Conway–Gordon type theorem for the complete four-partite graph $K_{3,3,1,1}$

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Abstract. We give a Conway–Gordon type formula for invariants of knots and links in a spatial complete four-partite graph $K_{3,3,1,1}$ in terms of the square of the linking number and the second coefficient of the Conway polynomial. As an application, we show that every rectilinear spatial $K_{3,3,1,1}$ contains a nontrivial Hamiltonian knot.

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1. Introduction

Throughout this paper we work in the piecewise linear category. Let $G$ be a finite graph. An embedding $f$ of $G$ into the Euclidean 3-space $\mathbb{R}^3$ is called a spatial embedding of $G$ and $f(G)$ is called a spatial graph. We denote the set of all spatial embeddings of $G$ by $\text{SE}(G)$. We call a subgraph $\gamma$ of $G$ which is homeomorphic to the circle a cycle of $G$ and denote the set of all cycles of $G$ by $\Gamma(G)$. We also call a cycle of $G$ a $k$-cycle if it contains exactly $k$ edges and denote the set of all $k$-cycles of $G$ by $\Gamma_k(G)$. In particular, a $k$-cycle is said to be Hamiltonian if $k$ equals the number of all vertices of $G$. For a positive integer $n$, $\Gamma^{(n)}(G)$ denotes the set of all cycles of $G$ (i.e., $\Gamma(G)$) if $n=1$ and the set of all unions of $n$ mutually disjoint cycles of $G$ if $n \geq 2$. For an element $\gamma$ in $\Gamma^{(n)}(G)$ and an element $f$ in $\text{SE}(G)$, $f(\gamma)$ is none other than a knot in $f(G)$ if $n=1$ and an $n$-component link in $f(G)$ if $n \geq 2$. In particular, we call $f(\gamma)$ a Hamiltonian knot in $f(G)$ if $\gamma$ is a Hamiltonian cycle.

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For an edge $e$ of a graph $G$, we denote the subgraph $G \setminus \text{int}e$ by $G - e$. Let $e = uv$ be an edge of $G$ which is not a loop, where $u$ and $v$ are distinct end vertices of $e$. Then we call the graph which is obtained from $G - e$ by identifying $u$ and $v$ the edge contraction of $G$ along $e$ and denote it by $G/e$.

A graph $H$ is called a minor of a graph $G$ if there exists a subgraph $G'$ of $G$ and the edges $e_1, e_2, \ldots, e_m$ of $G'$ each of which is not a loop such that $H$ is obtained from $G'$ by a sequence of edge contractions along $e_1, e_2, \ldots, e_m$. A minor $H$ of $G$ is called a proper minor if $H$ does not equal $G$. Let $P$ be a property of graphs which is closed under minor reductions; that is, for any graph $G$ which does not have $P$, all minors of $G$ also do not have $P$. A graph $G$ is said to be minor-minimal with respect to $P$ if $G$ has $P$ but all proper minors of $G$ do not have $P$. Then it is known that there exist finitely many minor-minimal graphs with respect to $P$ [RS].

Let $K_m$ be the complete graph on $m$ vertices, namely the simple graph consisting of $m$ vertices in which every pair of distinct vertices is connected by exactly one edge. Then the following are very famous in spatial graph theory, which are called the Conway–Gordon theorems.

**Theorem 1.1** (Conway–Gordon [CG]).

1. For any element $f$ in $\text{SE}(K_6)$,
   $\sum_{\gamma \in \Gamma^{(2)}(K_6)} \text{lk}(f(\gamma)) \equiv 1 \pmod{2},$
   where $\text{lk}$ denotes the linking number.

2. For any element $f$ in $\text{SE}(K_7)$,
   $\sum_{\gamma \in \Gamma^{(2)}(K_7)} a_2(f(\gamma)) \equiv 1 \pmod{2},$
   where $a_2$ denotes the second coefficient of the Conway polynomial.

A graph is said to be intrinsically linked if for any element $f$ in $\text{SE}(G)$, there exists an element $\gamma$ in $\Gamma^{(2)}(G)$ such that $f(\gamma)$ is a nonsplittable 2-component link, and to be intrinsically knotted if for any element $f$ in $\text{SE}(G)$, there exists an element $\gamma$ in $\Gamma(G)$ such that $f(\gamma)$ is a nontrivial knot. Theorem 1.1 implies that $K_6$ (resp. $K_7$) is intrinsically linked (resp. knotted). Moreover, the intrinsic linkedness (resp. knottedness) is closed under minor reductions [NeTh] (resp. [FL]), and $K_6$ (resp. $K_7$) is minor-minimal with respect to the intrinsically linkedness [S] (resp. knottedness [MRS]).

A $\Delta Y$-exchange is an operation to obtain a new graph $G_Y$ from a graph $G_{\Delta}$ by removing all edges of a 3-cycle $\Delta$ of $G_{\Delta}$ with the edges $\overline{uv}, \overline{vw}$ and $\overline{wu}$, and adding a new vertex $x$ and connecting it to each of the vertices $u, v$ and $w$ as illustrated in Figure 1.1 (we often denote $\overline{ux} \cup \overline{vx} \cup \overline{wx}$ by $Y$). A $Y \Delta$-exchange is the reverse of this operation. We call the set of all graphs obtained from a graph $G$ by a finite sequence of $\Delta Y$ and $Y \Delta$-exchanges the $G$-family and denote it by $\mathcal{F}(G)$. In particular, we denote
the set of all graphs obtained from $G$ by a finite sequence of $\Delta Y$-exchanges by $F_\Delta(G)$. For example, it is well known that the $K_6$-family consists of exactly seven graphs as illustrated in Figure 1.2, where an arrow between two graphs indicates the application of a single $\Delta Y$-exchange. Note that $F_\Delta(K_6) = F(K_6) \setminus \{P_7\}$. Since $P_{10}$ is isomorphic to the Petersen graph, the $K_6$-family is also called the Petersen family. It is also well known that the $K_7$-family consists of exactly twenty graphs, and there exist exactly six graphs in the $K_7$-family each of which does not belong to $F_\Delta(K_7)$. Then the intrinsic linkedness and the intrinsic knottedness behave well under $\Delta Y$-exchanges as follows.

**Proposition 1.2** (Sachs [S]).

(1) If $G_\Delta$ is intrinsically linked, then $G_Y$ is also intrinsically linked.

(2) If $G_\Delta$ is intrinsically knotted, then $G_Y$ is also intrinsically knotted.

Proposition 1.2 implies that any element in $F_\Delta(K_6)$ (resp. $F_\Delta(K_7)$) is intrinsically linked (resp. knotted). In particular, Robertson–Seymour–Thomas showed that the set of all minor-minimal intrinsically linked graphs equals the $K_6$-family, so the converse of Proposition 1.2(1) is also true [RST].
On the other hand, it is known that any element in $F_\Delta(K_7)$ is minor-minimal with respect to the intrinsic knottedness [KS], but any element in $F(K_7) \setminus F_\Delta(K_7)$ is not intrinsically knotted [FN], [HNTY], [GMN], so the converse of Proposition 1.2(2) is not true. Moreover, there exists a minor-minimal intrinsically knotted graph which does not belong to $F_\Delta(K_7)$ as follows. Let $K_{n_1,n_2,...,n_m}$ be the complete $m$-partite graph, namely the simple graph whose vertex set can be decomposed into $m$ mutually disjoint nonempty sets $V_1, V_2, \ldots, V_m$ where the number of elements in $V_i$ equals $n_i$ such that no two vertices in $V_i$ are connected by an edge and every pair of vertices in the distinct sets $V_i$ and $V_j$ is connected by exactly one edge, see Figure 1.3 which illustrates $K_{3,3}$, $K_{3,3,1}$ and $K_{3,3,1,1}$. Note that $K_{3,3,1}$ is isomorphic to $P_7$ in the $K_6$-family, namely $K_{3,3,1}$ is a minor-minimal intrinsically linked graph. On the other hand, Motwani–Raghunathan–Saran claimed in [MRS] that it may be proven that $K_{3,3,1,1}$ is intrinsically knotted by using the same technique of Theorem 1.1, namely, by showing that for any element in $SE(K_{3,3,1,1})$, the sum of $a_2$ over all of the Hamiltonian knots is always congruent to one modulo two. But Kohara–Suzuki showed in [KS] that the claim did not hold; that is, the sum of $a_2$ over all of the Hamiltonian knots is dependent to each element in $SE(K_{3,3,1,1})$. Actually, they demonstrated the specific two elements $f_1$ and $f_2$ in $SE(K_{3,3,1,1})$ as illustrated in Figure 1.4. Here $f_1(K_{3,3,1,1})$ contains exactly one nontrivial knot $f_1(\gamma_0)$ (= a trefoil knot, $a_2 = 1$) which is drawn by bold lines, where $\gamma_0$ is an element in $\Gamma_8(K_{3,3,1,1})$, and $f_2(K_{3,3,1,1})$ contains exactly two nontrivial knots $f_2(\gamma_1)$ and $f_2(\gamma_2)$ (= two trefoil knots) which are drawn by bold lines, where $\gamma_1$ and $\gamma_2$ are elements in $\Gamma_8(K_{3,3,1,1})$. Thus the situation of the case of $K_{3,3,1,1}$ is different from the case of $K_7$. By using another technique different from Conway–Gordon’s, Foisy proved the following.

**Theorem 1.3** (Foisy [F02]). For any element $f$ in $SE(K_{3,3,1,1})$, there exists an element $\gamma$ in $\bigcup_{k=4}^{8} \Gamma_k(K_{3,3,1,1})$ such that $a_2(f(\gamma)) \equiv 1 \pmod{2}$.

Theorem 1.3 implies $K_{3,3,1,1}$ is intrinsically knotted. Moreover, Proposition 1.2(2) and Theorem 1.3 imply that any element $G$ in $F_\Delta(K_{3,3,1,1})$ is
also intrinsically knotted. It is known that there exist exactly twenty six elements in $\mathcal{F}_\triangle(K_{3,3,1,1})$. Since Kohara–Suzuki pointed out that each of the proper minors of $G$ is not intrinsically knotted [KS], it follows that any element in $\mathcal{F}_\triangle(K_{3,3,1,1})$ is minor-minimal with respect to the intrinsic knottedness. Note that a $\triangle Y$-exchange does not change the number of edges of a graph. Since $K_7$ and $K_{3,3,1,1}$ have different numbers of edges, the families $\mathcal{F}_\triangle(K_7)$ and $\mathcal{F}_\triangle(K_{3,3,1,1})$ are disjoint.

Our first purpose in this article is to refine Theorem 1.3 by giving a kind of Conway–Gordon type formula for $K_{3,3,1,1}$ not over $\mathbb{Z}_2$, but integers $\mathbb{Z}$. In the following, $\Gamma_{k,l}^{(2)}(G)$ denotes the set of all unions of two disjoint cycles of a graph $G$ consisting of a $k$-cycle and an $l$-cycle, and $x$ and $y$ denotes the two vertices of $K_{3,3,1,1}$ with valency seven. Then we have the following.

**Theorem 1.4.**

(1) For any element $f$ in $\text{SE}(K_{3,3,1,1})$,

$$\sum_{\gamma \in \Gamma_{8}(K_{3,3,1,1})} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_7(K_{3,3,1,1})} a_2(f(\gamma))$$

$$- 4 \sum_{\gamma \in \Gamma'_6} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_5(K_{3,3,1,1})} a_2(f(\gamma))$$

$$= \sum_{\lambda \in \Gamma_{3,5}^{(2)}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 + 2 \sum_{\lambda \in \Gamma_{4,4}^{(2)}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 - 18,$$

where $\Gamma'_6$ is a specific proper subset of $\Gamma_6(K_{3,3,1,1})$ which does not depend on $f$ (see (2.31)).

(2) For any element $f$ in $\text{SE}(K_{3,3,1,1})$,

$$\sum_{\lambda \in \Gamma_{3,5}^{(2)}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 + 2 \sum_{\lambda \in \Gamma_{4,4}^{(2)}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 \geq 22.$$
We prove Theorem 1.4 in the next section. By combining the two parts of Theorem 1.4, we immediately obtain the following.

**Corollary 1.5.** For any element $f$ in $\text{SE}(K_{3,3,1,1})$, 

$$
\sum_{\gamma \in \Gamma_{8}(K_{3,3,1,1})} a_2(f(\gamma)) - \sum_{\gamma \in \Gamma_{7}(K_{3,3,1,1}) \setminus \{x,y\} \not\subset \gamma} a_2(f(\gamma)) 
- \sum_{\gamma \in \Gamma_{6}'} a_2(f(\gamma)) - \sum_{\gamma \in \Gamma_{5}(K_{3,3,1,1}) \setminus \{x,y\} \not\subset \gamma} a_2(f(\gamma)) \geq 1.
$$

Corollary 1.5 gives an alternative proof of the fact that $K_{3,3,1,1}$ is intrinsically knotted. Moreover, Corollary 1.5 refines Theorem 1.3 by identifying the cycles that might be nontrivial knots in $f(K_{3,3,1,1})$.

**Remark 1.6.** We see the left side of (1.5) is not always congruent to one modulo two by considering two elements $f_1$ and $f_2$ in $\text{SE}(K_{3,3,1,1})$ as illustrated in Figure 1.4. Thus Corollary 1.5 shows that the argument over $\mathbb{Z}$ has a nice advantage. In particular, $f_1$ gives the best possibility for (1.5), and therefore for (1.4) by Theorem 1.4(1). Actually $f_1(K_{3,3,1,1})$ contains exactly fourteen nontrivial links all of which are Hopf links, where the six of them are the images of elements in $\Gamma_{3,5}^{(2)}(K_{3,3,1,1})$ by $f_1$ and the eight of them are the images of elements in $\Gamma_{4,4}^{(2)}(K_{3,3,1,1})$ by $f_1$.

As we said before, any element $G$ in $\mathcal{F}_{\triangle}(K_7) \cup \mathcal{F}_{\triangle}(K_{3,3,1,1})$ is a minor-minimal intrinsically knotted graph. If $G$ belongs to $\mathcal{F}_{\triangle}(K_7)$, then it is known that Conway–Gordon type formula over $\mathbb{Z}_2$ as in Theorem 1.1 also holds for $G$ as follows.

**Theorem 1.7** (Nikkuni–Taniyama [NT]). Let $G$ be an element in $\mathcal{F}_{\triangle}(K_7)$. Then, there exists a map $\omega$ from $\Gamma(G)$ to $\mathbb{Z}_2$ such that for any element $f$ in $\text{SE}(G)$,

$$
\sum_{\gamma \in \Gamma(G)} \omega(\gamma)a_2(f(\gamma)) \equiv 1 \pmod{2}.
$$

Namely, for any element $G$ in $\mathcal{F}_{\triangle}(K_7)$, there exists a subset $\Gamma$ of $\Gamma(G)$ which depends on only $G$ such that for any element $f$ in $\text{SE}(G)$, the sum of $a_2$ over all of the images of the elements in $\Gamma$ by $f$ is odd. On the other hand, if $G$ belongs to $\mathcal{F}_{\triangle}(K_{3,3,1,1})$, we have a Conway–Gordon type formula over $\mathbb{Z}$ for $G$ as in Corollary 1.5 as follows. We prove it in Section 3.

**Theorem 1.8.** Let $G$ be an element in $\mathcal{F}_{\triangle}(K_{3,3,1,1})$. Then, there exists a map $\omega$ from $\Gamma(G)$ to $\mathbb{Z}$ such that for any element $f$ in $\text{SE}(G)$,

$$
\sum_{\gamma \in \Gamma(G)} \omega(\gamma)a_2(f(\gamma)) \geq 1.
$$
Our second purpose in this article is to give an application of Theorem 1.4 to the theory of rectilinear spatial graphs. A spatial embedding $f$ of a graph $G$ is said to be rectilinear if for any edge $e$ of $G$, $f(e)$ is a straight line segment in $\mathbb{R}^3$. We denote the set of all rectilinear spatial embeddings of $G$ by $\text{RSE}(G)$. We can see that any simple graph has a rectilinear spatial embedding by taking all of the vertices on the spatial curve $(t, t^2, t^3)$ in $\mathbb{R}^3$ and connecting every pair of two adjacent vertices by a straight line segment. Rectilinear spatial graphs appear in polymer chemistry as a mathematical model for chemical compounds, see [A] for example. Then by an application of Theorem 1.4, we have the following.

**Theorem 1.9.** For any element $f$ in $\text{RSE}(K_{3,3,1,1})$,

$$
\sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) \geq 1.
$$

We prove Theorem 1.9 in section 4. As a corollary of Theorem 1.9, we immediately have the following.

**Corollary 1.10.** For any element $f$ in $\text{RSE}(K_{3,3,1,1})$, there exists a Hamiltonian cycle $\gamma$ of $K_{3,3,1,1}$ such that $f(\gamma)$ is a nontrivial knot with $a_2(f(\gamma)) > 0$.

Corollary 1.10 is an affirmative answer to the question of Foisy–Ludwig [FL, QUESTION 5.8] which asks whether the image of every rectilinear spatial embedding of $K_{3,3,1,1}$ always contains a nontrivial Hamiltonian knot.

**Remark 1.11.**

1. In [FL, QUESTION 5.8], Foisy–Ludwig also asked that whether the image of every spatial embedding of $K_{3,3,1,1}$ (which may not be rectilinear) always contains a nontrivial Hamiltonian knot. As far as the authors know, it is still open.

2. In addition to the elements in $\mathcal{F}_\Delta(K_7) \cup \mathcal{F}_\Delta(K_{3,3,1,1})$, many minor-minimal intrinsically knotted graph are known [F04], [GMN]. In particular, it has been announced by Goldberg–Mattman–Naimi that all of the thirty two elements in $\mathcal{F}(K_{3,3,1,1}) \setminus \mathcal{F}_\Delta(K_{3,3,1,1})$ are minor-minimal intrinsically knotted graphs [GMN]. Note that their method is based on Foisy’s idea in the proof of Theorem 1.3 with the help of a computer.

2. Conway–Gordon type formula for $K_{3,3,1,1}$

To prove Theorem 1.4, we recall a Conway–Gordon type formula over $\mathbb{Z}$ for a graph in the $K_6$-family which is as below.

**Theorem 2.1.** Let $G$ be an element in $\mathcal{F}(K_6)$. Then there exist a map $\omega$ from $\Gamma(G)$ to $\mathbb{Z}$ such that for any element $f$ in $\text{SE}(G)$,

$$
2 \sum_{\gamma \in \Gamma(G)} \omega(\gamma)a_2(f(\gamma)) = \sum_{\gamma \in \Gamma^{(2)}(G)} \text{lk}(f(\gamma))^2 - 1.
$$


We remark here that Theorem 2.1 was shown by Nikkuni (for the case $G = K_6$) [N], O’Donnol ($G = P_7$) [O] and Nikkuni–Taniyama (for the others) [NT]. In particular, we use the following explicit formulae for $Q_8$ and $P_7$ in the proof of Theorem 1.4. For the other cases, see Hashimoto–Nikkuni [HN].

**Theorem 2.2.**

1. (Hashimoto–Nikkuni [HN]). For any element $f$ in $\text{SE}(Q_8)$,
   \[
   2 \sum_{\gamma \in \Gamma_7(P_7)} a_2(f(\gamma)) + 2 \sum_{\gamma \in \Gamma_6(Q_8), v, v' \notin \gamma} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_6(Q_8), \gamma \cap \{v, v'\} \neq \emptyset} a_2(f(\gamma))
   \]
   \[
   = \sum_{\gamma \in \Gamma^{(2)}_{1,4}(Q_8)} \text{lk}(f(\gamma))^2 - 1,
   \]
   where $v$ and $v'$ are exactly two vertices of $Q_8$ with valency three.

2. (O’Donnol [O]). For any element $f$ in $\text{SE}(P_7)$,
   \[
   2 \sum_{\gamma \in \Gamma_7(P_7)} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_6(P_7), u \notin \gamma} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_5(P_7)} a_2(f(\gamma))
   \]
   \[
   = \sum_{\gamma \in \Gamma^{(2)}_{1,4}(P_7)} \text{lk}(f(\gamma))^2 - 1,
   \]
   where $u$ is the vertex of $P_7$ with valency six.

By taking the modulo two reduction of (2.1), we immediately have the following fact containing Theorem 1.1(1).

**Corollary 2.3** (Sachs [S], Taniyama–Yasuhara [TY]). Let $G$ be an element in $\mathcal{F}(K_6)$. Then, for any element $f$ in $\text{SE}(G)$,
\[
\sum_{\gamma \in \Gamma^{(2)}(G)} \text{lk}(f(\gamma)) \equiv 1 \pmod{2}.
\]

Now we give labels for the vertices of $K_{3,3,1,1}$ as illustrated in the left figure in Figure 2.1. We also call the vertices 1, 3, 5 and 2, 4, 6 the black vertices and the white vertices, respectively. We regard $K_{3,3}$ as the subgraph of $K_{3,3,1,1}$ induced by all of the white and black vertices. Let $G_x$ and $G_y$ be two subgraphs of $K_{3,3,1,1}$ as illustrated in Figure 2.1(1) and (2), respectively. Since each of $G_x$ and $G_y$ is isomorphic to $P_7$, by applying Theorem 2.2(2) to $f|_{G_x}$ and $f|_{G_y}$ for an element $f$ in $\text{SE}(K_{3,3,1,1})$, it follows that

\[
2 \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_5(G_x)} a_2(f(\gamma))
\]
\[
= \sum_{\gamma \in \Gamma^{(2)}_{1,4}(G_x)} \text{lk}(f(\gamma))^2 - 1,
\]
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(2.3)  
$$2 \sum_{\gamma \in \Gamma_7(G_y)} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_5(G_y)} a_2(f(\gamma)) = \sum_{\gamma \in \Gamma_{4,4}^2(G_y)} \text{lk}(f(\gamma))^2 - 1.$$ 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{2.1.png}
\caption{(1) $G_x$, (2) $G_y$}
\end{figure}

Let $\gamma$ be an element in $\Gamma(K_{3,3,1,1})$ which is a 8-cycle or a 6-cycle containing $x$ and $y$. We will assign a type to $\gamma$ as follows:

- $\gamma$ is of Type A if the neighbor vertices of $x$ in $\gamma$ consist of both a black vertex and a white vertex (if and only if the neighbor vertices of $y$ in $\gamma$ consist of both a black vertex and a white vertex).
- $\gamma$ is of Type B if the neighbor vertices of $x$ in $\gamma$ consist of only black (resp. white) vertices and the neighbor vertices of $y$ in $\gamma$ consist of only white (resp. black) vertices.
- $\gamma$ is of Type C if $\gamma$ contains the edge $xy$.
- $\gamma$ is of Type D if $\gamma \in \Gamma_6(K_{3,3,1,1})$ and the neighbor vertices of $x$ and $y$ in $\gamma$ consist of only black or only white vertices.

Note that any element in $\Gamma_8(K_{3,3,1,1})$ is of Type A, B or C, and any element in $\Gamma_6(K_{3,3,1,1})$ containing $x$ and $y$ is of Type A, B, C or D.

On the other hand, let $\lambda$ be an element in $\Gamma_{4,4}^2(K_{3,3,1,1})$. We assign types to $\lambda$ as follows:

- $\lambda$ is of Type A if $\lambda$ does not contain the edge $xy$ and both $x$ and $y$ are contained in either connected component of $\lambda$.
- $\lambda$ is of Type B if $x$ and $y$ are contained in different connected components of $\lambda$.
- $\lambda$ is of Type C if $\lambda$ contains the edge $xy$.

Note that any element in $\Gamma_{4,4}^2(K_{3,3,1,1})$ is of Type A, B or C.

Then the following three lemmas hold.
Lemma 2.4. For any element \( f \) in \( SE(K_{3,3,1,1}) \),
\[
\begin{align*}
(2.4) \quad & \sum_{\lambda \in \Gamma_3^{(2)}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 + 2 \sum_{\lambda \in \Gamma_4^{(2)}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 \\
&= 4 \sum_{\gamma \in \Gamma_{6}(K_{3,3,1,1})} a_2(f(\gamma)) - 4 \left\{ \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_7(G_y)} a_2(f(\gamma)) \right\} \\
&+ 8 \sum_{\gamma \in \Gamma_{6}(K_{3,3,1,1})} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_{6}(K_{3,3,1,1})} a_2(f(\gamma)) \\
&- 4 \left\{ \sum_{\gamma \in \Gamma_5(G_x)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_5(G_y)} a_2(f(\gamma)) \right\} + 10.
\end{align*}
\]

Proof. For \( i = 1, 3, 5 \) and \( j = 2, 4, 6 \), let us consider subgraphs \( F^{(ij)}_x = (G_x - \overline{ij}) \cup \overline{iy} \cup \overline{xy} \) and \( F^{(ij)}_y = (G_y - \overline{ij}) \cup \overline{ix} \cup \overline{xy} \) of \( K_{3,3,1,1} \) as illustrated in Figure 2.2(1) and (2), respectively. Since each of \( F^{(ij)}_x \) and \( F^{(ij)}_y \) is homeomorphic to \( P_7 \), by applying Theorem 2.2(2) to \( f|_{F^{(ij)}_x} \) and \( f|_{F^{(ij)}_y} \), it follows that
\[
\begin{align*}
(2.5) \quad & \sum_{\lambda = \gamma \cup \gamma' \in \Gamma_3^{(2)}(F^{(ij)}_x)} \text{lk}(f(\lambda))^2 + \sum_{\lambda = \gamma \cup \gamma' \in \Gamma_4^{(2)}(F^{(ij)}_x)} \text{lk}(f(\lambda))^2 \\
&+ \sum_{\lambda = \gamma \cup \gamma' \in \Gamma_3^{(2)}(G_x)} \text{lk}(f(\lambda))^2 \\
&= 2 \left\{ \sum_{\gamma \in \Gamma_{5}(F^{(ij)}_x)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) \right\} \\
&- 4 \left\{ \sum_{\gamma \in \Gamma_7(F^{(ij)}_x)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) \right\} \\
&- 2 \left\{ \sum_{\gamma \in \Gamma_6(F^{(ij)}_x)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_5(G_x)} a_2(f(\gamma)) \right\} + 1.
\end{align*}
\]
Let us take the sum of both sides of (2.5) over $i = 1, 3, 5$ and $j = 2, 4, 6$. For an element $\gamma$ in $\Gamma_8(K_{3,3,1,1})$ of Type A, there uniquely exists $F_x^{(ij)}$ containing $\gamma$. This implies that

\[
\sum_{i,j} \left( \sum_{\gamma \in \Gamma_8(F_x^{(ij)})} a_2(f(\gamma)) \right) = \sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)).
\]

For an element $\gamma$ of $\Gamma_7(G_x)$, there exist exactly four edges of $K_{3,3}$ which are not contained in $\gamma$. Thus $\gamma$ is common for exactly four $F_x^{(ij)}$'s. This implies that

\[
\sum_{i,j} \left( \sum_{\gamma \in \Gamma_7(G_x) \setminus \gamma} a_2(f(\gamma)) \right) = 4 \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)).
\]

For an element $\gamma$ in $\Gamma_7(G_y)$, there uniquely exists $F_x^{(ij)}$ containing $\gamma$. This implies that

\[
\sum_{i,j} \left( \sum_{\gamma \in \Gamma_7(F_x^{(ij)})} a_2(f(\gamma)) \right) = \sum_{\gamma \in \Gamma_7(G_y)} a_2(f(\gamma)).
\]

For an element $\gamma$ in $\Gamma_6(K_{3,3})$, there exist exactly three edges of $K_{3,3}$ which are not contained in $\gamma$. Thus $\gamma$ is common for exactly three $F_x^{(ij)}$'s. This
implies that

\[(2.9) \quad \sum_{i,j} \left( \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) \right) = 3 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)). \]

For an element \(\gamma\) in \(\Gamma_6(K_{3,3,1,1})\) containing \(x\) and \(y\), if \(\gamma\) is of Type A, then there uniquely exists \(F^{(ij)}_x\) containing \(\gamma\). This implies that

\[(2.10) \quad \sum_{i,j} \left( \sum_{\gamma \in \Gamma_6(F^{(ij)}_x)} a_2(f(\gamma)) \right) = \sum_{\gamma \in \Gamma_6(K_{3,3,1,1})} a_2(f(\gamma)). \]

For an element \(\gamma\) in \(\Gamma_5(G_x)\), there exist exactly six edges of \(K_{3,3}\) which are not contained in \(\gamma\). Thus \(\gamma\) is common for exactly six \(F^{(ij)}_x\)'s. This implies that

\[(2.11) \quad \sum_{i,j} \left( \sum_{\gamma \in \Gamma_5(F^{(ij)}_x)} a_2(f(\gamma)) \right) = 6 \sum_{\gamma \in \Gamma_5(G_x)} a_2(f(\gamma)). \]

For an element \(\lambda = \gamma \cup \gamma'\) in \(\Gamma_{3,5}^{(2)}(K_{3,3,1,1})\) where \(\gamma\) is a 3-cycle and \(\gamma'\) is a 5-cycle, if \(\gamma\) contains \(x\) and \(\gamma'\) contains \(y\), then there uniquely exists \(F^{(ij)}_x\) containing \(\lambda\). This implies that

\[(2.12) \quad \sum_{i,j} \left( \sum_{\lambda = \gamma \cup \gamma' \in \Gamma_{3,5}^{(2)}(F^{(ij)}_x)} \frac{\text{lk}(f(\lambda))^2}{6} \right) = \sum_{\lambda = \gamma \cup \gamma' \in \Gamma_{3,5}^{(2)}(K_{3,3,1,1})} \frac{\text{lk}(f(\lambda))^2}{6}, \quad \gamma \in \Gamma_3(K_{3,3,1,1}), \gamma' \in \Gamma_5(K_{3,3,1,1}), \quad x \in \gamma, \quad y \in \gamma'. \]
For an element \( \lambda \) in \( \Gamma^{(2)}_{4,4}(K_{3,3,1,1}) \) of Type A, there uniquely exists \( F_{x}^{(ij)} \) containing \( \lambda \). This implies that

\[
\sum_{i,j} \left( \sum_{\lambda=\gamma\cup\gamma'\in F_{x}^{(ij)}} \text{lk}(f(\lambda))^2 \right) = \sum_{\lambda\in \Gamma^{(2)}_{4,4}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2.
\]

(2.13)

For an element \( \lambda \) in \( \Gamma^{(2)}_{3,4}(G_x) \), there exist exactly four edges of \( K_{3,3} \) which are not contained in \( \lambda \). Thus \( \lambda \) is common for exactly four \( F_{x}^{(ij)} \)'s. This implies that

\[
\sum_{i,j} \left( \sum_{\lambda=\gamma\cup\gamma'\in \Gamma^{(2)}_{3,4}(G_x)} \text{lk}(f(\lambda))^2 \right) = 4 \sum_{\lambda\in \Gamma^{(2)}_{3,4}(G_x)} \text{lk}(f(\lambda))^2.
\]

(2.14)

Thus by (2.5), (2.6), (2.7), (2.8), (2.9), (2.10), (2.11), (2.12), (2.13) and (2.14), we have

\[
\sum_{\lambda=\gamma\cup\gamma'\in \Gamma^{(2)}_{4,4}(G_x)} \text{lk}(f(\lambda))^2 + \sum_{\lambda\in \Gamma^{(2)}_{4,4}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2
\]

\[
+ 4 \sum_{\lambda\in \Gamma^{(2)}_{3,4}(G_x)} \text{lk}(f(\lambda))^2
\]

\[
= 2 \sum_{\gamma\in \Gamma_{3,3,1,1}(G_x)} a_2(f(\gamma)) + 8 \sum_{\gamma\in \Gamma_3(G_x)} a_2(f(\gamma))
\]

\[
- 4 \sum_{\gamma\in \Gamma_3(G_x)} a_2(f(\gamma)) - 12 \sum_{\gamma\in \Gamma_6(K_{3,3})} a_2(f(\gamma))
\]

\[
- 2 \sum_{\gamma\in \Gamma_6(K_{3,3,1,1})} a_2(f(\gamma)) + 9.
\]

(2.15)

Then by combining (2.15) and (2.2), we have
(2.16) \[ \sum_{\lambda=\gamma \cup \gamma' \in \Gamma_4^{(2)}(K_{3,3,1,1})} \operatorname{lk}(f(\lambda))^2 + \sum_{\lambda' \in \Gamma_4^{(2)}(K_{3,3,1,1})} \operatorname{lk}(f(\lambda))^2 \]

\[ = 2 \sum_{\gamma \in \Gamma_6(K_{3,3,1,1})} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_6(G_\gamma)} a_2(f(\gamma)) + 4 \sum_{\gamma \in \Gamma_6(K_{3,3,1})} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) + 5. \]

By applying Theorem 2.2(2) to \( f|_{F^{(ij)}_x} \) and combining the same argument as in the case of \( F^{(ij)}_x \) with (2.3), we also have

(2.17) \[ \sum_{\lambda=\gamma \cup \gamma' \in \Gamma_4^{(2)}(K_{3,3,1,1})} \operatorname{lk}(f(\lambda))^2 + \sum_{\lambda' \in \Gamma_4^{(2)}(K_{3,3,1,1})} \operatorname{lk}(f(\lambda))^2 \]

\[ = 2 \sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_7(G_\gamma)} a_2(f(\gamma)) + 4 \sum_{\gamma \in \Gamma_6(K_{3,3,1})} a_2(f(\gamma)) - 4 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) + 5. \]

Then by adding (2.16) and (2.17), we have the result. \( \square \)

Lemma 2.5. For any element \( f \) in \( \text{SE}(K_{3,3,1,1}) \),

(2.18) \[ \sum_{\lambda \in \Gamma_4^{(2)}(K_{3,3,1,1})} \operatorname{lk}(f(\lambda))^2 \]

\[ = 2 \sum_{\gamma \in \Gamma_6(K_{3,3,1,1})} a_2(f(\gamma)) + 4 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) - 2 \left\{ \sum_{\gamma \in \Gamma_6(G_\gamma)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_5(G_\gamma)} a_2(f(\gamma)) \right\} - 2 \sum_{\gamma \in \Gamma_6(K_{3,3,1,1})} a_2(f(\gamma)) + 2. \]
Proof. Let us consider subgraphs $Q_8^{(1)} = K_{3,3} \cup x \overline{x} \cup y \overline{y} \cup \overline{y} \overline{y} \cup y \overline{y}$ and $Q_8^{(2)} = K_{3,3} \cup x 4 \cup \overline{x} \overline{x} \cup y \overline{y} \cup \overline{y} 6$ of $K_{3,3,1,1}$ as illustrated in Figure 2.3(1) and (2), respectively. Since each of $Q_8^{(1)}$ and $Q_8^{(2)}$ is homeomorphic to $Q_8$, by applying Theorem 2.2(1) to $f_{Q_8^{(1)}}$ and $f_{Q_8^{(2)}}$, it follows that

$$\sum_{\lambda \in \Gamma_{4,4}(Q_8^{(i)})} \operatorname{lk}(f(\lambda))^2 = 2 \sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) + 2 \sum_{\gamma \in \Gamma_6(K_{3,3,1,1})} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_6(Q_8^{(i)})} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_6(Q_8^{(i)})} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_6(Q_8^{(i)})} a_2(f(\gamma)) + 1$$

for $i = 1, 2$. By adding (2.19) for $i = 1, 2$, we have the result. \hfill \box

![Figure 2.3. (1) $Q_8^{(1)}$, (2) $Q_8^{(2)}$](image)

**Lemma 2.6.** For any element $f$ in $\text{SE}(K_{3,3,1,1})$,

$$\sum_{\lambda \in \Gamma_{4,4}(K_{3,3,1,1})} \operatorname{lk}(f(\lambda))^2 = 2 \sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) - 8 \sum_{\gamma \in \Gamma_6(K_{3,3,1,1})} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_6(K_{3,3,1,1})} a_2(f(\gamma)) + 2.$$

**Proof.** For $k = 1, 2, \ldots, 6$, let us consider subgraphs

$$F_x^{(k)} = (G_x - xk) \cup xy \cup ky,$$
of $K_{3,3,1,1}$ as illustrated in Figure 2.4(1) and (2), respectively. Since each of $F_x^{(k)}$ and $F_y^{(k)}$ is also homeomorphic to $P_7$, by applying Theorem 2.2(2) to $f|_{F_x^{(k)}}$, it follows that

\[
\sum_{\lambda=\gamma\cup\gamma'\in\Gamma_{8}(K_{3,3,1,1})} \operatorname{lk}(f(\lambda))^2 + \sum_{\lambda\in\Gamma_{8}(G_x)} a_2(f(\lambda)) = \sum_{\gamma\in\Gamma_{8}(K_{3,3,1,1})^C} a_2(f(\gamma)).
\]
For an element $\gamma$ of $\Gamma_7(G_x)$, there exist exactly four edges which are incident to $x$ such that they are not contained in $\gamma$. Thus $\gamma$ is common for exactly four $F_x^{(k)}$'s. This implies that

$$\sum_k \left( \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) \right) = 4 \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)).$$

(2.23)

It is clear that any element $\gamma$ in $\Gamma_6(K_{3,3})$ is common for exactly six $F_x^{(k)}$'s. This implies that

$$\sum_k \left( \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) \right) = 6 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)).$$

(2.24)

For an element $\gamma$ in $\Gamma_6(K_{3,3,1,1})$ containing $x$ and $y$, if $\gamma$ is of Type C, then there uniquely exists $F_x^{(k)}$ containing $\gamma$. This implies that

$$\sum_k \left( \sum_{\gamma \in \Gamma_6(F_x^{(k)})} a_2(f(\gamma)) \right) = \sum_{\gamma \in \Gamma_6(K_{3,3,1,1}), x,y \in \gamma, \text{Type C}} a_2(f(\gamma)).$$

(2.25)

For an element $\gamma$ of $\Gamma_5(G_x)$, there exist exactly four edges which are incident to $x$ such that they are not contained in $\gamma$. Thus $\gamma$ is common for exactly four $F_x^{(k)}$'s. This implies that

$$\sum_k \left( \sum_{\gamma \in \Gamma_5(G_x)} a_2(f(\gamma)) \right) = 4 \sum_{\gamma \in \Gamma_5(G_x)} a_2(f(\gamma)).$$

(2.26)

For an element $\lambda = \gamma \cup \gamma'$ in $\Gamma_4^{(2)}(K_{3,3,1,1})$, if $\lambda$ is of Type C, then there uniquely exists $F_x^{(k)}$ containing $\lambda$. This implies that

$$\sum_k \left( \sum_{\lambda \in \Gamma_4^{(2)}(F_x^{(k)})} \text{lk}(f(\lambda))^2 \right) = \sum_{\lambda \in \Gamma_4^{(2)}(K_{3,3,1,1}), \text{Type C}} \text{lk}(f(\lambda))^2.$$
exactly four $F^{(k)}_x$'s. This implies that

\[
(2.28) \quad \sum_k \left( \sum_{\lambda \in \Gamma^{(2)}_{3,3,1,1}(G_x)} \text{lk}(f(\lambda))^2 \right) = 4 \sum_{\lambda \in \Gamma^{(2)}_{3,4}(G_x)} \text{lk}(f(\lambda))^2.
\]

Then by (2.21), (2.22), (2.23), (2.24), (2.25), (2.26), (2.27) and (2.28), we have

\[
(2.29) \quad \sum_{\lambda \in \Gamma^{(2)}_{4,4}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 + 4 \sum_{\lambda \in \Gamma^{(2)}_{3,4}(G_x)} \text{lk}(f(\lambda))^2
\]

\[
= 2 \sum_{\gamma \in \Gamma_6(K_{3,3,1,1})} a_2(f(\gamma)) + 8 \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) - 24 \sum_{\gamma \in \Gamma_6(K_{3,3})} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_5(G_x)} a_2(f(\gamma)) - 8 \sum_{x,y \in \gamma, \text{ Type C}} a_2(f(\gamma)) + 6.
\]

Then by combining (2.29) and (2.2), we have the result. We remark here that by applying Theorem 2.2 (2) to $f|_{F^{(k)}_x}$ combining the same argument as in the case of $F^{(k)}_x$ with (2.3), we also have (2.20).

\[\square\]

**Proof of Theorem 1.4.** (1) Let $f$ be an element in $\text{SE}(K_{3,3,1,1})$. Then by combining (2.4), (2.18) and (2.20), we have

\[
(2.30) \quad \sum_{\lambda \in \Gamma^{(2)}_{3,5}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 + 2 \sum_{\lambda \in \Gamma^{(2)}_{3,3,1,1}} \text{lk}(f(\lambda))^2
\]

\[
= 4 \sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) - 4 \left\{ \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_7(G_y)} a_2(f(\gamma)) \right\}
\]

\[
- 4 \left\{ \sum_{\gamma \in \Gamma_6(G_x)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_6(G_y)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_6(K_{3,3,1,1})} a_2(f(\gamma)) \right\}
\]

\[
- 4 \left\{ \sum_{\gamma \in \Gamma_5(G_x)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_5(G_y)} a_2(f(\gamma)) \right\} + 18.
\]
Note that
\[ \Gamma_k(G_x) \cup \Gamma_k(G_y) = \{ \gamma \in \Gamma_k(K_{3,3,1,1}) \mid \{x,y\} \not\subseteq \gamma \} \]
for \( k = 5, 7 \). Moreover, we define a subset \( \Gamma'_6 \) of \( \Gamma_6(K_{3,3,1,1}) \) by
\[ (2.31) \quad \Gamma'_6 = \{ \gamma \in \Gamma_6(G_x) \mid x \in \gamma \} \cup \{ \gamma \in \Gamma_6(G_y) \mid y \in \gamma \} \]
\[ \cup \{ \gamma \in \Gamma_6(K_{3,3,1,1}) \mid x, y \in \gamma, \ \gamma \text{ is of Type A, B or C} \} . \]
Then we see that (2.30) implies (1.3).

(2) Let \( f \) be an element in \( \text{SE}(K_{3,3,1,1}) \). Let us consider subgraphs \( H_1 = Q_8^{(1)} \cup \overline{xy} \) and \( H_2 = Q_8^{(2)} \cup \overline{xy} \) of \( K_{3,3,1,1} \) as illustrated in Figure 2.5(1) and (2), respectively. For \( i = 1, 2, \) \( H_i \) has the proper minor \( H'_i = H_i/\overline{xy} \) which is isomorphic to \( P_7 \). For a spatial embedding \( \lambda \) of \( H_i \), there exists a spatial embedding \( f' \) of \( H'_i \) such that \( f'(H'_i) \) is obtained from \( f(H_i) \) by contracting \( f(\overline{xy}) \) into one point. Note that this embedding is unique up to ambient isotopy in \( \mathbb{R}^3 \). Then by Corollary 2.3, there exists an element \( \mu'_i \) in \( \Gamma^{(2)}_{3,4}(H'_i) \) such that \( \text{lk}(f'(\mu'_i)) \equiv 1 \pmod{2} \) \( (i = 1, 2) \). Note that \( \mu'_i \) is mapped onto an element \( \mu_i \) in \( \Gamma_{4,4}(H_i) \) by the natural injection from \( \Gamma_{3,4}(H'_i) \) to \( \Gamma_{4,4}(H_i) \). Since \( f'(\mu'_i) \) is ambient isotopic to \( f(\mu_i) \), we have \( \text{lk}(f(\mu_i)) \equiv 1 \pmod{2} \) \( (i = 1, 2) \). We also note that both \( \mu_1 \) and \( \mu_2 \) are of Type C in \( \Gamma^{(2)}_{4,4}(K_{3,3,1,1}) \).

\[ \text{Figure 2.5. (1) } H_1, \ (2) \ H_2 \]

For \( v = x, y \) and \( i, j, k = 1, 2, \ldots, 6 \) \( (i \neq j) \), let \( P^{(k)}_8(v;ij) \) be the subgraph of \( K_{3,3,1,1} \) as illustrated in Figure 2.6(1) if \( v = y, \ k \in \{1, 3, 5\} \) and \( i, j \in \{2, 4, 6\} \), (2) if \( v = y, \ k \in \{2, 4, 6\} \) and \( i, j \in \{1, 3, 5\} \), (3) if \( v = x, \ k \in \{1, 3, 5\} \) and \( i, j \in \{2, 4, 6\} \) and (4) if \( v = x, \ k \in \{2, 4, 6\} \) and \( i, j \in \{1, 3, 5\} \). Note that there exist exactly thirty six \( P^{(k)}_8(v;ij) \)'s and they are isomorphic to \( P_8 \) in the \( K_9 \)-family. Thus by Corollary 2.3, there exists an element \( \lambda \) in \( \Gamma^{(2)}(P^{(k)}_8(v;ij)) \) such that \( \text{lk}(f(\lambda)) \equiv 1 \pmod{2} \). All elements in \( \Gamma^{(2)}(P^{(k)}_8(v;ij)) \) consist of exactly four elements in \( \Gamma^{(2)}_{3,8}(P^{(k)}_8(v;ij)) \).
and exactly four elements in $\Gamma_{4,4}^{(2)}(P_8^{(k)}(v;ij))$ of Type A or Type B because they do not contain the edge $xy$. It is not hard to see that any element in $\Gamma_{3,5}^{(2)}(K_{3,3,1,1})$ is common for exactly two $P_8^{(k)}(v;ij)$’s, and any element in $\Gamma_{4,4}^{(2)}(K_{3,3,1,1})$ of Type A or Type B is common for exactly four $P_8^{(k)}(v;ij)$’s.

By (2.4), there exist a nonnegative integer $m$ such that
\[ \sum_{\lambda \in \Gamma_{3,5}^{(2)}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 = 2m. \]

If $2m \geq 18$, since there exist at least two elements $\mu_1$ and $\mu_2$ in $\Gamma_{3,5}^{(2)}(K_{3,3,1,1})$ of Type C such that $\text{lk}(f(\mu_i)) \equiv 1 \pmod{2}$ $(i = 1, 2)$, we have
\[ \sum_{\lambda \in \Gamma_{3,5}^{(2)}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 + 2 \sum_{\lambda \in \Gamma_{4,4}^{(2)}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 \geq 18 + 4 = 22. \]

If $2m \leq 16$, then there exist at least $(36 - 4m)/4 = 9 - m$ elements in $\Gamma_{4,4}^{(2)}(K_{3,3,1,1})$ of Type A or Type B such that each of the corresponding 2-component links with respect to $f$ has an odd linking number. Then we have
\[ \sum_{\lambda \in \Gamma_{3,5}^{(2)}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 + 2 \sum_{\lambda \in \Gamma_{4,4}^{(2)}(K_{3,3,1,1})} \text{lk}(f(\lambda))^2 \geq 2m + 2 \{(9 - m) + 2\} = 22. \quad \square \]

3. $\triangle$Y-exchange and Conway–Gordon type formulae

In this section, we give a proof of Theorem 1.8. Let $G_\triangle$ and $G_Y$ be two graphs such that $G_Y$ is obtained from $G_\triangle$ by a single $\triangle$Y-exchange. Let $\gamma'$ be an element in $\Gamma(G_\triangle)$ which does not contain $\triangle$. Then there exists an
element $\Phi(\gamma')$ in $\Gamma(G_Y)$ such that $\gamma' \setminus \triangle = \Phi(\gamma') \setminus Y$. It is easy to see that the correspondence from $\gamma'$ to $\Phi(\gamma')$ defines a surjective map

$$\Phi : \Gamma(G_{\triangle}) \setminus \{\triangle\} \rightarrow \Gamma(G_Y).$$

The inverse image of an element $\gamma$ in $\Gamma(G_Y)$ by $\Phi$ contains at most two elements in $\Gamma(G_{\triangle}) \setminus \Gamma_{\triangle}(G_{\triangle})$. Figure 3.1 illustrates the case that the inverse image of $\gamma$ by $\Phi$ consists of exactly two elements. Let $\omega$ be a map from $\Gamma(G_{\triangle})$ to $\mathbb{Z}$. Then we define the map $\tilde{\omega}$ from $\Gamma(G_Y)$ to $\mathbb{Z}$ by

$$\tilde{\omega}(\gamma) = \sum_{\gamma' \in \Phi^{-1}(\gamma)} \omega(\gamma')$$

for an element $\gamma$ in $\Gamma(G_Y)$.

Let $f$ be an element in $\text{SE}(G_Y)$ and $D$ a 2-disk in $\mathbb{R}^3$ such that $D \cap f(G_Y) = f(Y)$ and $\partial D \cap f(G_Y) = \{f(u), f(v), f(w)\}$. Let $\varphi(f)$ be an element in $\text{SE}(G_{\triangle})$ such that $\varphi(f)(x) = f(x)$ for $x \in G_{\triangle} \setminus \triangle = G_Y \setminus Y$ and $\varphi(f)(G_{\triangle}) = (f(G_Y) \setminus f(Y)) \cup \partial D$. Thus we obtain a map

$$\varphi : \text{SE}(G_Y) \rightarrow \text{SE}(G_{\triangle}).$$

Then we immediately have the following.

**Proposition 3.1.** Let $f$ be an element in $\text{SE}(G_Y)$ and $\gamma$ an element in $\Gamma(G_Y)$. Then, $f(\gamma)$ is ambient isotopic to $\varphi(f)(\gamma')$ for each element $\gamma'$ in the inverse image of $\gamma$ by $\Phi$.

Then we have the following lemma which plays a key role to prove Theorem 1.8. This lemma has already been shown in [NT, Lemma 2.2] in more general form, but we give a proof for the reader’s convenience.
Lemma 3.2 (Nikkuni–Taniyama [NT]). For an element $f$ in $SE(G_Y)$,
\[
\sum_{\gamma \in \Gamma(G_Y)} \tilde{\omega}(\gamma) a_2(f(\gamma)) = \sum_{\gamma' \in \Gamma(G_\Delta)} \omega(\gamma') a_2(\varphi(f)(\gamma')).
\]

Proof. Since $\varphi(f)(\Delta)$ is the trivial knot, we have
\[
\sum_{\gamma' \in \Gamma(G_\Delta)} \omega(\gamma') a_2(\varphi(f)(\gamma')) = \sum_{\gamma' \in \Gamma(G_\Delta) \setminus \{\Delta\}} \omega(\gamma') a_2(\varphi(f)(\gamma')).
\]
Note that
\[
\Gamma(G_\Delta) \setminus \{\Delta\} = \bigcup_{\gamma \in \Gamma(G_Y)} \Phi^{-1}(\gamma).
\]
Then, by Proposition 3.1, we see that
\[
\sum_{\gamma' \in \Gamma(G_\Delta) \setminus \{\Delta\}} \omega(\gamma') a_2(\varphi(f)(\gamma')) = \sum_{\gamma \in \Gamma(G_Y)} \left( \sum_{\gamma' \in \Phi^{-1}(\gamma)} \omega(\gamma') a_2(\varphi(f)(\gamma'))\right) = \sum_{\gamma \in \Gamma(G_Y)} \left( \sum_{\gamma' \in \Phi^{-1}(\gamma)} \omega(\gamma') a_2(f(\gamma'))\right) = \sum_{\gamma \in \Gamma(G_Y)} \tilde{\omega}(\gamma) a_2(f(\gamma)). \quad \square
\]

Proof of Theorem 1.8. By Corollary 1.5, there exists a map
\[
\omega : \Gamma(K_{3,3,1,1}) \to \mathbb{Z}
\]
such that for any element $g$ in $SE(K_{3,3,1,1})$,
\[(3.2) \quad \sum_{\gamma' \in \Gamma(K_{3,3,1,1})} \omega(\gamma') a_2(g(\gamma')) \geq 1.
\]
Let $G$ be a graph which is obtained from $K_{3,3,1,1}$ by a single $\Delta Y$-exchange and $\tilde{\omega}$ the map from $\Gamma(G)$ to $\mathbb{Z}$ as in (3.1). Let $f$ be an element in $SE(G)$. Then by Lemma 3.2 and (3.2), we see that
\[
\sum_{\gamma \in \Gamma(G)} \tilde{\omega}(\gamma) a_2(f(\gamma)) = \sum_{\gamma' \in \Gamma(K_{3,3,1,1})} \omega(\gamma') a_2(\varphi(f)(\gamma')) \geq 1.
\]
By repeating this argument, we have the result. \quad \square

Remark 3.3. In Theorem 1.8, the proof of the existence of a map $\omega$ is constructive. It is also an interesting problem to give $\omega(\gamma)$ for each element $\gamma$ in $\Gamma(G)$ concretely.
4. Rectilinear spatial embeddings of $K_{3,3,1,1}$

In this section, we give a proof of Theorem 1.9. For an element $f$ in $\text{RSE}(G)$ and an element $\gamma$ in $\Gamma_k(G)$, the knot $f(\gamma)$ has a stick number less than or equal to $k$, where the stick number $s(K)$ of a knot $K$ is the minimum number of edges in a polygon which represents $K$. Then the following is well known.

**Proposition 4.1** (Adams [A], Negami [Ne]). For any nontrivial knot $K$, it follows that $s(K) \geq 6$. Moreover, $s(K) = 6$ if and only if $K$ is a trefoil knot.

We also show a lemma for a rectilinear spatial embedding of $P_7$ which is useful in proving Theorem 1.9.

**Lemma 4.2.** For an element $f$ in $\text{RSE}(P_7)$,

$$\sum_{\gamma \in \Gamma_7(P_7)} a_2(f(\gamma)) \geq 0.$$

**Proof.** Note that $a_2(\text{trivial knot}) = 0$ and $a_2(\text{trefoil knot}) = 1$. Thus by Proposition 4.1, $a_2(f(\gamma)) = 0$ for any element $\gamma$ in $\Gamma_5(P_7)$ and $a_2(f(\gamma)) \geq 0$ for any element $\gamma$ in $\Gamma_6(P_7)$. Moreover, by Corollary 2.3, we have

$$\sum_{\lambda \in \Gamma_5(2)(P_7)} \text{lk}(f(\lambda))^2 \geq 1.$$

Then Theorem 2.2(2) implies the result. □

**Proof of Theorem 1.9.** Let $f$ be an element in $\text{RSE}(K_{3,3,1,1})$. Since $G_x$ and $G_y$ are isomorphic to $P_7$, by Lemma 4.2, we have

$$\sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) \geq 0, \sum_{\gamma \in \Gamma_7(G_y)} a_2(f(\gamma)) \geq 0.$$

Then by Corollary 1.5 and (4.2), we have

$$\sum_{\gamma \in \Gamma_8(K_{3,3,1,1})} a_2(f(\gamma)) \geq \sum_{\gamma \in \Gamma_7(G_x)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_7(G_y)} a_2(f(\gamma)) + \sum_{\gamma \in \Gamma_6} a_2(f(\gamma)) + 1$$

$$\geq 0 + 0 + 0 + 1 = 1.$$ □

**Remark 4.3.** All of knots with $s \leq 8$ and $a_2 > 0$ are $3_1$, $5_1$, $5_2$, $6_3$, a square knot, a granny knot, $8_{19}$ and $8_{20}$ (Calvo [C]). Therefore, Theorem 1.9 implies that at least one of them appears in the image of every rectilinear spatial embedding of $K_{3,3,1,1}$. On the other hand, it is known that the image of every rectilinear spatial embedding of $K_7$ contains a trefoil knot (Brown [B]).
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It is still open whether the image of every rectilinear spatial embedding of $K_{3,3,1,1}$ contains a trefoil knot.

References


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