A modular description of $\mathcal{E}R(2)$

Romie Banerjee

Abstract. We give a description of the maximally unramified extension of completed second Real Johnson–Wilson theory using supersingular elliptic curves with $\Gamma_0(3)$-level structures.

Contents

1. Introduction 743
2. Setup 745
3. Real Johnson–Wilson theory from modular curves 752
References 757

1. Introduction

1.1. Real Johnson–Wilson theories. Complex conjugation acts on the complex $K$-theory spectrum $KU$ and the homotopy fixed points of this action is $KO$. In fact the complex orientation $MU \to KU$ is equivariant with respect to this $C_2$-action. Localizing at the prime 2, there is a $C_2$-equivariant orientation

$$BP \to KU(2)$$

with kernel $\langle v_i, i > 1 \rangle$. Hu and Kriz ([7],[8]) have realized this as a map of honest $C_2$-equivariant spectra

$$BP\mathbb{R} \to K\mathbb{R}$$

where $K\mathbb{R}$ is Atiyah’s real $K$-theory ([1]) localized at the prime 2. They have further generalized this to a map of equivariant spectra

$$BP\mathbb{R} \to \mathcal{E}R(n)$$

which is an equivariant refinement of the orientation $BP \to E(n)$ with kernel $\langle v_i, i > n \rangle$. Here $E(n)$ is the $2(2^n - 1)$-periodic Johnson–Wilson theory. It has coefficients $\mathbb{Z}(2)[v_1, \ldots, v_n^\pm 1]$ with $|v_i| = 2(2^i - 1)$.

The underlying nonequivariant spectrum of $\mathcal{E}R(n)$ is $E(n)$ and the homotopy fixed points with respect to the complex conjugation action is denoted by $\mathcal{E}R(n)$. This is $2^{n+2}(2^n - 1)$ periodic. The spectrum $\mathcal{E}R(1)$ is $KO(2)$. 

Received January 27, 2014.
2010 Mathematics Subject Classification. 55N20, 55N91, 14K10, 14H52.
Key words and phrases. Real Johnson–Wilson theories, Lubin–Tate spaces, topological modular forms, algebraic stacks.
ROMIE BANERJEE

Kitchloo and Wilson have done extensive computations with the spectrum $ER(2)$ ([11], [12]).

The spectrum $ER(n)$ is homotopy commutative, but its completion is an $E_\infty$-ring spectrum. In fact Averett ([2]) has shown that after completion the fixed points inclusion map $ER(n) \to E(n)$ is a higher chromatic generalization of the $C_2$-Galois extension (in the sense of [16]) $KO \to KU$.

More elaborately, let $E_n$ be the $n$-th Morava $E$-theory spectrum associated to the Lubin–Tate space of deformations of the Honda formal group $\Gamma_n$ over $\mathbb{F}_{2^n}$. This Lubin–Tate space is noncanonically isomorphic to $\text{Spf} \pi_0(E_n) = \text{Spf} W(\mathbb{F}_{2^n})[[u_1, \ldots, u_{n-1}]]$ where $W(k)$ denotes the Witt ring of $k$. Hopkins–Miller theory gives a unique $E_\infty$-ring structure on $E_n$, with coefficients

$$\pi_*E_n = \pi_0(E_n)[u^{\pm 1}]$$

where $|u| = 2$. The $n$-th Morava stabilizer group $S_n = \mathcal{O}^\times_{D_1, \mathbb{Q}_2}$ (the group of units in the maximal order of the division algebra over $\mathbb{Q}_2$ with Hasse invariant $\frac{1}{n}$) and the Galois group $\text{Gal} = \text{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2)$ act on the Lubin–Tate space. By Hopkins–Miller theory this lifts to an action of the extended stabilizer group $S_n \rtimes \text{Gal}$ on the spectrum $E_n$ through $E_\infty$-ring maps, up to contractible choices.

There is a subgroup $H(n) = \mathbb{F}_2^\times \rtimes \text{Gal}$ of the extended stabilizer and a central element $[-1]_{\Gamma_n} \in S_n$ corresponding to the formal inverse of the Honda formal group law. Averett shows that the standard equivalence $L_{K(n)}E(n) \simeq E_n^{hH(n)}$ is equivariant with respect to the $C_2$ action coming from $[-1]_{\Gamma_n}$. In other words there is an equivalence

$$L_{K(n)}ER(n) \simeq E_n^{h(C_2 \times H(n))}.$$  

1.2. Main result. In this paper we want to show that the Real Johnson–Wilson theory $ER(2)$ arises from certain modular curves in the same way the spectrum $TMF$ of topological modular forms arises from the moduli stack of smooth elliptic curves [6].

The stabilizer group $S_2$ has the maximal finite subgroup $\hat{A}_4 = Q_8 \rtimes C_3$ where $C_3$ acts on $Q_8$ by conjugation, and $H(2) = C_3 \rtimes \text{Gal} \subset \hat{A}_4 \rtimes \text{Gal}$.

Completion of $TMF$ at the chromatic height 2 admits the description (for the primes 2 and 3)

$$L_{K(2)}TMF = \left( \prod_{x \in \mathcal{E}ll^{ss}(\mathbb{F}_p)} E_2^{h\text{Aut}(x)} \right)^{h\text{Gal}(\mathbb{F}_p/\mathbb{F}_p)}.$$  

Here $\mathcal{E}ll^{ss}$ is the locus of supersingular elliptic curves at the prime $p$ and $E_2$ is the Hopkins–Miller spectrum associated to the Lubin–Tate space of deformations of the formal group law of $x$ over $\mathbb{F}_p$. Since there is a unique isomorphism class of elliptic curves $x$ at these primes and $\text{Aut}(x)$ is the
maximal finite subgroup of $\mathbb{S}_2$, the right hand side is the higher real $K$-
theory $EO^h_{\text{Gal}(\overline{\mathbb{F}_2}\mathbb{F}_2)}$.

We want to describe completed $ER(2)$ in terms of modular curves. This
raises the question: Given $p = 2$, does there exist an étale cover of $X$ of $\text{Ell}^{ss}$
so that there exists $x \in X$ for which $\text{Aut}(x)$ is the subgroup $C_2 \times C_3$ of $S_2$?

We state the main theorem here.

**Theorem 1.1.** For a $p$-local commutative $S$-algebra $A$, let $A(\mu_{\infty,p})$ denote
the $S$-algebra obtained by adjoining all the roots of unity of order prime to $p$. Then $L_{K(2)}ER(2)(\mu_{\infty,2})$ is an algebra over $\text{TMF}$.

There is a $K(2)$-local $C_2$-Galois extension

$$L_{K(2)}TMF_1(3)(\mu_{\infty,2}) \rightarrow L_{K(2)}TMF_0(3)(\mu_{\infty,2})$$

which is isomorphic to the extension $L_{K(2)}ER(2)(\mu_{\infty,2}) \rightarrow L_{K(2)}E(2)(\mu_{\infty,2})$.

It is worth pointing out that Behrens and Hopkins ([4]) have answered in
great detail the following question: Given a maximal finite subgroup $G$ of $S_n$
at the prime $p$, is the associated higher real $K$-theory $EO_n$ a summand of
the $K(n)$-localization of a $TAF$ spectrum associated to a unitary similitude
group of type $(1,n - 1)$? This paper is concerned with a similar question
where $p = 2$, $n = 2$ and $G$ is not maximal.

It is evident from the calculations of Mahowald and Rezk ([14]) that the
spectrum $TMF_1(3)$ is an example of a generalized $E(2)$ (see [3]). Presumably
there is a notion of a generalized $ER(n)$ so that $TMF_0(3)$ is an example of
a generalized $ER(2)$.

In the next section we reformulate our main theorem (as Theorem 2.3)
in algebraic-geometric language and review the various moduli spaces ap-
pearing in the formulation. The final section contains the proof of the main
theorem. In the rest of the paper the prime $p$ is 2.

**Acknowledgements.** The author has benefited greatly from discussions
with and comments received from Jack Morava, Niko Naumann and Andrew
Salch.

**2. Setup**

**2.1. Stacks from ring spectra.** Let $BP$ denote the $p$-local Brown–Peterson
spectrum. We can consider the associated flat Hopf algebroid:

$$V = BP_* \simeq \mathbb{Z}_p[v_1, \ldots]$$

$$VT = BP_*BP \simeq BP_*[t_1, \ldots].$$

Denote the stack associated to $(V, VT)$ by $\mathcal{M}$. This is an algebraic stack
in the $fpqc$-topology ([15]). Such stacks are representable by flat Hopf algebroids.
This differs from the standard notion of an algebraic stack in algebraic geometry (see [13, 4.1]) since its diagonal is not necessarily of
finite type.
The algebraic stack $\mathcal{M}$ is closely related to the moduli stack of one-dimensional commutative formal groups. A formal group over a scheme $S$ is a commutative group object in the category of formal schemes over $S$ that is fpqc-locally isomorphic to $(\hat{A}^1, 0)$ as schemes. Let’s denote this stack by $X_{1,fg}$. Given a ring $R$, the groupoid $X_{1,fg}(R)$ comprises formal groups $G/R$ over $\text{Spec } R$ and their isomorphisms.

The stack $X_{1,fg}$ carries a canonical line bundle $\omega$. For every $R$ we can construct the locally free rank one $R$-module $\omega_{G/R}$ of invariant 1-forms of $G$ over $\text{Spec } R$, and its formation is compatible with base change and therefore defines a line bundle $\omega$ over $X_{1,fg}$.

The stack $M = M(V, VT)$ is a $\mathbb{G}_m$ torsor over $X_{1,fg} \otimes \mathbb{Z}(p)$. Its points can be described as follows. For a $\mathbb{Z}(p)$-algebra $R$, the groupoid $M(R)$ consists of pairs $(G/R, \alpha : \omega_{G/R} \cong R)$ of a formal group and a trivialization of the $\omega$ section, and isomorphisms of $G$ that respect the trivializations.

**Definition 2.1.** Let $X$ be a $p$-local homotopy commutative ring spectrum so that $BP_*X$ is a commutative ring. Define $M_X$ to be the stack associated to the Hopf algebroid

$$V(X) = BP_*X$$
$$VT(X) = VT \otimes_V BP_*X.$$  

We now make clear the relation of $M_X$ with formal groups. The unit $S \to X$ gives a map of stacks $M_X \to M$. There is an algebraic stack $M_X$ over $X_{1,fg} \otimes \mathbb{Z}(p)$ along with a line bundle $\omega_X : M_X \to B\mathbb{G}_m$, so that $M_X$ is a $\mathbb{G}_m$-torsor over $M_X$. The $R$-points of $M_X$ can then be identified with pairs $(P \in M_X(R), \alpha : \omega_X(P/R) \cong R)$ of objects of $M_X$ over $\text{Spec } R$ and trivializations of their $\omega_X$ sections.

If $X$ is a homotopy $BP$-algebra then $M_X$ is an affine scheme and is $\text{Spec } X_*$. In this case the stack $M_X$ is defined to be the one associated to the Hopf algebroid

$$V(X) \cong VT \otimes_V X_*$$
$$VT(X) \cong VT \otimes_V VT \otimes_V X_*$$

which is a representation for $\text{Spec } X_* \times_M M$.

**Lemma 2.1.** If $X$ and $Y$ are $p$-local homotopy commutative ring spectra so that $BP_*X$ and $BP_*Y$ are commutative rings and $M_Y \to M$ is flat, then the stack associated to the smash product $X \wedge Y$ can be identified with the pullback.

$$M_{X \wedge Y} \cong M_X \times_M M_Y$$

**Proof.** Suppose $E$ is a $BP$-algebra and the map $M_E \to M$ is flat. This means that $E$ is a Landweber exact cohomology theory. Given an arbitrary $X$, the pullback $\text{Spec } E_* \times_M M_X$ is then the stack associated to the Hopf algebroid

$$E_* \otimes_V VT \otimes_V V(X) \cong V(E \wedge X)$$
$$E_* \otimes_V VT \otimes_V VT \otimes_V V(X) \cong VT(E \wedge X).$$
In other words the pullback is \( \mathcal{M}_{E\wedge X} \).

Suppose \( \mathcal{M}_Y \) is flat over \( \mathcal{M} \), then \( BP\wedge Y \) and \( BP\wedge BP\wedge Y \) are Landweber exact. Therefore there are equivalences of stacks:

\[
\text{Spec } V(Y) \times_\mathcal{M} \mathcal{M}_X \simeq \mathcal{M}_{BP\wedge Y\wedge X} \simeq \text{Spec } V(X \wedge Y),
\]

\[
\text{Spec } VT(Y) \times_\mathcal{M} \mathcal{M}_X \simeq \mathcal{M}_{BP\wedge BP\wedge Y\wedge X} \simeq \text{Spec } VT(X \wedge Y).
\]

The notion of height of a formal group gives a filtration of the moduli stack \( X^{fg} \) and this canonically lifts to a filtration of the \( \mathbb{G}_m \)-torsor \( \mathcal{M} \). One can give an explicit construction of the filtration. Let \( I_n = (p, v_1, \ldots, v_{n-1}) \) denote the invariant prime ideals of \( V \). The associated substacks correspond to formal groups of height at least \( n \).

\[
\mathcal{M}^{\geq n} = \text{Spec } (V/I_n, V/I_n \otimes_V VT \otimes_V V/I_n).
\]

There is a filtration on \( \mathcal{M} \) by closed substacks

\[
\mathcal{M} = \mathcal{M}^{\geq 0} \supseteq \mathcal{M}^{\geq 1} \supseteq \ldots \supseteq \mathcal{M}^{\geq \infty}.
\]

Let \( U^n = \mathcal{M} - \mathcal{M}^{\geq n+1} \) be the open substack of \( \mathcal{M} \) complementary to \( \mathcal{M}^{\geq n+1} \). There is an ascending chain on open immersions

\[
\emptyset = U^{-1} \subseteq U^0 \subseteq U^1 \subseteq \ldots \subseteq U^\infty \subseteq \mathcal{M}.
\]

Since for \( 0 \leq n < \infty \), \( I_n \) is finitely generated, the open immersion \( U^n \subseteq \mathcal{M} \) is quasi-compact and \( U^n \) is an algebraic stack. The \( R \)-points of \( U^n \) consist of formal groups of height at most \( n \) over \( \text{Spec } R \) along with trivializations of their corresponding cotangent bundles as before. The following corollary to [15, Theorem 26] gives an explicit atlas for \( U^n \).

**Proposition 2.1.** Let \( E(n) \) be the \( n \)-th Johnson–Wilson spectrum with \( E(n)_* = \mathbb{Z}(p)[v_1, \ldots, v_n^{-1}] \). Let

\[
(V_n, VT_n) := (E(n)_*, E(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(n)_*),
\]

be the Hopf algebroid induced from \( (BP_*, BP_*BP) \) by the Landweber exact map \( BP_* \rightarrow E(n)_* \). Then the Hopf algebroid \( (V_n, VT_n) \) is flat and its associated algebraic stack is \( U^n \).

By Lemma 2.1 the pullback \( U^n \times_\mathcal{M} \mathcal{M}_X \) is the stack associated with the Hopf algebroid

\[
(\mathcal{O}_n \otimes_V V(X), VT_n \otimes_V V(X)).
\]

Let \( \mathcal{M}^n = \mathcal{M}^{\geq n[v_n^{-1}]} \) denote the height \( n \)-layer. The stack \( \mathcal{M}^n \) is contained inside \( U^n \) as a closed substack. Define the formal neighborhood of \( \mathcal{M}^n \) by taking the completion of \( U^n \) at \( \mathcal{M}^n \).

**Definition 2.2.** \( \widehat{\mathcal{M}}^n = (U^n)^\wedge_{I_n} \)

This is the stack associated to the Hopf algebroid

\[
(V_n^\wedge, V_n^\wedge \otimes_{V_n} VT_n \otimes_{V_n} V_n^\wedge)
\]

where \( V_n^\wedge = (V_n)^\wedge \).
Let $\eta : \text{Spec} \, \mathbb{F}_p^n[u^\pm 1] \to \mathcal{M}^n$ be a lift of the the Honda formal group $\Gamma_n : \text{Spec} \, \mathbb{F}_p^n \to X_{1,fg}$ of height $n$. Then $\eta$ is a presentation for $\mathcal{M}^n$ and a pro-étale $\mathbb{S}_n \rtimes \text{Gal}(\mathbb{F}_p^n/\mathbb{F}_p)$-torsor. Let $\text{Def}(\Gamma_n, \mathbb{F}_p^n)$ be the Lubin–Tate space of deformations of $\Gamma_n$. Let $\text{Def}(\Gamma_n, \mathbb{F}_p^n)[u^\pm 1] := \text{Def}(\Gamma, \mathbb{F}_p^n) \times \text{Spec} \, \mathbb{Z}[u^\pm 1]$.

There is a map

$$\text{Def}(\Gamma_n, \mathbb{F}_p^n)[u^\pm 1] \to \hat{\mathcal{M}}^n$$

which is a presentation and a pro-étale $\mathbb{S}_n \rtimes \text{Gal}(\mathbb{F}_p^n/\mathbb{F}_p)$-torsor. Furthermore there is a pullback of stacks,

$$\begin{array}{ccc}
\text{Spec} \, \mathbb{F}_p^n[u^\pm 1] & \longrightarrow & \text{Def}(\Gamma_n, \mathbb{F}_p^n)[u^\pm 1] \\
\downarrow & & \downarrow \\
\hat{\mathcal{M}}^n & \longrightarrow & \hat{\mathcal{M}}^n
\end{array}$$

Let $E_n$ denote the Hopkins–Miller spectrum associated to $\text{Def}(\Gamma_n, \mathbb{F}_p^n)$. There is an isomorphism $\text{Def}(\Gamma_n, \mathbb{F}_p^n)[u^\pm 1] \simeq \text{Spf} \, \pi_* E_n$. The diagonal is

$$\text{Def}(\Gamma_n, \mathbb{F}_p^n) \times \hat{\mathcal{M}}^n \text{Def}(\Gamma_n, \mathbb{F}_p^n)[u^\pm 1]$$

$$\simeq \text{Def}(\Gamma_n, \mathbb{F}_p^n)[u^\pm 1] \times (\mathbb{S}_n \rtimes \text{Gal}(\mathbb{F}_p^n/\mathbb{F}_p))$$

$$\simeq \text{Spf} \, \pi_* L_{K(n)}(E_n \wedge E_n).$$

Therefore $\hat{\mathcal{M}}^n$ can also be represented by the Hopf algebroid $(E_n^*, \text{map}(\mathbb{S}_n \rtimes \text{Gal}(\mathbb{F}_p^n/\mathbb{F}_p), E_n^*))$ where map means the set of continuous maps.

**Definition 2.3.** $\hat{\mathcal{M}}^n \times_\mathcal{M} \mathcal{M}_X = (\mathcal{U}^n \times_\mathcal{M} \mathcal{M}_X)^{\hat{\mathcal{M}}^n}$

By Lemma 2.1 $\hat{\mathcal{M}}^n \times_\mathcal{M} \mathcal{M}_X$ is the stack associated to the Hopf algebroid $(E_n^* \otimes V(X), \text{map}(\mathbb{S}_n \rtimes \text{Gal}(\mathbb{F}_p^n/\mathbb{F}_p), E_n^* \otimes V(X)))$.

The remainder of this paper is concerned with the structure of $\hat{\mathcal{M}}^2 \times_\mathcal{M} \mathcal{M}_{E(2)}$.

**2.2. Elliptic curves.** A Weierstrass curve over $R$ is the closure in $\mathbb{P}^2_R$ of the affine curve

$$(3) \quad y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

over $R$. The curve is smooth if and only if $\Delta = \Delta(a_1, \ldots, a_6)$ is invertible in $R$. A strict isomorphism of Weierstrass curves is given by the change of coordinates

$$x' = x + r, \quad y' = y + sx + t.$$ 

The Weierstrass curves along with their coordinate changes form an algebraic stack $\mathcal{M}_{(A, \Gamma)}$ determined by the Hopf algebroid $(A, \Gamma)$ where $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6], \quad \Gamma = A[s, r, t]$. 
The Hopf algebroid structure maps are implicit in the definitions (see [6, section 3]).

We now explain how this stack $\mathcal{M}(A,\Gamma)$ is associated to elliptic curves. Let $\mathcal{E}ll$ denote the moduli stack of elliptic curves over $\text{Spec} \, \mathbb{Z}$. A morphism $\text{Spec}(R) \to \mathcal{E}ll$ classifies an elliptic curve $C \to R$, which is a smooth proper morphism whose geometric fibers are elliptic curves. Let $\mathcal{E}\mathcal{U}$ denote the compactified moduli stack classifying generalized elliptic curves. There exists a line bundle $\omega \to \mathcal{E}ll$ associated to the cotangent space at the identity section of a generalized elliptic curve. Given a smooth elliptic curve $C : \text{Spec} \, R \to \mathcal{E}ll$, the set of sections $\Gamma(\text{Spec} \, R, \omega(C))$ is the set of invariant 1-forms on $C$.

Let $\mathcal{E}ll$ be the $\mathbb{G}_m$-torsor over $\mathcal{E}ll$ whose $R$-points are given by pairs $(C/R, \alpha : \omega(C) \cong R)$. Here $C/R$ is an elliptic curve over $R$ and $\alpha$ is a choice of trivialization of the $R$-sections of $\omega$.

Any generalized elliptic curve $C \to S$ admits a presentation in the Weierstrass normal form locally over $S$ in the flat topology. The identity element of the elliptic curve is identified with the unique point at infinity of the Weierstrass curve. This gives a map of stacks $\mathcal{E}ll \to M(A,\Gamma)$ which is an equivalence on the substack of smooth elliptic curves, $\mathcal{E}ll \cong \mathcal{M}(A[\Delta^{-1}],\Gamma[\Delta^{-1}])$.

There is a natural map $\mathcal{E}ll \to X_{1,fg}$ classifying the formal group associated to an elliptic curve. This map lifts canonically to a map of the $\mathbb{G}_m$-torsors.

Consider the substack $\mathcal{E}ll_p \to \mathcal{E}ll$ which is the $p$-completion of $\mathcal{E}ll$. Note that $\mathcal{E}ll_p$ is a formal Deligne–Mumford stack. For any $p$-complete ring $R$ the map $\text{Spf}(R) \to \mathcal{E}ll$ classifies an ind-system $C_m/\text{Spec}(R/p^n)$ of compatible generalized elliptic curves.

Define $(\mathcal{E}ll)_p = \mathcal{E}ll \times_{\mathbb{Z}} \text{Spec}(\mathbb{F}_p)$. Let $(\mathcal{E}ll^{\text{ord}})_p \subset (\mathcal{E}ll)_p$ denote the locus of ordinary generalized elliptic curves in characteristic $p$, and let

$$(\mathcal{E}ll^{\text{ss}})_p = (\mathcal{E}ll)_p - (\mathcal{E}ll^{\text{ord}})_p$$

denote the locus of supersingular elliptic curves in characteristic $p$.

Consider the substack

$$(\mathcal{E}ll^{\text{ss}})_p \subset (\mathcal{E}ll)_p$$

where $\mathcal{E}ll^{\text{ss}}_p$ is the completion of $\mathcal{E}ll$ at $(\mathcal{E}ll^{\text{ss}})_p$.

Define the Hopf algebroid,

$$A' = A[\Delta^{-1}]_{(p,a_1)}, \quad \Gamma' = A' \otimes_A \Gamma \otimes_A A'.$$

Then,

$$\mathcal{E}ll^{\text{ss}}_p = \mathcal{M}(A',\Gamma').$$

The following is a special case of the Serre–Tate theorem for abelian schemes [9, Theorem 1.2.1].

**Theorem 2.1** (Serre, Tate). Let $R$ be a Noetherian ring with $p$ nilpotent and $I$ a nilpotent ideal in $R$, $R_0 = R/I$. Let $\mathcal{E}ll^{\text{ss}}_p(R)$ denote the category of
supersingular elliptic curves in characteristic $p$ over $R$ and $D(R, R_0)$ denote the category of triples

$$(C_0, G, \epsilon)$$

consisting of a supersingular elliptic curve $C_0$ over $R_0$, a formal group $G$ over $R$ and an isomorphism of formal groups over $R_0$, $\epsilon : G_0 \simeq C_0^\wedge$, where $G_0$ is reduction modulo $I$ of $G$. Then the functor

$$\mathcal{Ell}^{ss}_p(R) \to D(R, R_0)$$

$$C \mapsto (C_0, C^\wedge, \text{natural } \epsilon)$$

is an equivalence of categories.

The following is implied by the Serre–Tate theorem.

**Proposition 2.2.** There is an equivalence

$$\mathcal{Ell}^{ss}_p \simeq \widehat{\mathcal{M}}^2 \times_\mathcal{M} \overline{\mathcal{Ell}}$$

of formal Deligne–Mumford stacks.

Let $E$ be a supersingular elliptic curve over a field $k$ of characteristic $p$ classifying a point $\eta : \text{Spec } k \to (\mathcal{Ell})_{\kappa_p}$. Serre–Tate theory gives an isomorphism of the formal neighborhood of $\eta$ with the universal deformation space for the formal group $\hat{E}$. Therefore there is an equivalence of deformation spaces

$$\text{Def}(E, k) \simeq \text{Def}(\hat{E}, k).$$

**2.3. Level structures.** In this section we review the modular curves. Let $S$ be a scheme over $\mathbb{Z}[1/N]$ and $C$ a smooth elliptic curve over $S$. Let $C[N]$ denote the $N$-torsion points of $C$. The group scheme $C[N]$ is étale locally over $S$ isomorphic to the discrete group scheme $(\mathbb{Z}/N\mathbb{Z})^2$ over $C$. Let $\mathcal{Ell}(N)$ denote the moduli stack of pairs $(C, \phi)$ where $C$ is a smooth elliptic curve and $\phi$ is a full level-$n$ structure, a choice of isomorphism

$$\phi : (\mathbb{Z}/N\mathbb{Z})^2 \simeq \overline{\mathbb{Z}[N]}.$$

Equivalently, the points of $\mathcal{Ell}(N)$ are triples $(C, P, Q)$ where $P$ and $Q$ are a pair of sections $S \to C[N]$ that are, locally over $S$, linearly independent.

Let $\mathcal{Ell}_1(N)$ denote the moduli stack of pairs $(C, P)$ where $P$ is a primitive $N$-torsion point over $C$. Finally, let $\mathcal{Ell}_0(N)$ denote the moduli stack of pairs $(C, H)$ where $H$ is a choice of a subgroup scheme $H \subset C[N]$ locally isomorphic to $\mathbb{Z}/N\mathbb{Z}$.

In this paper we’ll work with stacks $\mathcal{Ell}(N)$, $\mathcal{Ell}_1(N)$ and $\mathcal{Ell}_0(N)$ which are $\mathbb{G}_m$-torsors over the modular curves $\mathcal{Ell}(N)$, $\mathcal{Ell}_1(N)$ and $\mathcal{Ell}_0(N)$. The $R$-points of $\mathcal{Ell}(N)$ consists of triples $(C, \eta, (P, Q))$, where $C$ is an elliptic curve over $R$, $\eta$ is a choice of a nowhere vanishing invariant 1-form of $C$ and $(P, Q)$ is a full level-$N$ structure described as before.
**Theorem 2.2** (Deligne, Rapoport ([10])). The moduli stack $\text{Ell}(N)$ is a smooth affine scheme over $\text{Spec} \, \mathbb{Z}[1/N]$ for $N \geq 3$. For $N \geq 4$ the moduli stack $\text{Ell}_1(N)$ is a smooth affine scheme over $\text{Spec} \, \mathbb{Z}[1/N]$.

The maps forgetting level structures induce a diagram

$$
\begin{array}{ccc}
\text{Ell}(N) & \xrightarrow{\Gamma_1(N)} & \text{Ell}_1(N) \\
\downarrow & & \downarrow \\
\text{Ell}_0(N) & \xrightarrow{(\mathbb{Z}/N\mathbb{Z})^\times} & \text{Ell} \times \text{Spec} \, \mathbb{Z}[1/N]
\end{array}
$$

where all the arrows are finite étale and the labeled ones are Galois.

The Galois groups are defined as follows:

$$
\Gamma_1(N) = \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL(2, N) \right\},
$$

$$
\Gamma_0(N) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL(2, N) \right\}.
$$

Let $\text{Ell}(N)_p$, $\text{Ell}_1(N)_p$ and $\text{Ell}_0(N)_p$ denote the completions at $p$. Let $\text{Ell}(N)^{ss}_p$ denote the pullback

$$
\begin{array}{ccc}
\text{Ell}(N)^{ss}_p & \longrightarrow & \text{Ell}(N)_p \\
\downarrow & & \downarrow \\
\text{Ell}^{ss}_p & \longrightarrow & \text{Ell}_p.
\end{array}
$$

Since $\text{Ell}(N)_p$ is formal affine (assuming $N \geq 3$) and the right vertical arrow is an étale $GL(2, N)$-torsor, Serre–Tate theory implies there is an equivalence

$$
\text{Ell}(N)^{ss}_p \simeq \prod_i \text{Spf} \, W(k_i)[[u_1]][u_1^{\pm 1}]
$$

for a finite set of fields $k_i$ (depending on $N$).

**2.4. Restatement of the main theorem.** Given a formal Deligne–Mumford stack $\mathcal{S}$ over $\mathbb{Z}_p$, define its maximal unramified cover $\mathcal{S}^{nr}$ to be the pullback $\mathcal{S} \times_{\mathbb{Z}_p} \text{Spf} \, W(\overline{\mathbb{F}}_p)$. The map $\mathcal{S}^{nr} \rightarrow \mathcal{S}$ is an pro-étale cover with Galois group $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \hat{\mathbb{Z}}$. Our main Theorem 1.1 can be restated in the following way.
Theorem 2.3. There is a map of stacks over \( \widehat{\mathcal{M}}^2 \),
\[ \widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{ER(2)}^{nr} \to \text{Ell}^2_{ss}. \]
The map over \( \widehat{\mathcal{M}}^2 \) induced by forgetting level structure
\[ (\text{Ell}_1(3))_{ss}^{nr} \to (\text{Ell}_0(3))_{ss}^{nr} \]
is equivalent to the map
\[ \widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{E(2)}^{nr} \to \widehat{\mathcal{M}}^2 \times_{\mathcal{M}} \mathcal{M}_{ER(2)}^{nr} \]
over \( \widehat{\mathcal{M}}^2 \) induced by the inclusion of fixed points \( ER(2) \to E(2) \).

3. Real Johnson–Wilson theory from modular curves

3.1. Level 3-structures at the prime 2. Consider the supersingular elliptic curve
\[ C : x^3 + y^2 + y = 0 \in \mathbb{P}^2_{\mathbb{F}_4}. \]

The automorphism group \( G_{24} = \text{Aut}_{\mathbb{F}_4}(C) \) is the group of units in the maximal order of a rational quaternion algebra \( \mathbb{Q}\{i,j,k\} \). It’s isomorphic to the binary tetrahedral group \( \hat{A}_4 = Q_8 \rtimes C_3 \) of order 24, where \( C_3 \) acts on \( Q_8 \) by conjugation. It contains the quaternion group \( Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \) and 16 other elements \((\pm 1 \pm i \pm j \pm k)/2 \).

Let \( \hat{C} \) be the completion of \( C \) at the identity section. \( \hat{C} \) is a formal group of height 2 over \( \mathbb{F}_4 \). The automorphism group \( \text{Aut}(\hat{C}) = O_{D_1/\mathbb{Q}_2} \) is the group of units in the maximal order of the 2-adic quaternion algebra
\[ D_{1/\mathbb{Q}_2} = \mathbb{Q}_2\{i,j,k\}. \]

Abstractly this is the completion of the Hurwitz lattice
\[ \mathbb{Z}(\pm 1, \pm i, \pm j, \pm k) \coprod (\pm 1 \pm i \pm j \pm k)/2 \]
at the ideal \((2)\). Notice that \( G_{24} \) is the maximal finite subgroup.

Proposition 3.1. The map \( \text{Spec} \mathbb{F}_4 \to (\text{Ell}^ss)_{\mathbb{F}_2} \) classifying \( C \) is a presentation and an étale \( G_{24} \rtimes \text{Gal}(\mathbb{F}_4/\mathbb{F}_2) \)-torsor.

It follows that the map \( \text{Def}(C, \mathbb{F}_4) \to \text{Ell}^ss_2 \) classifying the universal deformation of \( C \) is an étale \( G_{24} \rtimes \text{Gal}(\mathbb{F}_4/\mathbb{F}_2) \)-torsor. By Serre–Tate theory there is an isomorphism of deformation spaces
\[ \text{Def}(C, \mathbb{F}_4) \simeq \text{Spf} W(\mathbb{F}_4)[[a_1]]. \]
The curve \( C \) lifts to \( W(\mathbb{F}_4)[[a_1]] \) as \( \tilde{C} : y^2 + a_1 xy + y = x^3 \), which is the universal deformation curve over \( \text{Def}(C, \mathbb{F}_4) \). \( \tilde{C} \) lifts further to
\[ y^2 + a_1 ux y + u^3 y = x^3 \]
over \( \text{Spf} W(\mathbb{F}_4)[[a_1]][u^{\pm 1}] \). We shall call this \( \tilde{C} \) from now.
Proposition 3.2 ([6, Theorem 3.1]). The map
\[
\text{Spf } W(\mathbb{F}_4)[[a_1]][u^\pm 1] \rightarrow \text{Ell}^s_2 \simeq \mathcal{M}_{(A',\Gamma')} \]
classifying the curve
\[
\bar{C} : y^2 + a_1 u x y + u^3 y = x^3
\]
is a presentation and an étale \(G_{24} \times \text{Gal}(\mathbb{F}_4/\mathbb{F}_2)\)-torsor.

This is a restatement of
\[
L_{K(2)} TMF = EO_2.
\]

The level 3-structures on elliptic curves and their associated moduli stacks are related by finite étale morphisms
\[
\text{Ell}(3) \xrightarrow{6} \text{Ell}_1(3) \xrightarrow{2} \text{Ell}_0(3) \xrightarrow{4} \text{Ell} \times \text{Spec } \mathbb{Z}[1/3]
\]
of degrees 6, 2 and 4. The related modular groups are as follows:
\[
\Gamma_1(3) = \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL(2,3) \right\} \simeq C_3 \rtimes C_2,
\]
\[
\Gamma_0(3) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL(2,3) \right\} \simeq C_6 \rtimes C_2.
\]

The point \((0,0)\) on the Weierstrass normal form of a smooth elliptic curve is a point of order 3 if it is of the form
\[
C(a_1,a_3) : y^2 + a_1 x y + a_3 y = x^3
\]
where \(a_3\) is a unit. The curve \(C(a_1,a_3)\) comes with the invariant 1-form
\[
\eta = \frac{dx}{2y + a_1 x + a_3} = \frac{dy}{3x^2 - a_1 y}.
\]

Furthermore, if \((C',P,\eta')\) is a smooth elliptic curve with a point \(P\) of order 3 and an invariant 1-form \(\eta'\) then there exists a unique isomorphism of the triple (see [14, Prop. 3.2])
\[
(C(a_1,a_3), (0,0), \eta) \simeq (C', P, \eta').
\]

This gives us the following equivalence of stacks:
\[
\text{Ell}_1(3) = \text{Spec } \mathbb{Z}[1/3][a_1,a_3^{\pm 1}, \Delta^{-1}],
\]
\[
\text{Ell}_1(3)^ss = \text{Spf } \mathbb{Z}_2[[a_1]][a_3^{\pm 1}],
\]
where \(\Delta = a_3^3(a_1^3 - 27a_3^2).

Proposition 3.3. The map
\[
\text{Spf } W(\mathbb{F}_4)[[a_1]][u^\pm 1] \rightarrow \text{Ell}_1(3)^ss
\]
classifying the point \((\bar{C}, (0,0), \frac{dx}{2y + a_1 u x + a^3})\) is a presentation and an étale \(C_3 \rtimes \text{Gal}(\mathbb{F}_4/\mathbb{F}_2)\)-torsor. The embedding \(C_3 \subset S_2\) is induced by lifting the 2-adic Teichmüller character \(\mathbb{F}_4^\times \rightarrow W(\mathbb{F}_4)^\times\).
There is a $C_2$-étale map forgetting level structure,
\[
\text{Ell}_1(3) \to \text{Ell}_0(3),
\]
\[(C, P, \eta) \mapsto (C, (P), \eta).\]
The group $C_2$ acts on $\text{Ell}_1(3)$ by $(C, P, \eta) \mapsto (C, -P, \eta)$.

**Proposition 3.4.** The map
\[
\text{Spf} \, W(\mathbb{F}_4)[[a_1]][u^{\pm 1}] \to \text{Ell}_0(3)_2^{ss}
\]
classifying the point $(\widetilde{C}, (0, 0)), \frac{dx}{2y + a_1ux + u^3}$ is a presentation and an étale $C_2 \times C_3 \rtimes \text{Gal} (\mathbb{F}_4/\mathbb{F}_2)$-torsor. The group $C_2$ is the normal subgroup of $\text{Q}_8 \rtimes C_3$ generated by the formal inverse $[-1]_{C\wedge}$.

The following diagram shows how the various modular curves at the supersingular locus in characteristic 2 are related. All the arrows are finite étale and the labeled ones are Galois. Notation: $\text{Gal} = \text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$.

As a consequence there is a diagram of $K(2)$-local elliptic spectra.

\[
L_{K(2)} TMF_1(3) \xrightarrow{\simeq} E_2^{h(C_3 \rtimes \text{Gal})}
\]
\[
L_{K(2)} TMF_0(3) \xrightarrow{\simeq} E_2^{h(C_2 \times C_3 \rtimes \text{Gal})}
\]
\[
L_{K(2)} TMF \xrightarrow{\simeq} E_2^{h(\hat{A}_4 \rtimes \text{Gal})}.
\]

Here $E_2$ is the Hopkins–Miller Morava $E$-theory spectrum associated to the deformation space $\text{Def}(C, \mathbb{F}_4)$.

**3.2. The structure of $\mathcal{M}_{E(2)}$ near the height 2 point.** Let the point $f : \text{Spec} \, \mathbb{F}_4 \to L^2$ classify the formal group $C^\wedge$ associated to the supersingular elliptic curve $C : y^2 + y + x^3 = 0$. Let $\widetilde{C}$ denote the universal deformation of the curve $C$ over $\text{Spf} \, W(\mathbb{F}_4)[[a_1]][u^{\pm 1}]$ as in equation (5).
A MODULAR DESCRIPTION OF ER(2) 755

Since \( U^2 \) denotes the moduli stack of formal groups of height at most 2 we know from Prop. 2.1 there is a faithfully flat presentation

\[ \text{Spec } \mathbb{Z}(2)[v_1, v_2^\pm 1] \rightarrow U^2. \]

As a consequence there exists an extension \( K \) of \( \mathbb{F}_4 \) such that

\[ f : \text{Spf } W(K)[[a_1]][u^\pm 1] \rightarrow \text{Spf } W(\mathbb{F}_4)[[a_1]][u^\pm 1] \]

is an fpqc cover and the pullback \( f^*(\tilde{C}^\wedge) \) is isomorphic to a 2-typical formal group law \( \psi : \text{Spf } W(K)[[a_1]][u^\pm 1] \rightarrow \text{Spec } \mathbb{Z}(2)[v_1, v_2^\pm 1] \).

\[ \alpha : f^*(\tilde{C}^\wedge) \simeq \psi. \]

Consider the map

\[ \text{Spf } W(K)[[a_1]][u^\pm 1] (f^*(\tilde{C}^\wedge), \alpha, \psi) \rightarrow \widehat{M}^2 \times_M \mathcal{M}_{E(2)} \]

classifying the formal group \( f^*(\tilde{C}^\wedge) \), \( \psi \) and the isomorphism \( \alpha \). This map is a surjection and an étale presentation.

The Teichmüller character map \( \mathbb{F}_4^\times \rightarrow W(\mathbb{F}_4)^\times \) lifts to give an embedding \( \mathbb{F}_4^\times \subset \text{Aut}(f^*(C^\wedge)) \). For a \( \omega \in \mathbb{F}_4^\times \) the linear formal power series \( g(t) = \omega t \) gives an element of \( S_2 \). Since the action of \( \text{Aut}(f^*(C^\wedge)) \) extends to an action on the deformation space \( \text{Spf } W(K)[[a_1]][u^\pm 1] \), we obtain an action of \( g \) on the ring \( W(K)[[a_1]][u^\pm 1] \) by

\[ g(u) = \omega u, \]

\[ g(ua_1) = \omega^2 ua_1, \]

which leaves \( v_1 \) and \( v_2 \) invariant. Moreover these are the only elements of \( S_2 \) which act invariantly on \( v_1 \) and \( v_2 \). Since

\[ \widehat{M}^2 \times_M \text{Spec } \mathbb{Z}(2)[v_1, v_2^\pm 1] \simeq \text{Spf } \mathbb{Z}_2[[v_1]][v_2^\pm 1] \]

we can identify the pullback in the following diagram of stacks:

\[ \begin{array}{ccc}
\text{Spf } W(K)[[a_1]][u^\pm 1] & \times (C_3 \rtimes \text{Gal}(K, \mathbb{F}_2)) & \rightarrow \text{Spf } W(K)[[a_1]][u^\pm 1] \\
\downarrow & & \downarrow \\
\text{Spf } W(K)[[a_1]][u^\pm 1] & \rightarrow & \widehat{M}^2 \times_M \mathcal{M}_{E(2)}. \\
\end{array} \]

Therefore, the map \( \text{Spf } W(K)[[a_1]][u^\pm 1] \rightarrow \widehat{M}^2 \times_M \mathcal{M}_{E(2)} \) is an étale \( C_3 \rtimes \text{Gal}(K, \mathbb{F}_2) \)-torsor. The conjugation action of \( \text{Gal}(K/\mathbb{F}_2) \) on \( C_3 \) factors through the quotient \( \text{Gal}(K/\mathbb{F}_2) \rightarrow \text{Gal}(\mathbb{F}_4/\mathbb{F}_2) \) since every automorphism of \( \tilde{C}^\wedge \) is already defined over \( \mathbb{F}_4 \). In terms of \( K(2) \)-localization,

\[ L_{K(2)} E(2) = E_2^HH \]

where \( H = \mathbb{F}_4^\times \rtimes \text{Gal}(K/\mathbb{F}_2) \) and \( E_2 \) is the Hopkins–Miller spectrum associated to \( \text{Def}(C^\wedge, K) \).
More generally, we can use the maximal unramified extension $E_{nr}^2$, the Hopkins–Miller spectrum associated to the deformation space $\text{Def}(C, \overline{\mathbb{F}}_2)$. The extended Morava stabilizer group in this case is

$$S_2 \rtimes \text{Gal}(\mathbb{F}_2/\mathbb{F}_2) = S_2 \rtimes \hat{\mathbb{Z}}.$$ 

**Proposition 3.5.** There is a map of stacks $\text{Def}(C, \overline{\mathbb{F}}_2) \rightarrow \widehat{\mathcal{M}}^2 \times_\mathcal{M} \mathcal{M}_{E(2)}$ which is a surjection and an étale $C_3 \rtimes \hat{\mathbb{Z}}$-torsor.

The maximal unramified cover of the deformation space $\text{Def}(C, \mathbb{F}_4)$ can be identified with $\text{Def}(C, \overline{\mathbb{F}}_2)$. There is a natural map of stacks (we use the notation $//C$ for quotient stacks)

$$\text{Def}(C, \overline{\mathbb{F}}_2) \rightarrow \text{Def}(C, \mathbb{F}_4)//\text{Gal}(\mathbb{F}_4/\mathbb{F}_2),$$

which realizes as a $K(n)$-local $\mathbb{Z}$-Galois extension of $S$-algebras

$$E^\text{Gal}_2 \rightarrow E_{nr}^2.$$ 

The $\mathbb{Z}$-extension is obtained by adjoining all the roots of unity of order prime to $p$ to the $p$-complete spectrum $E^\text{Gal}_2$ and $p$-completing ([16, 5.4.6]).

The map (6) factors through the quotient

$$\text{Def}(C, \overline{\mathbb{F}}_2)//C_3 \rightarrow \text{Def}(C, \mathbb{F}_4)//\hat{\mathbb{A}}_4 \rtimes \text{Gal}(\mathbb{F}_4/\mathbb{F}_2).$$

In terms of $K(2)$-localization there is a map

$$L_{K(2)}TMF \simeq (E_2)^{\hat{\mathbb{A}}_4 \times \text{Gal}} \rightarrow (E_{nr}^2)^{hC_3} \simeq L_{K(2)}E(2)(\mu_{\infty,2}).$$

Combining Proposition 3.3 and Proposition 3.5 proves the following.

**Proposition 3.6.** There are equivalences of stacks over $\widehat{\mathcal{M}}^2$.

$$\text{Def}(C, \overline{\mathbb{F}}_2)//C_3 \simeq \widehat{\mathcal{M}}^2 \times_\mathcal{M} \mathcal{M}_{E(2)}^{nr} \simeq (\text{Ell}_1(3)^{ss})^{nr}.$$ 

### 3.3. The structure of $\mathcal{M}_{ER(2)}$ near the height 2 point.

**Proposition 3.7.** The map of stacks

$$\widehat{\mathcal{M}}^2 \times_\mathcal{M} \mathcal{M}_{E(2)} \rightarrow \widehat{\mathcal{M}}^2 \times_\mathcal{M} \mathcal{M}_{ER(2)}$$

is a surjection and an étale $C_2$-torsor.

**Proof.** Combining Averett’s formula (1) and [16, Theorem 5.4.4] we see that the map of ring spectra $L_{K(2)}ER(2) \rightarrow L_{K(2)}E(2)$ is a $K(2)$-local $C_2$-Galois extension of 2-complete commutative $S$-algebras.

Given the description of $\widehat{\mathcal{M}}^2 \times_\mathcal{M} \mathcal{M}_{E(2)}$ as the quotient stack

$$\text{Def}(C, \overline{\mathbb{F}}_2)//C_3 \rtimes \hat{\mathbb{Z}},$$

the action of $C_2$ must come from the inverse map $[-1]_C$ on the curve $C$. 

The induced action on the deformation space is as follows:

$$[-1]_C : W(F_4)[[a_1]][u^\pm 1] \longrightarrow W(F_4)[[a_1]][u^\pm 1]$$

$$[-1]_C(u) = -u,$$

$$[-1]_C(ua_1) = ua_1.$$

**Proposition 3.8.** There is a map of stacks $\text{Def}(C, F_2) \to \hat{M}^2 \times_M \mathcal{M}_{ER(2)}$ which is an étale surjection and a $C_2 \times C_3 \times \hat{Z}$-torsor. The $C_2$-action on $\text{Def}(C, F_2)$ comes from the inverse $[-1]_C$.

This produces a map

$$\text{Def}(C, F_2) \to \hat{M}^2 \times_M \mathcal{M}_{ER(2)}$$

In terms of $K(2)$-localization the map of spectra (7) factors through the real part:

$$L_{K(2)} TMF \to (E_2^{nr})^{h(C_2 \times C_3)} \simeq L_{K(2)} ER(2)(\mu_{\infty, 2}) \to L_{K(2)} E(2)(\mu_{\infty, 2}).$$

This proves the first part of Theorem 2.3. Combining Proposition 3.4 with Proposition 3.8 proves the following which together with Proposition 3.6 proves the rest of Theorem 2.3.

**Proposition 3.9.** There are equivalences of stacks over $\hat{M}^2$,

$$\text{Def}(C, F_2) \to \hat{M}^2 \times_M \mathcal{M}_{ER(2)}^{nr} \simeq (\text{Ell}_0(3)^{ss})^{nr}.$$

**References**


School of Mathematics, Tata Institute of Fundamental Research, Mumbai 400005, India

Current address: Indian Institute of Science Education and Research, Bhopal, India 462066

romie@iiserb.ac.in

This paper is available via http://nyjm.albany.edu/j/2014/20-37.html.