exactly four $F^{(k)}_x$'s. This implies that

$$\sum_k \left( \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) \right) = 4 \sum_{\Gamma_7(G_x)} a_2(f(\gamma)).$$

(2.23)

It is clear that any element $\gamma$ in $\Gamma_6(K_{3,3})$ is common for exactly six $F^{(k)}_x$'s. This implies that

$$\sum_k \left( \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) \right) = 6 \sum_{\Gamma_6(K_{3,3})} a_2(f(\gamma)).$$

(2.24)

For an element $\gamma$ in $\Gamma_6(K_{3,3,1,1})$ containing $x$ and $y$, if $\gamma$ is of Type C, then there uniquely exists $F^{(k)}_x$ containing $\gamma$. This implies that

$$\sum_k \left( \sum_{\gamma \in \Gamma_6(F^{(k)}_x)} a_2(f(\gamma)) \right) = \sum_{\Gamma_6(K_{3,3,1,1})} a_2(f(\gamma)).$$

(2.25)

For an element $\gamma$ of $\Gamma_5(G_x)$, there exist exactly four edges which are incident to $x$ such that they are not contained in $\gamma$. Thus $\gamma$ is common for exactly four $F^{(k)}_x$'s. This implies that

$$\sum_k \left( \sum_{\gamma \in \Gamma_5(G_x)} a_2(f(\gamma)) \right) = 4 \sum_{\Gamma_5(G_x)} a_2(f(\gamma)).$$

(2.26)

For an element $\lambda = \gamma \cup \gamma'$ in $\Gamma^{(2)}_{4,4}(K_{3,3,1,1})$, if $\lambda$ is of Type C, then there uniquely exists $F^{(k)}_x$ containing $\lambda$. This implies that

$$\sum_k \left( \sum_{\gamma \in \Gamma^{(2)}_{4,4}(F^{(k)}_x)} \text{lk}(f(\lambda))^2 \right) = \sum_{\Gamma^{(2)}_{4,4}(K_{3,3,1,1})} \text{Type C} \lambda \in \Gamma^{(2)}_{4,4}(K_{3,3,1,1}) \text{lk}(f(\lambda))^2.$$

(2.27)
exactly four $F_x^{(k)}$'s. This implies that

$$\sum_k (\sum_{\lambda \in \Gamma^{(2)}_{3,4}(G_x)} \frac{\sum_{\lambda \in \Gamma^{(2)}_{3,4}(G_x)} \lambda}{x \notin \lambda})^2 = 4 \sum_{\lambda \in \Gamma^{(2)}_{3,4}(G_x)} \lambda.$$

Then by (2.21), (2.22), (2.23), (2.24), (2.25), (2.26), (2.27) and (2.28), we have

$$\sum_{\lambda \in \Gamma^{(2)}_{3,4}(K_{3,3,1,1})} a_2(f(\lambda)) + 4 \sum_{\lambda \in \Gamma^{(2)}_{3,4}(G_x)} \lambda = 2 \sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) + 8 \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) - 24 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma))$$

Then by combining (2.29) and (2.2), we have the result. We remark here that by applying Theorem 2.2 (2) to $f|_{F^y}$ and combining the same argument as in the case of $F_x^{(k)}$ with (2.3), we also have (2.20).

**Proof of Theorem 1.4.** (1) Let $f$ be an element in $SE(K_{3,3,1,1})$. Then by combining (2.4), (2.18) and (2.20), we have

$$\sum_{\lambda \in \Gamma^{(2)}_{3,5}(K_{3,3,1,1})} \lambda + 2 \sum_{\lambda \in \Gamma^{(2)}_{3,4}(K_{3,3,1,1})} \lambda = 4 \sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) - 4 \left\{ \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_7(G_y)} a_2(f(\gamma)) \right\}$$

Then by combining (2.3), we have the result. We remark here that by applying Theorem 2.2 (2) to $f|_{F^y}$ and combining the same argument as in the case of $F_x^{(k)}$ with (2.3), we also have (2.20).
Note that
\[ \Gamma_k(G_x) \cup \Gamma_k(G_y) = \{ \gamma \in \Gamma_k(K_{3,3,1,1}) \mid \{x, y\} \not\subset \gamma \} \]
for \( k = 5, 7 \). Moreover, we define a subset \( \Gamma'_6 \) of \( \Gamma_6(K_{3,3,1,1}) \) by
\[ \Gamma'_6 = \{ \gamma \in \Gamma_6(G_x) \mid x \in \gamma \} \cup \{ \gamma \in \Gamma_6(G_y) \mid y \in \gamma \} \]
\[ \cup \{ \gamma \in \Gamma_6(K_{3,3,1,1}) \mid x, y \in \gamma, \gamma \text{ is of Type A, B or C} \} . \]

Then we see that (2.30) implies (1.3).

(2) Let \( f \) be an element in \( SE(K_{3,3,1,1}) \). Let us consider subgraphs \( H_1 = Q_8(1) \cup \overline{xy} \) and \( H_2 = Q_8(2) \cup \overline{xy} \) of \( K_{3,3,1,1} \) as illustrated in Figure 2.5(1) and (2), respectively. For \( i = 1, 2, \) \( H_i \) has the proper minor \( H'_i = H_i/\overline{xy} \) which is isomorphic to \( P_7 \). For a spatial embedding \( f|_{H_i} \) of \( H_i \), there exists a spatial embedding \( f' \) of \( H'_i \) such that \( f'(H'_i) \) is obtained from \( f(H_i) \) by contracting \( f(xy) \) into one point. Note that this embedding is unique up to ambient isotopy in \( \mathbb{R}^3 \). Then by Corollary 2.3, there exists an element \( \mu'_i \) in \( \Gamma^{(2)}_{3,4}(H'_i) \) such that \( \text{lk}(f'(\mu'_i)) \equiv 1 \pmod{2} \) \((i = 1, 2)\). Note that \( \mu'_i \) is mapped onto an element \( \mu_i \) in \( \Gamma_{4,4}(H_i) \) by the natural injection from \( \Gamma_{3,4}(H'_i) \) to \( \Gamma_{4,4}(H_i) \). Since \( f'(\mu'_i) \) is ambient isotopic to \( f(\mu_i) \), we have \( \text{lk}(f(\mu_i)) \equiv 1 \pmod{2} \) \((i = 1, 2)\). We also note that both \( \mu_1 \) and \( \mu_2 \) are of Type C in \( \Gamma^{(2)}_{4,4}(K_{3,3,1,1}) \).

\[ \text{Figure 2.5. (1) } H_1, \text{ (2) } H_2 \]

For \( v = x, y \) and \( i, j, k = 1, 2, \ldots, 6 \) \((i \neq j)\), let \( P_8^{(k)}(v; ij) \) be the subgraph of \( K_{3,3,1,1} \) as illustrated in Figure 2.6 (1) if \( v = y, k \in \{1, 3, 5\} \) and \( i, j \in \{2, 4, 6\} \), (2) if \( v = y, k \in \{2, 4, 6\} \) and \( i, j \in \{1, 3, 5\} \), (3) if \( v = x, k \in \{1, 3, 5\} \) and \( i, j \in \{2, 4, 6\} \) and (4) if \( v = x, k \in \{2, 4, 6\} \) and \( i, j \in \{1, 3, 5\} \). Note that there exist exactly thirty six \( P_8^{(k)}(v; ij) \)'s and they are isomorphic to \( P_8 \) in the \( K_6 \)-family. Thus by Corollary 2.3, there exists an element \( \lambda \) in \( \Gamma^{(2)}(P_8^{(k)}(v; ij)) \) such that \( \text{lk}(f(\lambda)) \equiv 1 \pmod{2} \). All elements in \( \Gamma^{(2)}(P_8^{(k)}(v; ij)) \) consist of exactly four elements in \( \Gamma_{3,8}^{(2)}(P_8^{(k)}(v; ij)) \).
and exactly four elements in $\Gamma^{(2)}_{4,4}(P^k_8(v;ij))$ of Type A or Type B because they do not contain the edge $xy$. It is not hard to see that any element in $\Gamma^{(2)}_{3,5}(K_{3,3,1,1})$ is common for exactly two $P^k_8(v;ij)$'s, and any element in $\Gamma^{(2)}_{4,4}(K_{3,3,1,1})$ of Type A or Type B is common for exactly four $P^k_8(v;ij)$'s.

Figure 2.6. $P^k_8(v;ij)$

By (2.4), there exist a nonnegative integer $m$ such that

$$\sum_{\lambda \in \Gamma^{(2)}_{3,5}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 = 2m.$$ 

If $2m \geq 18$, since there exist at least two elements $\mu_1$ and $\mu_2$ in $\Gamma^{(2)}_{3,5}(K_{3,3,1,1})$ of Type C such that $\text{lk}(f(\mu_i)) \equiv 1 \pmod{2}$ ($i = 1, 2$), we have

$$\sum_{\lambda \in \Gamma^{(2)}_{3,5}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 + 2 \sum_{\lambda \in \Gamma^{(2)}_{4,4}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 \geq 18 + 4 = 22.$$ 

If $2m \leq 16$, then there exist at least $(36 - 4m)/4 = 9 - m$ elements in $\Gamma^{(2)}_{4,4}(K_{3,3,1,1})$ of Type A or Type B such that each of the corresponding 2-component links with respect to $f$ has an odd linking number. Then we have

$$\sum_{\lambda \in \Gamma^{(2)}_{3,5}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 + 2 \sum_{\lambda \in \Gamma^{(2)}_{4,4}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 \geq 2m + 2 \{(9 - m) + 2\} = 22. \quad \square$$

3. $\triangle Y$-exchange and Conway–Gordon type formulae

In this section, we give a proof of Theorem 1.8. Let $G_\triangle$ and $G_Y$ be two graphs such that $G_Y$ is obtained from $G_\triangle$ by a single $\triangle Y$-exchange. Let $\gamma'$ be an element in $\Gamma(G_\triangle)$ which does not contain $\triangle$. Then there exists an
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element $Φ(γ′)$ in $Γ(G_Y)$ such that $γ′ \setminus Δ = Φ(γ′) \setminus Y$. It is easy to see that the correspondence from $γ′$ to $Φ(γ′)$ defines a surjective map

$$Φ : Γ(G_Δ) \setminus \{Δ\} → Γ(G_Y).$$

The inverse image of an element $γ$ in $Γ(G_Y)$ by $Φ$ contains at most two elements in $Γ(G_Δ) \setminus Γ(Δ(G_Δ))$. Figure 3.1 illustrates the case that the inverse image of $γ$ by $Φ$ consists of exactly two elements. Let $ω$ be a map from $Γ(G_Δ)$ to $Z$. Then we define the map $\tilde{ω}$ from $Γ(G_Y)$ to $Z$ by

$$\tilde{ω}(γ) = \sum_{γ′ ∈ Φ^{-1}(γ)} ω(γ′) \quad (3.1)$$

for an element $γ$ in $Γ(G_Y)$.

Let $f$ be an element in $SE(G_Y)$ and $D$ a 2-disk in $R^3$ such that $D \cap f(G_Y) = f(Y)$ and $\partial D \cap f(G_Y) = \{f(u), f(v), f(w)\}$. Let $φ(f)$ be an element in $SE(G_Δ)$ such that $φ(f)(x) = f(x)$ for $x ∈ G_Δ \setminus Δ = G_Y \setminus Y$ and $φ(f)(G_Δ) = (f(G_Y) \setminus f(Y)) \cup \partial D$. Thus we obtain a map

$$φ : SE(G_Y) → SE(G_Δ).$$

Then we immediately have the following.

**Proposition 3.1.** Let $f$ be an element in $SE(G_Y)$ and $γ$ an element in $Γ(G_Y)$. Then, $f(γ)$ is ambient isotopic to $φ(f)(γ′)$ for each element $γ′$ in the inverse image of $γ$ by $Φ$.

Then we have the following lemma which plays a key role to prove Theorem 1.8. This lemma has already been shown in [NT, Lemma 2.2] in more general form, but we give a proof for the reader’s convenience.
Lemma 3.2 (Nikkuni–Taniyama [NT]). For an element $f$ in $\text{SE}(G_Y)$,
\[\sum_{\gamma \in \Gamma(G_Y)} \tilde{\omega}(\gamma)a_2(f(\gamma)) = \sum_{\gamma' \in \Gamma(G_{\Delta})} \omega(\gamma')a_2(\varphi(f)(\gamma')).\]

**Proof.** Since $\varphi(f)(\Delta)$ is the trivial knot, we have
\[\sum_{\gamma' \in \Gamma(G_{\Delta})} \omega(\gamma')a_2(\varphi(f)(\gamma')) = \sum_{\gamma' \in \Gamma(G_{\Delta}) \setminus \{\Delta\}} \omega(\gamma')a_2(\varphi(f)(\gamma')).\]

Note that
\[\Gamma(G_{\Delta}) \setminus \{\Delta\} = \bigcup_{\gamma \in \Gamma(G_Y)} \Phi^{-1}(\gamma).\]

Then, by Proposition 3.1, we see that
\[\sum_{\gamma' \in \Gamma(G_{\Delta}) \setminus \{\Delta\}} \omega(\gamma')a_2(\varphi(f)(\gamma')) = \sum_{\gamma \in \Gamma(G_Y)} \left( \sum_{\gamma' \in \Phi^{-1}(\gamma)} \omega(\gamma')a_2(f(\gamma')) \right)\]
\[= \sum_{\gamma \in \Gamma(G_Y)} \left( \sum_{\gamma' \in \Phi^{-1}(\gamma)} \omega(\gamma)a_2(f(\gamma)) \right)\]
\[= \sum_{\gamma \in \Gamma(G_Y)} \tilde{\omega}(\gamma)a_2(f(\gamma)).\]

**Proof of Theorem 1.8.** By Corollary 1.5, there exists a map $\omega : \Gamma(K_{3,3,1,1}) \to \mathbb{Z}$ such that for any element $g$ in $\text{SE}(K_{3,3,1,1})$,
\[\sum_{\gamma' \in \Gamma(K_{3,3,1,1})} \omega(\gamma')a_2(g(\gamma')) \geq 1.\]

Let $G$ be a graph which is obtained from $K_{3,3,1,1}$ by a single $\Delta Y$-exchange and $\tilde{\omega}$ the map from $\Gamma(G)$ to $\mathbb{Z}$ as in (3.1). Let $f$ be an element in $\text{SE}(G)$. Then by Lemma 3.2 and (3.2), we see that
\[\sum_{\gamma \in \Gamma(G)} \tilde{\omega}(\gamma)a_2(f(\gamma)) = \sum_{\gamma' \in \Gamma(K_{3,3,1,1})} \omega(\gamma')a_2(\varphi(f)(\gamma')) \geq 1.\]

By repeating this argument, we have the result. \(\Box\)

**Remark 3.3.** In Theorem 1.8, the proof of the existence of a map $\omega$ is constructive. It is also an interesting problem to give $\omega(\gamma)$ for each element $\gamma$ in $\Gamma(G)$ concretely.
4. Rectilinear spatial embeddings of $K_{3,3,1,1}$

In this section, we give a proof of Theorem 1.9. For an element $f$ in $\text{RSE}(G)$ and an element $\gamma$ in $\Gamma_k(G)$, the knot $f(\gamma)$ has stick number less than or equal to $k$, where the stick number $s(K)$ of a knot $K$ is the minimum number of edges in a polygon which represents $K$. Then the following is well known.

**Proposition 4.1** (Adams [A], Negami [Ne]). For any nontrivial knot $K$, it follows that $s(K) \geq 6$. Moreover, $s(K) = 6$ if and only if $K$ is a trefoil knot.

We also show a lemma for a rectilinear spatial embedding of $P_7$ which is useful in proving Theorem 1.9.

**Lemma 4.2.** For an element $f$ in $\text{RSE}(P_7)$,

$$\sum_{\gamma \in \Gamma_7(P_7)} a_2(f(\gamma)) \geq 0.$$ 

**Proof.** Note that $a_2(\text{trivial knot}) = 0$ and $a_2(\text{trefoil knot}) = 1$. Thus by Proposition 4.1, $a_2(f(\gamma)) = 0$ for any element $\gamma$ in $\Gamma_5(P_7)$ and $a_2(f(\gamma)) \geq 0$ for any element $\gamma$ in $\Gamma_6(P_7)$. Moreover, by Corollary 2.3, we have

$$\sum_{\lambda \in \Gamma_5^{(2)}(P_7)} \text{lk}(f(\lambda))^2 \geq 1.$$ 

Then Theorem 2.2(2) implies the result. $\square$

**Proof of Theorem 1.9.** Let $f$ be an element in $\text{RSE}(K_{3,3,1,1})$. Since $G_x$ and $G_y$ are isomorphic to $P_7$, by Lemma 4.2, we have

$$\sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) \geq 0, \quad \sum_{\gamma \in \Gamma_7(G_y)} a_2(f(\gamma)) \geq 0.$$ 

Then by Corollary 1.5 and (4.2), we have

$$\sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) \geq \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_7(G_y)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_6} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_5(K_{3,3,1,1})} a_2(f(\gamma)) + 1$$

$$\geq 0 + 0 + 0 + 0 + 1 = 1. \quad \square$$

**Remark 4.3.** All of knots with $s \leq 8$ and $a_2 > 0$ are $3_1$, $5_1$, $5_2$, $6_3$, a square knot, a granny knot, $8_{19}$ and $8_{20}$ (Calvo [C]). Therefore, Theorem 1.9 implies that at least one of them appears in the image of every rectilinear spatial embedding of $K_{3,3,1,1}$. On the other hand, it is known that the image of every rectilinear spatial embedding of $K_7$ contains a trefoil knot (Brown
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It is still open whether the image of every rectilinear spatial embedding of $K_{3,3,1,1}$ contains a trefoil knot.

References


CONWAY–GORDON TYPE THEOREM FOR $K_{3,3,1,1}$


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This paper is available via http://nyjm.albany.edu/j/2014/20-27.html.