A sequence of inclusions whose colimit is not a homotopy colimit

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Abstract. It is known that the homotopy colimit of a sequence of inclusions of T1 spaces is weakly equivalent with the actual colimit. We show that the assumption of T1 is essential by providing a counterexample for non-T1 spaces.

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1. Introduction

It is well known that the canonical map from the homotopy colimit (telescope, [1]) of a sequence of inclusions

$X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots$

of T1 topological spaces to the actual colimit is a weak equivalence:

$(1) \quad \text{hocolim}_n X_n \xrightarrow{\sim} \bigcup_n X_n.$

The reason is simply that for any compact space $K$ (using the covering definition, regardless of separation axioms), the image of a continuous map

$f : K \to \bigcup_n X_n$

is contained in one of the spaces $X_n$. This is easily seen as follows: Assume otherwise and pick points $s_n \in f(K) \setminus X_n$. Let

$S_m = \{s_m, s_{m+1}, \ldots\}.$

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Then $S_m \cap X_n$ is closed in $X_n$ for each $n$, since the spaces are T1, and hence $S_m$ is closed in $\bigcup X_n$. Hence, the sets $f^{-1}(S_m)$ are closed in $K$, $f^{-1}(S_m) \supseteq f^{-1}(S_{m+1})$, while

$$\bigcap_m f^{-1}(S_m) = f^{-1}\left(\bigcap_m S_m\right) = f^{-1}(\emptyset) = \emptyset.$$ 

This is a contradiction to $K$ being compact.

The authors do not know an original reference for this simple argument, which however plays a key role in homotopy theory (cf. [2]). Clearly, the assumption that the spaces $X_n$ are T1 is essential to the argument.

The first author noticed, however, that (1) also holds for so-called quasi-discrete spaces, which means spaces in which an intersection of any (possibly infinite) number of open sets is open. Finite spaces are examples of quasi-discrete spaces. Quasi-discrete spaces are, in some sense, the opposites of T1 spaces. For any T0 space $X$, there is a partial ordering on the set $X$ where $x \leq y$ if and only if the closure of $x$ contains $y$. For T1 spaces, this partial ordering is trivial. On the other hand, for quasi-discrete spaces, the ordering determines the topology completely: For a quasi-discrete space $X$, a subset $S \subseteq X$ is closed if and only if

$$x \in S, \ x \leq y \Rightarrow y \in S.$$

McCord [3] exhibited, for a quasi-discrete space $X$, a continuous map from the classifying space of the poset $(X_{\mathrm{disc}}, \leq)$ (where $X_{\mathrm{disc}}$ denotes $X$ with the discrete topology) to $X$ which is a weak equivalence, and is functorial under inclusions. This implies (1).

The authors then began asking whether (1) is true for all topological spaces. Eventually, they found a counterexample, which is the subject of the present note. It remains an open problem if (1) is true for some reasonable separation axiom weaker than T1, such as TD spaces (a space is TD if every point $x$ contains an open neighborhood $U$ such that $U \setminus \{x\}$ is open). While such follow-up questions may fall into the realm of curiosities, the example presented here is an important cautionary tale on the role of the T1 axiom in the foundations of homotopy theory.

2. The example

For $m \in \mathbb{N}$, let $Q_m \subseteq \mathbb{R}^2$ be defined as

$$(\{0\} \times [-1, 1]) \cup \left(\left\{x, \sin\left(\frac{1}{x}\right)\right\} : x \in \left(0, \frac{1}{m}\right)\right),$$

with the subspace topology induced by $\mathbb{R}^2$. Observe that for $i \leq j$, $Q_j \subseteq Q_i$, and $Q_j$ is open in $Q_i$. Also observe that

$$\bigcap_{i=1}^{\infty} Q_i = \{0\} \times [-1, 1].$$
Let \( X \) be the set
\[
\{x_1, x_2, \ldots, y\} \times Q_1
\]
with topology generated by the basis
\[
\beta = \{(\{x_k\} \times U) \cup (\{x_{k+1}, x_{k+2}, \ldots\} \times Q_k) : k \in \mathbb{N} \text{ and } V \subseteq U \subseteq Q_k \text{ and } U \text{ and } V \text{ are open in } Q_k\}
\]

We must check that \( \beta \) is actually a basis of topology.

**Lemma 1.** \( \beta \) is closed under finite intersections.

**Proof.** We consider two separate cases. In the first case, let
\[
A = (\{x_k\} \times U) \cup (\{x_{k+1}, \ldots\} \times Q_k) \cup (\{y\} \times V)
\]
\[
B = (\{x_k\} \times W) \cup (\{x_{k+1}, \ldots\} \times Q_k) \cup (\{y\} \times Z)
\]
for some \( k \in \mathbb{N}, \) and some open sets \( U, V, W, \) and \( Z \) such that \( V \subseteq U \subseteq Q_k \) and \( Z \subseteq W \subseteq Q_k. \) Then
\[
A \cap B = (\{x_k\} \times U \cap W) \cup (\{x_{k+1}, \ldots\} \times Q_k) \cup (\{y\} \times V \cap Z)
\]
\( U \cap W \) and \( V \cap Z \) are both open subsets of \( Q_k, \) and \( (V \cap Z) \subseteq (U \cap W). \) Hence \( A \cap B \) is in \( \beta. \)

In the second case, let
\[
A = (\{x_j\} \times U) \cup (\{x_{j+1}, \ldots\} \times Q_j) \cup (\{y\} \times V)
\]
\[
B = (\{x_k\} \times W) \cup (\{x_{k+1}, \ldots\} \times Q_k) \cup (\{y\} \times Z)
\]
for some \( j, k \in \mathbb{N}, \) and some open sets \( U, V, W, \) and \( Z \) such that \( V \subseteq U \subseteq Q_j \) and \( Z \subseteq W \subseteq Q_k. \) Without loss of generality we may assume \( j < k. \) Then
\[
A \cap B = ((\{x_{j+1}, \ldots\} \times Q_j) \cap (\{x_k\} \times W))
\]
\[
\cup ((\{x_{j+1}, \ldots\} \times Q_j) \cap (\{x_{k+1}, \ldots\} \times Q_k))
\]
\[
\cup ((\{y\} \times V) \cap (\{y\} \times Z))
\]
\[
= (\{x_k\} \times W) \cup (\{x_{k+1}, \ldots\} \times Q_k) \cup (\{y\} \times Z \cap V),
\]
so \( A \cap B \) is in \( \beta. \)

Let \( X_n \) be the set \( \{x_1, \ldots, x_n, y\} \times Q_1, \) with the subspace topology inherited from \( X. \) Observe that \( X_n \) has basis
\[
\beta_n = \{(\{x_k\} \times U) \cup (\{x_{k+1}, x_{k+2}, \ldots, x_n\} \times Q_k) : k \in \mathbb{N}, k \leq n \text{ and } V \subseteq U \subseteq Q_k \text{ and } U \text{ and } V \text{ are open in } Q_k\}
\]

Also observe that \( X_n \) has the subspace topology inherited from \( X_{n+1}. \)

**Theorem 2.** \( X = \bigcup_{n=1}^{\infty} X_n, \) and \( X \) has the union topology, i.e., a subset \( U \subseteq X \) is open if and only if for every \( n, \) \( U \cap X_n \) is open in \( X_n. \)
**Proof.** We want to show that a set $U$ is open in $X$ if and only if $U \cap X_n$ is open in $X_n$ for all $n \in \mathbb{N}$.

One direction is trivial: if $U$ is open in $X$, then by definition $U \cap X_n$ is open in $X_n$ for all $n \in \mathbb{N}$.

Now suppose that $U \subseteq X$, and that $U \cap X_n$ is open in $X_n$ for all $n \in \mathbb{N}$. Fix a point $q \in U$. We will exhibit an $X$-open neighborhood of $q$ in $U$. Note that $q$ is either of the form $(x_k, z)$ or $(y, z)$, for some $k \in \mathbb{N}$ and $z \in Q_1$, and these two cases need to be handled separately.

**Case 1.** Suppose $q = (x_k, z)$. Either $z \in Q_k$, or $z \in Q_1 \setminus Q_k$. These subcases again need to be handled separately.

**Subcase 1a.** Suppose $z \in Q_k$. Then for $m \geq k + 1$, any basis element of $\beta_m$ containing $q = (x_k, z)$ also contains $\{x_{k+1}, \ldots, x_m\} \times Q_k$. Furthermore, there exists an open neighborhood $V$ of $z$ such that $V \subseteq Q_k$ and $\{x_k\} \times V \subseteq U \cap X_k$. Thus $U$ contains $\{(x_k) \times V\} \cup \{(x_{k+1}, \ldots) \times Q_k\}$, which is an open set in $X$ containing $q$.

**Subcase 1b.** Suppose $z \notin Q_k$. Let $j < k$ be the unique integer such that $z \in Q_j \setminus Q_{j+1}$. Then for $m > k$, every element of $\beta_m$ which contains $(x_k, z)$ also contains $\{x_{j+1}, \ldots, x_m\} \times Q_j$. Hence $U$ contains $\{x_{j+1}, \ldots\} \times Q_j$, which is an open set in $X$ containing $q$.

**Case 2.** Suppose $q = (y, z)$. Again, we must distinguish two subcases:

**Subcase 2a.** $z \in \{0\} \times [-1, 1]$. By hypothesis $U \cap X_1$ is open in $X_1$, so it contains $\{y\} \times V$ for some $V$ open in $Q_1$ and $z \in V$. Because $U$ is open in $Q_1$, it must contain points not in $\{0\} \times [-1, 1]$. Thus, there exists a $k \in \mathbb{N}$ such that $V \subseteq Q_k$ but $V \not\subseteq Q_{k+1}$. Let $t$ be a point in $V \cap (Q_k \setminus Q_{k+1})$. Now, for $m \geq k$, $U \cap X_m$ is open in $X_m$ by hypothesis. But the definition of $\beta_m$ implies that any open set in $X_m$ containing $(y, t)$ must also contain
\[\{x_{k+1}, \ldots, x_m\} \times Q_k.\]

Thus $U$ contains
\[\{(x_{k+1}, \ldots) \times Q_k\} \cup \{(y) \times (V \cap Q_{k+1})\},\]
which is open in $X$ and contains $q$.

**Subcase 2b.** $z \in Q_k \setminus Q_{k+1}$ for some $k \in \mathbb{N}$. Then any element of $\beta_k$ containing $q = (y, z)$ contains a set of the form
\[\{(x_k) \times U\} \cup \{(y) \times V\}\]
for some $V \subseteq U \subseteq Q_k$ open, $z \in V$. On the other hand, any element of $\beta_m$ for $m \geq k$ which contains $q = (y, z)$ also contains
\[\{x_{k+1}, \ldots, x_m\} \times Q_k.\]

Hence, $U$ contains
\[\{(x_k) \times U\} \cup \{(x_{k+1}, x_{k+1}, \ldots) \times Q_k\} \cup \{(y) \times V\}\]
which is open in $X$ and contains $q$. □

3. Proof that $X$ is not the homotopy colimit of $X_n$

**Theorem 3.** The weak equivalence (1) is false for the spaces $X_n$ constructed in the previous section.

The theorem is a consequence of the following two propositions.

**Proposition 4.** There exists a continuous path in $X$ from $(x_1, (0, 0))$ to $(y, (0, 0))$.

**Proof.** Let $f : [0, 1] \to X$; $f(0) = (x_1, (0, 0))$, $f(t) = (x_{n+1}, (0, 0))$, for $n \in (1 - \frac{1}{n}, 1 - \frac{1}{n+1}]$, $n \in \mathbb{N}$, and $f(1) = (y, (0, 0))$.

If $U$ is open in $X$ and $(x_n, (0, 0)) \in U$, then for all $k \geq n$, $(x_k, (0, 0)) \in U$. Hence, if $y \in U$, then $f^{-1}(U)$ is either $\{1\}$, all of $[0, 1]$, or of the form

$$\left(1 - \frac{1}{n}, 1\right)$$

for some $n \in \mathbb{N}$. On the other hand, if $y \not\in U$, then $f^{-1}(U)$, $[0, 1)$, or of the form

$$\left(1 - \frac{1}{n}, 1\right)$$

for some $n \in \mathbb{N}$. Hence $f$ is continuous. □

**Proposition 5.** For $n \in \mathbb{N}$, there does not exist a continuous path in $X_n$ from $(x_1, (0, 0))$ to $(y, (0, 0))$.

This proposition will be proved in a sequence of lemmas.

**Lemma 6.** Let $A$ be the set $\{y\} \times \{0\} \times [-1, 1]$. Then for $n \in \mathbb{N}$, $A$ is closed in $X_n$.

**Proof.** Observe that the complement of $A$ in $X_n$ is the union

$$(\{x_1, \ldots, x_n\} \times Q_1) \cup (\{x_1, \ldots, x_n, y\} \times (Q_1 \setminus \{0\} \times [-1, 1])))$$

$$= (\{x_1\} \times Q_1) \cup (\{x_2, \ldots, x_n\} \times Q_1)$$

$$\cup (\{x_1, \ldots, x_n, y\} \times (Q_1 \setminus \{0\} \times [-1, 1])))$$

This is a basis element in $X_n$, hence $A$ is closed in $X_n$. □

**Lemma 7.** For all $n \in \mathbb{N}$, the space $\{y\} \times Q_1$ with the subspace topology inherited from $X_n$ is homeomorphic to $Q_1$.

**Proof.** Taking the intersection of each element of $\beta_n$ with $\{y\} \times Q_1$ gives

$$\{y\} \times U : U \text{ is an open subset of } Q_1$$

as a basis for the inherited topology. Thus, the map sending $(y, z)$ to $z$, for $z \in Q_1$, is trivially a homeomorphism from $\{y\} \times Q_1$ onto $Q_1$. □
Lemma 8. For $n \in \mathbb{N}$, the set $A$ is a path-component of $X_n$.

Proof. The set $A = \{y\} \times \{0\} \times [-1, 1]$ is clearly path-connected. Now suppose $\omega : [0, 1] \to X_n$ is a path such that $\omega(0) = (y, (0, 0))$. We wish to show that $\omega([0, 1]) \subseteq A$. By Lemma 6, $A$ is closed in $Q_n$, so $\omega^{-1}(A)$ is closed in $[-1, 1]$. Observe from the definition of $\beta_n$ that $\{y\} \times Q_{n+1}$ is open in $X_n$. Thus $\omega^{-1}(\{y\} \times Q_{n+1})$ is an open subset of $[-1, 1]$, and hence a disjoint union of relatively open intervals. By Lemma 7, $A$ is a path-component of $\{y\} \times Q_{n+1}$. Thus $\omega^{-1}(A)$ is a union of path components of $\omega^{-1}(\{y\} \times Q_{n+1})$. But the path-components of $\omega^{-1}(\{y\} \times Q_{n+1})$ are just disjoint relatively open intervals in $[0, 1]$, hence $\omega^{-1}(A)$ is also a disjoint union of relatively open intervals in $[0, 1]$, so $\omega^{-1}(A)$ is open in $[0, 1]$.

By hypothesis, $\omega(0) \in A$, so $\omega^{-1}(A)$ is nonempty. Thus $\omega^{-1}(A)$ is a nonempty, closed, and open subset of $[-1, 1]$, hence it is all of $[-1, 1]$. □

Note that Proposition 5 is a formal consequence of Lemma 8.

Remark. The spaces $X_n$ are, in fact, compact and hence they, and the space $X$, are compactly generated. Therefore, our example also applies to the version of the category of compactly generated spaces [4] without the T1 axiom.

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