C*-algebras associated with textile dynamical systems

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Abstract. A C*-symbolic dynamical system \((A, \rho, \Sigma)\) is a finite family \(\{\rho_\alpha\}_{\alpha \in \Sigma}\) of endomorphisms of a C*-algebra \(A\) with some conditions. It yields a C*-algebra \(O_\rho\) from an associated Hilbert C*-bimodule. In this paper, we will extend the notion of C*-symbolic dynamical system to C*-textile dynamical system \((A, \rho, \eta, \Sigma_\rho, \Sigma_\eta, \kappa)\) which consists of two C*-symbolic dynamical systems \((A, \rho, \Sigma_\rho)\) and \((A, \eta, \Sigma_\eta)\) with certain commutation relations \(\kappa\) between their endomorphisms \(\{\rho_\alpha\}_{\alpha \in \Sigma_\rho}\) and \(\{\eta_a\}_{a \in \Sigma_\eta}\). C*-textile dynamical systems yield two-dimensional subshifts and C*-algebras \(O_{\rho, \eta}\). We will study their structure of the algebras \(O_{\rho, \eta}\) and present its K-theory formulae.

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1. Introduction

In [24], the author has introduced a notion of $\lambda$-graph system as presentations of subshifts. The $\lambda$-graph systems are labeled Bratteli diagram with shift transformation. They yield $C^*$-algebras so that its K-theory groups are related to topological conjugacy invariants of the underlying symbolic dynamical systems. The class of these $C^*$-algebras include the Cuntz–Krieger algebras. He has extended the notion of $\lambda$-graph system to $C^*$-symbolic dynamical system, which is a generalization of both a $\lambda$-graph system and an automorphism of a unital $C^*$-algebra. It is a finite family $\{\rho_\alpha\}_{\alpha \in \Sigma}$ of endomorphisms of a unital $C^*$-algebra $A$ such that $\rho_\alpha(Z_A) \subset Z_A, \alpha \in \Sigma$ and $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$ where $Z_A$ denotes the center of $A$. A finite labeled graph $G$ gives rise to a $C^*$-symbolic dynamical system $(A_G, \rho_G, \Sigma)$ such that $A_G = C_N$ for some $N \in \mathbb{N}$. A $\lambda$-graph system $L$ is a generalization of a finite labeled graph and yields a $C^*$-symbolic dynamical system $(A_L, \rho_L, \Sigma)$ such that $A_L$ is $C(\Omega_L)$ for some compact Hausdorff space $\Omega_L$ with $\dim \Omega_L = 0$. It also yields a $C^*$-algebra $O_L$. A $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$ provides a subshift $\Lambda_\rho$ over $\Sigma$ and a Hilbert $C^*$-bimodule $H_\rho$ over $A$. The $C^*$-algebra $O_\rho$ for $(A, \rho, \Sigma)$ may be realized as a Cuntz–Pimsner algebra from the Hilbert $C^*$-bimodule $H_\rho$ over $A$. The $C^*$-algebra $O_\rho$ is the $C^*$-symbolic crossed product of $A$ by the subshift $\Lambda_\rho$. If $A = C(X)$ with $\dim X = 0$, there exists a $\lambda$-graph system $\mathcal{L}$ such that the subshift $\Lambda_\rho$ is the subshift $\Lambda_\mathcal{L}$ presented by $\mathcal{L}$ and the $C^*$-algebra $O_\rho$ is the $C^*$-algebra $O_{\mathcal{L}}$ associated with $\mathcal{L}$. If in particular, $A = C^0$, the subshift $\Lambda_\rho$ is a sofic shift and $O_\rho$ is a Cuntz–Krieger algebra. If $\Sigma = \{\alpha\}$ an automorphism $\alpha$ of a unital $C^*$-algebra $A$, the $C^*$-algebra $O_\rho$ is the ordinary crossed product $A \times_{\alpha} \mathbb{Z}$.

G. Robertson–T. Steger [43] have initiated a certain study of higher dimensional analogue of Cuntz–Krieger algebras from the view point of tiling systems of 2-dimensional plane. After their work, A. Kumjian–D. Pask [19] have generalized their construction to introduce the notion of higher rank graphs and its $C^*$-algebras. The $C^*$-algebras constructed from higher rank graphs are called the higher rank graph $C^*$-algebras. Since then, there have been many studies on these $C^*$-algebras by many authors (cf. [1], [9], [10], [11], [13], [16], [19], [36], [42], [43], etc.).

M. Nasu in [34] has introduced the notion of textile system which is useful in analyzing automorphisms and endomorphisms of topological Markov shifts. A textile system also gives rise to a two-dimensional tiling called Wang tiling. Among textile systems, LR textile systems have specific properties that consist of two commuting symbolic matrices. In [28], the author has extended the notion of textile systems to $\lambda$-graph systems and has defined a notion of textile systems on $\lambda$-graph systems, which are called textile $\lambda$-graph systems for short. $C^*$-algebras associated to textile systems have been initiated by V. Deaconu ([9]).
In this paper, we will extend the notion of \(C^\ast\)-symbolic dynamical system to \(C^\ast\)-textile dynamical system which is a higher dimensional analogue of \(C^\ast\)-symbolic dynamical system. The \(C^\ast\)-textile dynamical system 
\((\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)\) consists of two \(C^\ast\)-symbolic dynamical systems 
\((\mathcal{A}, \rho, \Sigma^\rho)\) and 
\((\mathcal{A}, \eta, \Sigma^\eta)\) with the following commutation relations between \(\rho\) and \(\eta\) through \(\kappa\). Set
\[
\Sigma^\rho \eta = \{(\alpha, b) \in \Sigma^\rho \times \Sigma^\eta \mid \eta_b \circ \rho_{\alpha} \neq 0\}, \\
\Sigma^\eta \rho = \{(a, \beta) \in \Sigma^\eta \times \Sigma^\rho \mid \rho_{\beta} \circ \eta_{\alpha} \neq 0\}.
\]
We require that there exists a bijection \(\kappa : \Sigma^\rho \eta \rightarrow \Sigma^\eta \rho\), which we fix and call a specification. Then the required commutation relations are
\[
\kappa(\alpha, b) = (a, \beta)\quad \text{if} \quad \kappa(\alpha, b) = (a, \beta).
\]
A \(C^\ast\)-textile dynamical system provides a two-dimensional subshifts and a
\(C^\ast\)-algebra \(\mathcal{O}_{\rho, \eta}^\kappa\). The \(C^\ast\)-algebra \(\mathcal{O}_{\rho, \eta}^\kappa\) is defined to be the universal \(C^\ast\)-algebra 
\(C^\ast(x, S_\alpha, T_a; x \in \mathcal{A}, \alpha \in \Sigma^\rho, a \in \Sigma^\eta)\) generated by \(x \in \mathcal{A}\) and two 
families of partial isometries \(S_\alpha, \alpha \in \Sigma^\rho, T_a, a \in \Sigma^\eta\) subject to the following 
relations called \((\rho, \eta; \kappa)\):
\[
\sum_{\beta \in \Sigma^\rho} S_\beta S_\beta^* = 1, \quad x S_\alpha S_\alpha^* = S_\alpha S_\alpha^* x, \quad S_\alpha^* x S_\alpha = \rho_{\alpha}(x), \\
\sum_{b \in \Sigma^\eta} T_b T_b^* = 1, \quad x T_a T_a^* = T_a T_a^* x, \quad T_a^* x T_a = \eta_{\alpha}(x), \\
S_a T_b = T_a S_\beta \quad \text{if} \quad \kappa(\alpha, b) = (a, \beta)
\]
for all \(x \in \mathcal{A}\) and \(\alpha \in \Sigma^\rho, a \in \Sigma^\eta\).

In Section 3, we will construct a tiling system in the plane from a \(C^\ast\)-textile dynamical system. The resulting tiling system is a two-dimensional 
subshift. In Section 4, we will study some basic properties of the \(C^\ast\)-algebra \(\mathcal{O}_{\rho, \eta}^\kappa\). In Section 5, we will introduce a condition called \((I)\) on 
\((\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)\) which will be studied as a generalization of the condition 
\((I)\) on \(C^\ast\)-symbolic dynamical system [26] (cf. [8], [25]). In Section 6, we will realize the \(C^\ast\)-algebra \(\mathcal{O}_{\rho, \eta}^\kappa\) as a Cuntz–Pimsner algebra associated with a certain Hilbert \(C^\ast\)-bimodule in a concrete way. We will have the 
following theorem.

**Theorem 1.1.** Let \((\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)\) be a \(C^\ast\)-textile dynamical system satisfying condition \((I)\). Then the \(C^\ast\)-algebra \(\mathcal{O}_{\rho, \eta}^\kappa\) is a unique concrete \(C^\ast\)-algebra subject to the relations \((\rho, \eta; \kappa)\). If \((\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)\) is irreducible, \(\mathcal{O}_{\rho, \eta}^\kappa\) is simple.

A \(C^\ast\)-textile dynamical system \((\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)\) is said to form square 
if the \(C^\ast\)-subalgebra of \(\mathcal{A}\) generated by the projections \(\rho_{\alpha}(1), \alpha \in \Sigma^\rho\) and 
the \(C^\ast\)-subalgebra of \(\mathcal{A}\) generated by the projections \(\eta_{\alpha}(1), a \in \Sigma^\eta\) coincide. 
It is said to have trivial \(K_1\) if \(K_1(\mathcal{A}) = \{0\}\). In Section 7 and Section 8, we
will restrict our interest to the $C^*$-textile dynamical systems forming square
to prove the following K-theory formulae:

**Theorem 1.2.** Suppose that $(A, \rho, \eta, \Sigma^A, \Sigma^B, \kappa)$ forms square and has trivial $K_1$. Then there exist short exact sequences for $K_0(O^\kappa_{\rho, \eta})$ and $K_1(O^\kappa_{\rho, \eta})$ such that

$$0 \to K_0(A)/((\text{id} - \lambda_\gamma)K_0(A) + (\text{id} - \lambda_\rho)K_0(A))$$
$$\to K_0(O^\kappa_{\rho, \eta})$$
$$\to \text{Ker}(\text{id} - \lambda_\gamma) \cap \text{Ker}(\text{id} - \lambda_\rho) \text{ in } K_0(A) \to 0$$

and

$$0 \to (\text{Ker}(\text{id} - \lambda_\gamma) \text{ in } K_0(A))/((\text{id} - \lambda_\rho)(\text{Ker}(\text{id} - \lambda_\gamma) \text{ in } K_0(A))$$
$$\to K_1(O^\kappa_{\rho, \eta})$$
$$\to \text{Ker}(\text{id} - \bar{\lambda}_\rho) \text{ in } (K_0(A)/(\text{id} - \lambda_\gamma)K_0(A)) \to 0$$

where the endomorphisms $\lambda_\rho, \lambda_\gamma : K_0(A) \to K_0(A)$ are defined by

$$\lambda_\rho([p]) = \sum_{\alpha \in \Sigma^B} [\rho_\alpha(p)] \in K_0(A) \text{ for } [p] \in K_0(A),$$

$$\lambda_\gamma([p]) = \sum_{\alpha \in \Sigma^B} [\eta_\alpha(p)] \in K_0(A) \text{ for } [p] \in K_0(A)$$

and $\bar{\lambda}_\rho$ denotes an endomorphism on $K_0(A)/(1 - \lambda_\gamma)K_0(A)$ induced by $\lambda_\rho$.

Let $A, B$ be mutually commuting $N \times N$ matrices with entries in non-negative integers. Let $G_A = (V_A, E_A), G_B = (V_B, E_B)$ be directed graphs
with common vertex set $V_A = V_B$, whose transition matrices are $A, B$ respectively. Let $M_A, M_B$ denote symbolic matrices for $G_A, G_B$ whose components consist of formal sums of the directed edges of $G_A, G_B$ respectively. Let $\Sigma^{AB}, \Sigma^{BA}$ be the sets of the pairs of the concatenated directed edges in $E_A \times E_B, E_B \times E_A$ respectively. By the condition $AB = BA$, one may take a bijection $\kappa : \Sigma^{AB} \to \Sigma^{BA}$ which gives rise to a specified equivalence $M_A M_B \cong M_B M_A$. We then have a $C^*$-textile dynamical system
written as $(A, \rho^A, \rho^B, \Sigma^A, \Sigma^B, \kappa)$. The associated $C^*$-algebra is denoted by $O^\kappa_{A,B}$. The $C^*$-algebra $O^\kappa_{A,B}$ is realized as a 2-graph $C^*$-algebra constructed by Kumjian–Pask ([19]). It is also seen in Deaconu’s paper [9]. We will see the following proposition in Section 9.

**Proposition 1.3.** Keep the above situations. There exist short exact sequences for $K_0(O^\kappa_{A,B})$ and $K_1(O^\kappa_{A,B})$ such that

$$0 \to \mathbb{Z}^N/((1 - A)\mathbb{Z}^N + (1 - B)\mathbb{Z}^N)$$
$$\to K_0(O^\kappa_{A,B})$$
$$\to \text{Ker}(1 - A) \cap \text{Ker}(1 - B) \text{ in } \mathbb{Z}^N \to 0$$
and
\[ 0 \to (\text{Ker}(1 - B) \text{ in } \mathbb{Z}^N)/(1 - A)(\text{Ker}(1 - B) \text{ in } \mathbb{Z}^N) \to K_1(\mathcal{O}_{A,B}^x) \to \text{Ker}(1 - A) \text{ in } (\mathbb{Z}^N/(1 - B)\mathbb{Z}^N) \to 0, \]
where \( \tilde{A} \) is an endomorphism on the abelian group \( \mathbb{Z}^N/(1 - B)\mathbb{Z}^N \) induced by the matrix \( A \).

Throughout the paper, we will denote by \( \mathbb{Z}_+ \) the set of nonnegative integers and by \( \mathbb{N} \) the set of positive integers.

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2. \( \lambda \)-graph systems, \( C^* \)-symbolic dynamical systems and their \( C^* \)-algebras

In this section, we will briefly review \( \lambda \)-graph systems and \( C^* \)-symbolic dynamical systems. Throughout the section, \( \Sigma \) denotes a finite set with its discrete topology, that is called an alphabet. Each element of \( \Sigma \) is called a symbol. Let \( \Sigma^\mathbb{Z} \) be the infinite product space \( \prod_{i \in \mathbb{Z}} \Sigma_i \), where \( \Sigma_i = \Sigma \), endowed with the product topology. The transformation \( \sigma \) on \( \Sigma^\mathbb{Z} \) given by \( \sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}} \) is called the full shift over \( \Sigma \). Let \( \Lambda \) be a shift invariant closed subset of \( \Sigma^\mathbb{Z} \) i.e. \( \sigma(\Lambda) = \Lambda \). The topological dynamical system \( (\Lambda, \sigma|_\Lambda) \) is called a two-sided subshift, written as \( \Lambda \) for brevity. A word \( \mu = (\mu_1, \ldots, \mu_k) \) of \( \Sigma \) is said to be admissible for \( \Lambda \) if there exists \( (x_i)_{i \in \mathbb{Z}} \in \Lambda \) such that \( \mu_1 = x_1, \ldots, \mu_k = x_k \). Let us denote by \( |\mu| \) the length \( k \) of \( \mu \). Let \( B_k(\Lambda) \) be the set of admissible words of \( \Lambda \) with length \( k \). The union \( \bigcup_{k=0}^{\infty} B_k(\Lambda) \) is denoted by \( B_*(\Lambda) \) where \( B_0(\Lambda) \) denotes the empty word. For two words \( \mu = (\mu_1, \ldots, \mu_k), \nu = (\nu_1, \ldots, \nu_n) \), we write a new word \( \mu \nu = (\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_n) \).

There is a class of subshifts called sofic shifts, that are presented by finite labeled graphs ([14], [17], [18]). \( \lambda \)-graph systems are generalization of finite labeled graphs. Any subshift is presented by a \( \lambda \)-graph system. Let
\[ \mathfrak{L} = (V, E, \lambda, \iota) \]
be a \( \lambda \)-graph system over \( \Sigma \) with vertex set \( V = \cup_{l \in \mathbb{Z}_+} V_l \) and edge set \( E = \cup_{l \in \mathbb{Z}_+} E_{l,l+1} \) that is labeled with symbols in \( \Sigma \) by a map \( \lambda : E \to \Sigma \), and that is supplied with surjective maps \( \iota = (\iota_{l,l+1}) : V_{l+1} \to V_l \) for \( l \in \mathbb{Z}_+ \). Here the vertex sets \( V_l, l \in \mathbb{Z}_+ \) and the edge sets \( E_{l,l+1}, l \in \mathbb{Z}_+ \) are finite disjoint sets for each \( l \in \mathbb{Z}_+ \). An edge \( e \) in \( E_{l,l+1} \) has its source vertex \( s(e) \) in \( V_l \) and its terminal vertex \( t(e) \) in \( V_{l+1} \) respectively. Every vertex in \( V \) has a successor and every vertex in \( V_l \) for \( l \in \mathbb{N} \) has a predecessor. It is then required that for vertices \( u \in V_{l-1} \) and \( v \in V_{l+1} \), there exists a bijective correspondence between the set of edges \( e \in E_{l,l+1} \) such that \( t(e) = v, \iota(s(e)) = u \) and the set of edges \( f \in E_{l-1,l} \) such that \( s(f) = u, t(f) = \iota(v) \), preserving their labels (24). We assume that \( \mathfrak{L} \) is left-resolving, which means that \( t(e) \neq t(f) \).
whenever $\lambda(e) = \lambda(f)$ for $e, f \in E_{l,l+1}$. Let us denote by $\{v^1_l, \ldots, v^m_{l(m)}\}$ the vertex set $V_l$ at level $l$. For $i = 1, 2, \ldots, m(l), j = 1, 2, \ldots, m(l+1), \alpha \in \Sigma$ we put

$$A_{l,l+1}(i, \alpha, j) = \begin{cases} 1 & \text{if } s(e) = v^1_i, \lambda(e) = \alpha, \ t(e) = v^{l+1}_j \text{ for some } e \in E_{l,l+1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$I_{l,l+1}(i, j) = \begin{cases} 1 & \text{if } t_{l,l+1}(v^{l+1}_j) = v^1_i, \\ 0 & \text{otherwise}. \end{cases}$$

The $C^*$-algebra $O_{\Sigma}$ associated with $\Sigma$ is the universal $C^*$-algebra generated by partial isometries $S_\alpha, \alpha \in \Sigma$ and projections $E^l_i, i = 1, 2, \ldots, m(l), \ l \in \mathbb{Z}_+$ subject to the following operator relations called ($\Sigma$):

\begin{align*}
(2.1) \quad & \sum_{\beta \in \Sigma} S_\beta S_\beta^* = 1, \\
(2.2) \quad & \sum_{i=1}^{m(l)} E^l_i = 1, \quad E^l_i = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i, j) E^{l+1}_j, \\
(2.3) \quad & S_\alpha S_\alpha^* E^l_i = E^l_i S_\alpha S_\alpha^*, \\
(2.4) \quad & S_\alpha^* E^l_i S_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \alpha, j) E^{l+1}_j,
\end{align*}

for $i = 1, 2, \ldots, m(l), \ l \in \mathbb{Z}_+, \alpha \in \Sigma$. If $\Sigma$ satisfies $\lambda$-condition (1) and is $\lambda$-irreducible, the $C^*$-algebra $O_{\Sigma}$ is simple and purely infinite ([25], [26]).

Let $A_{\Sigma,l}$ be the $C^*$-subalgebra of $O_{\Sigma}$ generated by the projections $E^l_i, i = 1, 2, \ldots, m(l)$. We denote by $A_{\Sigma}$ the $C^*$-subalgebra of $O_{\Sigma}$ generated by all the projections $E^l_i, i = 1, 2, \ldots, m(l), \ l \in \mathbb{Z}_+$. As $A_{\Sigma,l} \subset A_{\Sigma,l+1}$ and $\cup_{l \in \mathbb{Z}_+} A_{\Sigma,l}$ is dense in $A$, the algebra $A_{\Sigma}$ is a commutative AF-algebra. For $\alpha \in \Sigma$, put

$$\rho_{\alpha}^\Sigma(X) = S_\alpha^* XS_\alpha \quad \text{for} \quad X \in A_{\Sigma}.$$  

Then $\{\rho_{\alpha}^\Sigma\}_{\alpha \in \Sigma}$ yields a family of $*$-endomorphisms of $A_{\Sigma}$ such that $\rho_{\alpha}^\Sigma(1) \neq 0, \sum_{\alpha \in \Sigma} \rho_{\alpha}^\Sigma(1) \geq 1$ and for any nonzero $x \in A_{\Sigma}, \rho_{\alpha}^\Sigma(x) \neq 0$ for some $\alpha \in \Sigma$.

The situations above are generalized to $C^*$-symbolic dynamical systems as follows. Let $\mathcal{A}$ be a unital $C^*$-algebra. In what follows, an endomorphism of $\mathcal{A}$ means a $*$-endomorphism of $\mathcal{A}$ that does not necessarily preserve the unit $1_\mathcal{A}$ of $\mathcal{A}$. The unit $1_\mathcal{A}$ is denoted by 1 unless we specify. Denote by $Z_\mathcal{A}$ the center of $\mathcal{A}$. Let $\rho_\alpha, \alpha \in \Sigma$ be a finite family of endomorphisms of $\mathcal{A}$ indexed by symbols of a finite set $\Sigma$. We assume that $\rho_\alpha(Z_\mathcal{A}) \subset Z_\mathcal{A}, \alpha \in \Sigma$. The family $\rho_\alpha, \alpha \in \Sigma$ of endomorphisms of $\mathcal{A}$ is said to be essential if $\rho_\alpha(1) \neq 0$ for all $\alpha \in \Sigma$ and $\sum_{\alpha} \rho_\alpha(1) \geq 1$. It is said to be faithful if for any nonzero $x \in \mathcal{A}$ there exists a symbol $\alpha \in \Sigma$ such that $\rho_\alpha(x) \neq 0$. 


Definition 2.1 (cf. [27]). A $C^*$-symbolic dynamical system is a triplet $(\mathcal{A}, \rho, \Sigma)$ consisting of a unital $C^*$-algebra $\mathcal{A}$ and an essential and faithful finite family $\{\rho_\alpha\}_{\alpha \in \Sigma}$ of endomorphisms of $\mathcal{A}$.

As in the above discussion, we have a $C^*$-symbolic dynamical system $(\mathcal{A}_L, \rho_L, \Sigma)$ from a $\lambda$-graph system $L$. In [27], [29], [30], we have defined a $C^*$-symbolic dynamical system in a less restrictive way than the above definition. Instead of the above condition $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$ with $\rho_\alpha(Z_A) \subset Z_A, \alpha \in \Sigma$, we have used the condition in the papers that the closed ideal generated by $\rho_\alpha(1), \alpha \in \Sigma$ coincides with $\mathcal{A}$. All of the examples appeared in the papers [27], [29], [30] satisfy the condition $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$ with $\rho_\alpha(Z_A) \subset Z_A, \alpha \in \Sigma$, and all discussions in the papers well work under the above new definition.

A $C^*$-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ yields a subshift $\Lambda_\rho$ over $\Sigma$ such that a word $(\alpha_1, \ldots, \alpha_k)$ of $\Sigma$ is admissible for $\Lambda_\rho$ if and only if $(\rho_{\alpha_k} \circ \cdots \circ \rho_{\alpha_1})(1) \neq 0$ ([27, Proposition 2.1]). We say that a subshift $\Lambda$ acts on a $C^*$-algebra $\mathcal{A}$ if there exists a $C^*$-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ such that the associated subshift $\Lambda_\rho$ is $\Lambda$.

The $C^*$-algebra $\mathcal{O}_\rho$ associated with a $C^*$-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ has been originally constructed in [27] as a $C^*$-algebra by using the Pimsner’s general construction of $C^*$-algebras from Hilbert $C^*$-bimodules [39] (cf. [15] etc.). It is realized as the universal $C^*$-algebra $C^*(x, S_\alpha; x \in \mathcal{A}, \alpha \in \Sigma)$ generated by $x \in \mathcal{A}$ and partial isometries $S_\alpha, \alpha \in \Sigma$ subject to the following relations called $(\rho)$:

$$\sum_{\beta \in \Sigma} S_\beta S^*_\beta = 1, \quad xS_\alpha S^*_\alpha = S_\alpha S^*_\alpha x, \quad S^*_\alpha xS_\alpha = \rho_\alpha(x)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma$. The $C^*$-algebra $\mathcal{O}_\rho$ is a generalization of the $C^*$-algebra $\mathcal{O}_\Sigma$ associated with the $\lambda$-graph system $\Sigma$.

A $C^*$-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ is said to be free if there exists a unital increasing sequence $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}$ of $C^*$-subalgebras of $\mathcal{A}$ such that:

1. $\rho_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$ for all $l \in \mathbb{Z}_+$ and $\alpha \in \Sigma$.
2. $\bigcup_{l \in \mathbb{Z}_+} \mathcal{A}_l$ is dense in $\mathcal{A}$.
3. For $j \leq l$ there exists a projection $q \in \mathcal{D}_\rho \cap \mathcal{A}_l'$ such that:
   (i) $qx \neq 0$ for $0 \neq x \in \mathcal{A}_l$,
   (ii) $\phi_n^{\rho}(q)q = 0$ for all $n = 1, 2, \ldots, j$,

where $\mathcal{D}_\rho$ is the $C^*$-subalgebra of $\mathcal{O}_\rho$ generated by elements

$$S_{\mu_1} \cdots S_{\mu_k} x S^*_{\mu_k} \cdots S^*_{\mu_1}$$
for \((\mu_1, \ldots, \mu_k) \in B_+((\Lambda_\rho))\) and \(x \in A\), and
\[
\phi_\rho(X) = \sum_{\alpha \in \Sigma} S_\alpha XS_\alpha^*, \quad X \in D_\rho.
\]
The freeness has been called condition (I) in [30]. If in particular, one may take the above subalgebras \(A_l \subset A\), \(l = 0, 1, 2, \ldots\) to be of finite dimensional, then \((A, \rho, \Sigma)\) is said to be \(AF\)-free. \((A, \rho, \Sigma)\) is said to be irreducible if there is no nontrivial ideal of \(A\) invariant under the positive operator \(\lambda_\rho\) on \(A\) defined by \(\lambda_\rho(x) = \sum_{\alpha \in \Sigma} \rho_\alpha(x)\), \(x \in A\). It has been proved that if \((A, \rho, \Sigma)\) is free and irreducible, then the \(C^*\)-algebra \(O_\rho\) is simple ([30]).

3. \(C^*\)-textile dynamical systems and two-dimensional subshifts

Let \(\Sigma\) be a finite set. The two-dimensional full shift over \(\Sigma\) is defined to be
\[
\Sigma^{Z^2} = \{(x_{i,j})_{i,j} \in \Sigma^2 | x_{i,j} \in \Sigma\}.
\]
An element \(x \in \Sigma^{Z^2}\) is regarded as a function \(x : Z^2 \rightarrow \Sigma\) which is called a configuration on \(Z^2\). For \(x \in \Sigma^{Z^2}\) and \(F \subset Z^2\), let \(x_F\) denote the restriction of \(x\) to \(F\). For a vector \(m = (m_1, m_2) \in Z^2\), let \(\sigma^m : \Sigma^{Z^2} \rightarrow \Sigma^{Z^2}\) be the translation along vector \(m\) defined by
\[
\sigma^m((x_{i,j})_{i,j} \in \Sigma^2) = (x_{i+m_1,j+m_2})_{i,j} \in \Sigma^2.
\]
A subset \(X \subset \Sigma^{Z^2}\) is said to be translation invariant if \(\sigma^m(X) = X\) for all \(m \in Z^2\). It is obvious to see that a subset \(X \subset \Sigma^{Z^2}\) is translation invariant if and only if \(X\) is invariant only both horizontally and vertically, that is, \(\sigma^{(1,0)}(X) = X\) and \(\sigma^{(0,1)}(X) = X\). For \(k \in Z_+, \) put
\[
[-k,k]^2 = \{(i, j) \in Z^2 | -k \leq i, j \leq k\} = [-k,k] \times [-k,k].
\]
A metric \(d\) on \(\Sigma^{Z^2}\) is defined by for \(x, y \in \Sigma^{Z^2}\) with \(x \neq y\)
\[
d(x, y) = \frac{1}{2^k} \quad \text{if} \quad x_{(0,0)} = y_{(0,0)},
\]
where \(k = \max\{k \in Z_+ | x_{[-k,k]^2} = y_{[-k,k]^2}\}\). If \(x_{(0,0)} \neq y_{(0,0)}\), put \(k = -1\) on the above definition. If \(x = y\), we set \(d(x, y) = 0\). A two-dimensional subshift \(X\) is defined to be a closed, translation invariant subset of \(\Sigma^{Z^2}\) (cf. [21, p.467]). A finite subset \(F \subset Z^2\) is said to be a shape. A pattern \(f\) on a shape \(F\) is a function \(f : F \rightarrow \Sigma\). For a list \(\mathcal{F}\) of patterns, put
\[
X_\mathcal{F} = \{(x_{i,j})_{i,j} \in \Sigma^2 | \sigma^m(x)|_F \notin \mathcal{F} \text{ for all } m \in Z^2 \text{ and } F \subset Z^2\}.
\]
It is well-known that a subset \(X \subset \Sigma^{Z^2}\) is a two-dimensional subshift if and only if there exists a list \(\mathcal{F}\) of patterns such that \(X = X_\mathcal{F}\).

We will define a certain property of two-dimensional subshift as follows:
Definition 3.1. A two-dimensional subshift $X$ is said to have the diagonal property if for $(x_{i,j})_{i,j \in \mathbb{Z}^2}$, $(y_{i,j})_{i,j \in \mathbb{Z}^2} \in X$, the conditions

$$x_{i,j} = y_{i,j}, \quad x_{i+1,j-1} = y_{i+1,j-1}$$

imply

$$x_{i,j-1} = y_{i,j-1}, \quad x_{i+1,j} = y_{i+1,j}.$$ 

A two-dimensional subshift having the diagonal property is called a textile dynamical system.

Lemma 3.2. If a two dimensional subshift $X$ has the diagonal property, then for $x \in X$ and $(i, j) \in \mathbb{Z}^2$, the configuration $x$ is determined by the diagonal line $(x_{i+n,j-n})_{n \in \mathbb{Z}}$ through $(i, j)$.

Proof. By the diagonal property, the sequence $(x_{i+n,j-n})_{n \in \mathbb{Z}}$ determines both the sequences $(x_{i+1+n,j-n})_{n \in \mathbb{Z}}$ and $(x_{i-1+n,j-n})_{n \in \mathbb{Z}}$. Repeating this way, the sequence $(x_{i+n,j-n})_{n \in \mathbb{Z}}$ determines the whole configuration $x$. □

Let $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^n, \kappa)$ be a $C^*$-textile dynamical system. It consists of two $C^*$-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^n)$ with common unital $C^*$-algebra $\mathcal{A}$ and commutation relations between their endomorphisms $ho_\alpha, \alpha \in \Sigma^\rho, \eta_a, a \in \Sigma^n$ through a bijection $\kappa$ between the following sets $\Sigma^{\rho n}$ and $\Sigma^{n\rho}$, where

$$\Sigma^{\rho n} = \{(\alpha, b) \in \Sigma^\rho \times \Sigma^n | \eta_b \circ \rho_\alpha \neq 0\},$$
$$\Sigma^{n\rho} = \{(a, \beta) \in \Sigma^n \times \Sigma^\rho | \rho_\beta \circ \eta_a \neq 0\}.$$

The given bijection $\kappa : \Sigma^{\rho n} \longrightarrow \Sigma^{n\rho}$ is called a specification. The required commutation relations are

$$\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a \quad \text{if} \; \kappa(\alpha, b) = (a, \beta).$$

A $C^*$-textile dynamical system will yield a two-dimensional subshift $X_{\rho,\eta}^\kappa$. We set

$$\Sigma_\kappa = \{\omega = (\alpha, b, a, \beta) \in \Sigma^\rho \times \Sigma^n \times \Sigma^\rho \times \Sigma^n | \kappa(\alpha, b) = (a, \beta)\}.$$

For $\omega = (\alpha, b, a, \beta)$, since $\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a$ as endomorphisms on $\mathcal{A}$, one may identify the quadruplet $(\alpha, b, a, \beta)$ with the endomorphism $\eta_b \circ \rho_\alpha(= \rho_\beta \circ \eta_a)$ on $\mathcal{A}$ which we will denote by simply $\omega$. Define maps $t(= \text{top}), b(= \text{bottom}) : \Sigma_\kappa \longrightarrow \Sigma^\rho$ and $l(= \text{left}), r(= \text{right}) : \Sigma_\kappa \longrightarrow \Sigma^\rho$ by setting

$$t(\omega) = \alpha, \quad b(\omega) = \beta, \quad l(\omega) = a, \quad r(\omega) = b.$$

\[
\begin{array}{c}
\alpha \xrightarrow{t(\omega)} \\
\downarrow a = t(\omega) \\
\beta \xrightarrow{b = r(\omega)} \\
\downarrow b \xrightarrow{l(\omega)} \\
\end{array}
\]
A configuration \((\omega_{i,j})_{(i,j) \in \mathbb{Z}^2} \in \Sigma_{\kappa}^2\) is said to be \emph{paved} if the conditions
\[
\begin{align*}
l(\omega_{i,j}) &= b(\omega_{i,j+1}), & r(\omega_{i,j}) &= l(\omega_{i+1,j}), \\
l(\omega_{i,j}) &= r(\omega_{i-1,j}), & b(\omega_{i,j}) &= l(\omega_{i,j-1})
\end{align*}
\]
hold for all \((i, j) \in \mathbb{Z}^2\). We set
\[
X_{\rho,\eta}^\kappa = \{(\omega_{i,j})_{(i,j) \in \mathbb{Z}^2} \in \Sigma_{\kappa}^2 \mid (\omega_{i,j})_{(i,j) \in \mathbb{Z}^2} \text{ is paved and} \omega_{i+n,j-n} \circ \omega_{i+n-1,j-n+1} \circ \cdots \circ \omega_{i+1,j-1} \circ \omega_{i,j} \neq 0 \text{ for all } (i, j) \in \mathbb{Z}^2, n \in \mathbb{N}\},
\]
where \(\omega_{i+n,j-n} \circ \omega_{i+n-1,j-n+1} \circ \cdots \circ \omega_{i+1,j-1} \circ \omega_{i,j}\) is the compositions as endomorphisms on \(\mathcal{A}\).

**Lemma 3.3.** Suppose that a configuration \((\omega_{i,j})_{(i,j) \in \mathbb{Z}^2} \in \Sigma_{\kappa}^2\) is paved. Then \((\omega_{i,j})_{(i,j) \in \mathbb{Z}^2} \in X_{\rho,\eta}^\kappa\) if and only if
\[
\rho_b(\omega_{i+n,j-m}) \circ \cdots \circ \rho_b(\omega_{i+1,j-m}) \circ \rho_b(\omega_{i,j-m}) \circ \eta_l(\omega_{i,j-m}) \circ \cdots \circ \eta_l(\omega_{i,j}) \neq 0
\]
for all \((i, j) \in \mathbb{Z}^2, n, m \in \mathbb{Z}_+\).

**Proof.** Suppose that \((\omega_{i,j})_{(i,j) \in \mathbb{Z}^2} \in X_{\rho,\eta}^\kappa\). For \((i, j) \in \mathbb{Z}^2, n, m \in \mathbb{Z}_+\), we may assume that \(m \geq n\). Since
\[
0 \neq \omega_{i+m,j-m} \circ \cdots \circ \omega_{i+n+1,j-m} \circ \omega_{i+n,j-m} \circ \cdots \circ \omega_{i,j-m}
\]
\[
= \omega_{i+m,j-m} \circ \cdots \circ \omega_{i+n+1,j-m} \circ \rho_b(\omega_{i+n,j-m}) \circ \cdots \circ \rho_b(\omega_{i+1,j-m}) \circ \rho_b(\omega_{i,j-m})
\]
\[
\circ \eta_l(\omega_{i,j-m}) \circ \cdots \circ \eta_l(\omega_{i,j-1}) \circ \eta_l(\omega_{i,j});
\]
one has
\[
\rho_b(\omega_{i+n,j-m}) \circ \cdots \circ \rho_b(\omega_{i+1,j-m}) \circ \rho_b(\omega_{i,j-m}) \circ \eta_l(\omega_{i,j-m}) \circ \cdots \circ \eta_l(\omega_{i,j-1}) \circ \eta_l(\omega_{i,j}) \neq 0.
\]
The converse implication is clear by the equality:

\[ ω_{i+n,j-n} \circ \cdots \circ ω_{i,j-n} \circ \cdots \circ ω_{i,j-1} \circ ω_{i,j} = ρ_b(ω_{i+n,j-n}) \circ \cdots \circ ρ_b(ω_{i,j-n}) \circ η_l(ω_{i,j-1}) \circ η_l(ω_{i,j}) \cdot \ □ \]

**Proposition 3.4.** \( X^\kappa_{ρ,η} \) is a two-dimensional subshift having diagonal property, that is, \( X^\kappa_{ρ,η} \) is a textile dynamical system.

**Proof.** It is easy to see that the set

\[ E = \{ (ω_{i,j})_{(i,j)\in \mathbb{Z}^2} \in Σ^\mathbb{Z}^2 \mid (ω_{i,j})_{(i,j)\in \mathbb{Z}^2 \in X^\kappa_{ρ,η}} \} \]

is closed, because its complement is open in \( Σ^\mathbb{Z}^2 \). The following set

\[ U = \{ (ω_{i,j})_{(i,j)\in \mathbb{Z}^2} \in Σ^\mathbb{Z}^2 \mid ω_{k+n,l-n} \circ ω_{k+n-1,l-n+1} \circ \cdots \circ ω_{k+1,l-1} \circ ω_{k,l} = 0 \text{ for some } (k,l) \in \mathbb{Z}^2, n \in \mathbb{N} \} \]

is open in \( Σ^\mathbb{Z}^2 \). As the equality \( X^\kappa_{ρ,η} = E \cap U^c \) holds, the set \( X^\kappa_{ρ,η} \) is closed. It is also obvious that \( X^\kappa_{ρ,η} \) is translation invariant so that \( X^\kappa_{ρ,η} \) is a two-dimensional subshift. It is easy to see that \( X^\kappa_{ρ,η} \) has diagonal property. □

We call \( X^\kappa_{ρ,η} \) the textile dynamical system associated with

\( (A, ρ, η, Σ^ρ, Σ^η, κ) \).

Let us now define a (one-dimensional) subshift \( X^δ_{ρ,η} \) over \( Σ_κ \), which consists of diagonal sequences of \( X^\kappa_{ρ,η} \) as follows:

\[ X^δ_{ρ,η} = \{ (ω_{n,-n})_{n \in \mathbb{Z}} \in Σ^\mathbb{Z} \mid (ω_{i,j})_{(i,j)\in \mathbb{Z}^2} \in X^\kappa_{ρ,η} \} \].

By Lemma 3.2, an element \( (ω_{n,-n})_{n \in \mathbb{Z}} \) of \( X^δ_{ρ,η} \) may be extended to

\[ (ω_{i,j})_{(i,j)\in \mathbb{Z}^2} \in X^\kappa_{ρ,η} \]

in a unique way. Hence the one-dimensional subshift \( X^δ_{ρ,η} \) determines the two-dimensional subshift \( X^\kappa_{ρ,η} \). Therefore we have:

**Lemma 3.5.** The two-dimensional subshift \( X^\kappa_{ρ,η} \) is not empty if and only if the one-dimensional subshift \( X^δ_{ρ,η} \) is not empty.

For \( (A, ρ, η, Σ^ρ, Σ^η, κ) \), we will have a \( C^* \)-symbolic dynamical system \( (A, δ^κ, Σ_κ) \) in Section 4. It presents the subshift \( X^δ_{ρ,η} \). Since a subshift presented by a \( C^* \)-symbolic dynamical system is always not empty, one sees

**Proposition 3.6.** The two-dimensional subshift \( X^\kappa_{ρ,η} \) is not empty.
4. $C^*$-textile dynamical systems and their $C^*$-algebras

The $C^*$-algebra $O^\kappa_{\rho, \eta}$ is defined to be the universal $C^*$-algebra

$$C^*(x, S_\alpha, T_a; x \in \mathcal{A}, \alpha \in \Sigma^\rho, a \in \Sigma^\eta)$$
generated by $x \in \mathcal{A}$ and partial isometries $S_\alpha, \alpha \in \Sigma^\rho, T_a, a \in \Sigma^\eta$ subject to the following relations called $(\rho, \eta; \kappa)$:

\begin{align}
(4.1) & \quad \sum_{\beta \in \Sigma^\rho} S_{\beta} S_{\beta}^* = 1, \quad xS_\alpha S_{\alpha}^* = S_\alpha S_{\alpha}^* x, \quad S_{\alpha}^* x S_\alpha = \rho_\alpha(x), \\
(4.2) & \quad \sum_{b \in \Sigma^\eta} T_b T_b^* = 1, \quad xT_a T_a^* = T_a T_a^* x, \quad T_a^* x T_a = \eta_a(x), \\
(4.3) & \quad S_\alpha T_b = T_a S_{\beta} \quad \text{if} \quad \kappa(\alpha, b) = (a, \beta)
\end{align}

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$. We will study the algebra $O^\kappa_{\rho, \eta}$. For $(\alpha, b, a, \beta) \in \Sigma^\rho \times \Sigma^\eta \times \Sigma^\eta \times \Sigma^\rho$, we set

$$RB(\alpha, a) = \{(b, \beta) \in \Sigma^\eta \times \Sigma^\rho \mid \kappa(\alpha, b) = (a, \beta)\},$$
$$R(\alpha, a, \beta) = \{b \in \Sigma^\eta \mid \kappa(\alpha, b) = (a, \beta)\},$$
$$R(\alpha, a) = \bigcup_{\beta \in \Sigma^\rho} R(\alpha, a, \beta).$$

**Lemma 4.1.** For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, one has $T_a^* S_\alpha \neq 0$ if and only if $RB(\alpha, a) \neq \emptyset$.

**Proof.** Suppose that $T_a^* S_\alpha \neq 0$. As $T_a^* S_\alpha = \sum_{b' \in \Sigma^\eta} T_{a}^* S_{a} T_{b'} T_{b'}^*$, there exists $b' \in \Sigma^\eta$ such that $T_{a}^* S_{a} T_{b'}^* \neq 0$. Hence $\eta_{b'} \circ \rho_\alpha \neq 0$ so that $(\alpha, b') \in \Sigma^\rho$. Then one may find $(a', b') \in \Sigma^\rho$ such that $\kappa(\alpha, b') = (a', b')$ and hence $S_a T_{b'} = T_{a'} S_{b'}$. Since $0 \neq T_{a}^* S_{a} T_{b'} = T_{a}^* T_{a'} S_{b'}$, one sees that $a = a'$ so that $(b', \beta') \in RB(\alpha, a)$.

Suppose next that $\kappa(\alpha, b) = (a, \beta)$ for some $(b, \beta) \in \Sigma^\eta \times \Sigma^\rho$. Since $\eta_{b} \circ \rho_\alpha = \rho_\beta \circ \eta_a \neq 0$, one has $0 \neq S_a T_{b} = T_{a} S_{\beta}$. It follows that

$$S_b T_a^* S_{\alpha} T_{b} = (T_{a} S_{\beta})^* T_{a} S_{\beta}$$

so that $T_a^* S_\alpha \neq 0$. \hfill \Box

**Lemma 4.2.** For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, we have

\begin{equation}
T_a^* S_\alpha = \sum_{(b, \beta) \in RB(\alpha, a)} S_{\beta} \eta_{b}(\rho_\alpha(1))T_{b}^*
\end{equation}

and hence

\begin{equation}
S_a^* T_a = \sum_{(b, \beta) \in RB(\alpha, a)} T_{b} \rho_\beta(\eta_a(1))S_{\beta}^*.
\end{equation}

**Proof.** We may assume that $T_a^* S_\alpha \neq 0$. One has

$$T_a^* S_\alpha = \sum_{b' \in \Sigma^\eta} T_{a}^* S_a T_{b'} T_{b'}^*.$$
For \( b' \in \Sigma^\eta \) with \((\alpha, b') \in \Sigma^{\eta\eta} \), take \((a', \beta') \in \Sigma^{\eta\eta} \) such that \( \kappa(\alpha, b') = (a', \beta') \) so that

\[
T^*_a S_{\alpha} T^*_a T^*_{b'} = T^*_a S_{\beta} T^*_{b'}.
\]

Hence \( T^*_a S_{\alpha} T^*_a T^*_{b'} \neq 0 \) implies \( a = a' \). Since \( T^*_a T_a = \eta_a(1) \) which commutes with \( S_{\beta} S^*_{\beta'} \), we have

\[
T^*_a T_a S_{\beta} T^*_a T^*_{b'} = S_{\beta} S^*_{\beta'} T_a S_{\alpha} T^*_a T^*_{b'} = S_{\beta} \rho_{\beta'}(\eta_a(1)) T^*_b = S_{\beta} \eta_{\beta'}(\rho_\alpha(1)) T^*_b.
\]

It follows that

\[
T^*_a S_{\alpha} = \sum_{(b', \beta') \in RB(\alpha, a)} T^*_a T_a S_{\beta} T^*_a = \sum_{(b', \beta') \in RB(\alpha, a)} S_{\beta} \eta_{\beta'}(\rho_\alpha(1)) T^*_b. \quad \square
\]

Hence we have:

**Lemma 4.3.** For \( \alpha \in \Sigma^\rho, a \in \Sigma^\eta \), we have

\[
T_a T^*_a S_{\alpha} S^*_a = \sum_{b \in R(\alpha, a)} S_{\alpha} T_b T^*_b S^*_a.
\]

Hence \( T_a T^*_a \) commutes with \( S_{\alpha} S^*_a \).

**Proof.** By (4.4), we have

\[
T_a T^*_a S_{\alpha} S^*_a = \sum_{(b, \beta) \in RB(\alpha, a)} T_a S_{\beta} \eta_{\beta}(\rho_\alpha(1)) T^*_b S^*_a
\]

\[
= \sum_{b \in R(\alpha, a)} S_{\alpha} T_b \eta_{\beta}(\rho_\alpha(1)) T^*_b S^*_a
\]

\[
= \sum_{b \in R(\alpha, a)} S_{\alpha} \rho_\alpha(1) T_b T^*_b S^*_a
\]

\[
= \sum_{b \in R(\alpha, a)} S_{\alpha} T_b T^*_b S^*_a. \quad \square
\]

Recall that \( Z_A \) denotes the center of \( A \) which consists of elements of \( A \) commuting with all elements of \( A \).

**Lemma 4.4.** For \( \alpha \in \Sigma^\rho, a \in \Sigma^\eta \) and \( x, y \in Z_A \), \( T_a y T^*_a \) commutes with \( S_{\alpha} x S^*_a \).
Therefore we have
\[
T_\alpha yT_\alpha^* S_\alpha x S_\alpha^* = T_\alpha y \sum_{(b,\beta) \in RB(\alpha,\alpha)} S_\beta \eta_\beta(\rho_\alpha(1)) T_b^* x S_\alpha^*
\]
\[
= \sum_{(b,\beta) \in RB(\alpha,\alpha)} T_\alpha S_\beta S_\beta^* y S_\beta \eta_\beta(\rho_\alpha(1)) T_b^* x T_b T_b^* S_\alpha^*
\]
\[
= \sum_{(b,\beta) \in RB(\alpha,\alpha)} S_\alpha T_b \rho_\beta(y) \eta_\beta(\rho_\alpha(1)) \eta_\beta(x) S_\beta^* T_b^* S_\alpha^*
\]
\[
= \sum_{(b,\beta) \in RB(\alpha,\alpha)} S_\alpha T_\beta \eta_\beta(x) \eta_\beta(\rho_\alpha(1)) \rho_\beta(y) S_\beta^* T_b^* S_\alpha^*
\]
\[
= \sum_{(b,\beta) \in RB(\alpha,\alpha)} S_\alpha x \rho_\alpha(1) T_b S_\beta^* y T_a^*
\]
\[
= \sum_{(b,\beta) \in RB(\alpha,\alpha)} S_\alpha x S_\alpha^* S_\alpha T_b S_\beta^* S_\alpha^* T_a y T_a^*
\]
\[
= \sum_{b \in R(\alpha,\alpha)} S_\alpha^* S_\alpha T_b T_b^* S_\alpha^* T_a = \sum_{b \in \Sigma^\eta} S_\alpha^* S_\alpha T_b T_b^* S_\alpha^* T_a = S_\alpha^* T_a.
\]
Now if \((\alpha, b') \not\in \Sigma^\rho \eta\), then \(S_\alpha T_{b'} = 0\). Hence
\[
\sum_{b \in R(\alpha,\alpha)} S_\alpha^* S_\alpha T_b T_b^* S_\alpha^* T_a = \sum_{b \in \Sigma^\eta} S_\alpha^* S_\alpha T_b T_b^* S_\alpha^* T_a = S_\alpha^* T_a.
\]
Therefore we have
\[
T_\alpha y T_\alpha^* S_\alpha x S_\alpha^* = S_\alpha^* T_a y T_a^*. \quad \square
\]

For words \(\mu = (\mu_1, \ldots, \mu_j) \in B_j(\Lambda_\rho), \zeta = (\zeta_1, \ldots, \zeta_k) \in B_k(\Lambda_\eta)\), we set
\[
S_\mu = S_{\mu_1} \cdots S_{\mu_j}, \quad T_\zeta = T_{\zeta_1} \cdots T_{\zeta_k}.
\]
For a subset \(F\) of \(O_\rho^\infty \eta^\infty\), denote by \(C^*(F)\) the \(C^*\)-subalgebra of \(O_\rho^\infty \eta^\infty\) generated by the elements of \(F\). We define \(C^*\)-subalgebras \(D_{\rho, \eta}, D_{j, k}\) of \(O_\rho^\infty \eta^\infty\) by
\[
D_{\rho, \eta} = C^*(S_\mu T_\zeta x T_\zeta^* S_\mu^*: \mu \in B_*(\Lambda_\rho), \zeta \in B_*(\Lambda_\eta), x \in A),
\]
\[
D_{j, k} = C^*(S_\mu T_\zeta x T_\zeta^* S_\mu^*: \mu \in B_j(\Lambda_\rho), \zeta \in B_k(\Lambda_\eta), x \in A) \quad \text{for } j, k \in \mathbb{Z}_+.
\]
By the commutation relation (4.3), one sees that
\[
D_{j, k} = C^*(T_\xi S_\nu x S_\nu^* T_\xi^*: \nu \in B_j(\Lambda_\rho), \xi \in B_k(\Lambda_\eta), x \in A).
\]
The identities
\[
S_\mu T_\zeta x T_\zeta^* S_\mu^* = \sum_{a \in \Sigma^\eta} S_\mu T_{\zeta a} \eta_a(x) T_{\zeta a}^* S_\mu^*,
\]
\[
T_\zeta S_\nu x S_\nu^* T_\zeta^* = \sum_{a \in \Sigma^\rho} T_{\zeta a} S_{\nu a} \eta_a(x) S_{\nu a}^* T_{\zeta a}
\]
for \(x \in A\) and \(\mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta)\) yield the embeddings
\[
D_{j, k} \hookrightarrow D_{j, k+1}, \quad D_{j, k} \hookrightarrow D_{j+1, k}
\]
respectively such that $\bigcup_{j,k \in \mathbb{Z}_+} D_{j,k}$ is dense in $D_{\rho,\eta}$.

**Proposition 4.5.** If $\mathcal{A}$ is commutative, so is $D_{\rho,\eta}$.

**Proof.** The preceding lemma tells us that $D_{1,1}$ is commutative. Suppose that the algebra $D_{j,k}$ is commutative for fixed $j, k \in \mathbb{N}$. We will show that the both algebras $D_{j+1,k}$ and $D_{j,k+1}$ are commutative. The algebra $D_{j+1,k}$ consists of the linear span of elements of the form:

$$S_\alpha x S_\alpha^* \quad \text{for } x \in D_{j,k}, \alpha \in \Sigma^\rho.$$  

For $x, y \in D_{j,k}$, $\alpha, \beta \in \Sigma^\rho$, we will show that $S_\alpha x S_\alpha^*$ commutes with both $S_\beta y S_\beta^*$ and $y$. If $\alpha = \beta$, it is easy to see that $S_\alpha x S_\alpha^*$ commutes with $S_\alpha y S_\alpha^*$, because $\rho_\alpha(1) \in A \subset D_{j,k}$. If $\alpha \neq \beta$, both $S_\alpha x S_\alpha^* S_\beta y S_\beta^*$ and $S_\beta y S_\beta^* S_\alpha x S_\alpha^*$ are zeros. Since $S_\alpha^* y S_\alpha \in D_{j-1,k} \subset D_{j,k}$, one sees $S_\alpha^* y S_\alpha$ commutes with $x$. One also sees that $S_\alpha^* S_\alpha = D_{j,k}$ commutes with $y$. It follows that

$$S_\alpha x S_\alpha^* y = S_\alpha x S_\alpha^* y S_\alpha^* S_\alpha = S_\alpha^* S_\alpha y S_\alpha x S_\alpha^* = y S_\alpha x S_\alpha^*.$$ 

Hence the algebra $D_{j+1,k}$ is commutative, and similarly so is $D_{j,k+1}$. By induction, the algebras $D_{j,k}$ are all commutative for all $j, k \in \mathbb{N}$. Since $\bigcup_{j,k \in \mathbb{N}} D_{j,k}$ is dense in $D_{\rho,\eta}$, $D_{\rho,\eta}$ is commutative.

**Proposition 4.6.** Let $\mathcal{O}_{\rho,\eta}^{alg}$ be the dense $*$-subalgebra of $\mathcal{O}_{\rho,\eta}^{\text{alg}}$ algebraically generated by elements $x \in \mathcal{A}$, $S_\alpha, \alpha \in \Sigma^\rho$ and $T_a, a \in \Sigma^n$. Then each element of $\mathcal{O}_{\rho,\eta}^{alg}$ is a finite linear combination of elements of the form:

$$(4.6) \quad S_\mu T_\zeta x S_\nu T_\xi S_\rho^* \quad \text{for } x \in \mathcal{A}, \mu, \nu \in B_*(\Lambda_\rho), \zeta, \xi \in B_*(\Lambda_\eta).$$ 

**Proof.** For $\alpha, \beta \in \Sigma^\rho$, $a, b \in \Sigma^n$ and $x \in \mathcal{A}$, we have

$$S_\alpha^* S_\beta = \begin{cases} \rho_\alpha(1) \in A & \text{if } \alpha = \beta, \\ 0 & \text{otherwise,} \end{cases} \quad T_a^* T_b = \begin{cases} \eta_\alpha(1) \in A & \text{if } a = b, \\ 0 & \text{otherwise,} \end{cases} \quad T_a^* T_a = \sum_{(b,\beta) \in RB(\alpha,a)} T_b \rho_\beta(\eta_\alpha(1)) S_\beta^*,$$ 

$$T_a^* x = \rho_\alpha(x) S_\alpha, \quad T_a^* = \eta_\alpha(x) T_a^*.$$ 

And also

$$S_\beta^* T_a^* = \begin{cases} T_b^* S_\alpha^* & \text{if } (a,\beta) \in \Sigma^{\eta_\rho} \text{ and } (a,\beta) = \kappa(\alpha,b), \\ 0 & \text{if } (a,\beta) \notin \Sigma^{\eta_\rho}. \end{cases}$$

Therefore we conclude that any element of $\mathcal{O}_{\rho,\eta}^{alg}$ is a finite linear combination of elements of the form of $(4.6)$.

Similarly we have:

**Proposition 4.7.** Each element of $\mathcal{O}_{\rho,\eta}^{alg}$ is a finite linear combination of elements of the form:

$$(4.7) \quad T_\zeta S_\mu x S_\nu^* T_\xi S_\rho^* \quad \text{for } x \in \mathcal{A}, \mu, \nu \in B_*(\Lambda_\rho), \zeta, \xi \in B_*(\Lambda_\eta).$$
In the rest of this section, we will have a $C^*$-symbolic dynamical system $(\mathcal{A}, \delta^\kappa, \Sigma_\kappa)$ from $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, which presents the one-dimensional subshift $X_{\delta^\kappa}$ described in the previous section. For $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, define an endomorphism $\delta^\kappa_\omega$ on $\mathcal{A}$ for $\omega \in \Sigma_\kappa$ by setting

$$\delta^\kappa_\omega(x) = \eta_b(\rho_a(x))(= \rho_b(\eta_a(x))), \quad x \in \mathcal{A}, \quad \omega = (\alpha, b, a, \beta) \in \Sigma_\kappa.$$

**Lemma 4.8.** $(\mathcal{A}, \delta^\kappa, \Sigma_\kappa)$ is a $C^*$-symbolic dynamical system that presents $X_{\delta^\kappa}$.

**Proof.** We will show that $\delta^\kappa$ is essential and faithful. Now both $C^*$-symbolic dynamical systems $(\mathcal{A}, \eta, \Sigma^\eta)$ and $(\mathcal{A}, \rho, \Sigma^\rho)$ are essential. Since $\rho_\alpha(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}$ and $\eta_\lambda(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}$, it is clear that $\delta^\kappa_\omega(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}$. By the inequalities

$$\sum_{\omega \in \Sigma_\kappa} \delta^\kappa_\omega(1) = \sum_{b \in \Sigma^\eta} \sum_{\alpha \in \Sigma^\rho} \eta_b(\rho_\alpha(1)) \geq \sum_{b \in \Sigma^\eta} \eta_b(1) \geq 1,$$

$\{\delta^\kappa_\omega\}_{\omega \in \Sigma_\kappa}$ is essential. For any nonzero $x \in \mathcal{A}$, there exists $\alpha \in \Sigma^\rho$ such that $\rho_\alpha(x) \neq 0$ and there exists $b \in \Sigma^\eta$ such that $\eta_b(\rho_\alpha(x)) \neq 0$. Hence $\delta^\kappa$ is faithful so that $(\mathcal{A}, \delta^\kappa, \Sigma_\kappa)$ is a $C^*$-symbolic dynamical system. It is obvious that the subshift presented by $(\mathcal{A}, \delta^\kappa, \Sigma_\kappa)$ is $X_{\delta^\kappa}$. \hfill \Box

Put

$$\tilde{X}_{\rho, \eta}^\kappa = \{(\omega_{i,j})_{(i,j) \in \mathbb{Z}^2} \in \Sigma_{\kappa}^{\mathbb{N}^2} \mid (\omega_{i,j})_{(i,j) \in \mathbb{Z}^2} \in X_{\rho, \eta}^\kappa\}$$

and

$$\tilde{X}_{\delta^\kappa} = \{(\omega_{n,-n})_{n \in \mathbb{N}} \in \Sigma_{\kappa}^{\mathbb{N}} \mid (\omega_{i,j})_{(i,j) \in \mathbb{Z}^2} \in \tilde{X}_{\rho, \eta}^\kappa\}.$$ The latter set $\tilde{X}_{\delta^\kappa}$ is the right one-sided subshift for $X_{\delta^\kappa}$.

**Lemma 4.9.** A configuration $(\omega_{i,j})_{(i,j) \in \mathbb{Z}^2} \in \tilde{X}_{\rho, \eta}^\kappa$ extends to a whole configuration $(\omega_{i,j})_{(i,j) \in \mathbb{Z}^2} \in X_{\rho, \eta}^\kappa$.

**Proof.** For $(\omega_{i,j})_{(i,j) \in \mathbb{Z}^2} \in \tilde{X}_{\rho, \eta}^\kappa$, put $x_i = \omega_{i,-i}$, $i \in \mathbb{N}$ so that $x = (x_i)_{i \in \mathbb{N}} \in \tilde{X}_{\delta^\kappa}$. Since $\tilde{X}_{\delta^\kappa}$ is a one-sided subshift, there exists an extension $\tilde{x} \in X_{\delta^\kappa}$ to two-sided sequence such that $\tilde{x}_i = x_i$ for $i \in \mathbb{N}$. By the diagonal property, $\tilde{x}$ determines a whole configuration $\tilde{\omega}$ to $\mathbb{Z}^2$ such that $\tilde{\omega} \in X_{\delta^\kappa}$ and $(\omega_{i,-i})_{i \in \mathbb{N}} = \tilde{x}$. Hence $\omega_{i,j} = \omega_{i,j}$ for all $i, j \in \mathbb{N}$. \hfill \Box

Let $\mathcal{D}_{\rho, \eta}$ be the $C^*$-subalgebra of $\mathcal{D}_{\rho, \eta}$ defined by

$$\mathcal{D}_{\rho, \eta} = C^*(S_{\mu}T_{\zeta}T_{\xi}^*S_{\mu}^* : \mu \in B_s(\Lambda_\rho), \zeta \in B_s(\Lambda_\eta)) = C^*(T_{\xi}S_{\mu}S_{\delta}T_{\xi}^* : \nu \in B_s(\Lambda_\rho), \xi \in B_s(\Lambda_\eta))$$

which is a commutative $C^*$-subalgebra of $\mathcal{D}_{\rho, \eta}$. Put for $\mu = (\mu_1, \ldots, \mu_n) \in B_s(\Lambda_\rho)$, $\zeta = (\zeta_1, \ldots, \zeta_m) \in B_s(\Lambda_\eta)$ the cylinder set

$$U_{\mu, \zeta} = \{(\omega_{i,j})_{(i,j) \in \mathbb{N}^2} \in \tilde{X}_{\rho, \eta}^\kappa \mid t(\omega_{i,-1}) = \mu_i, i = 1, \ldots, n, r(\omega_{n,j}) = \zeta_j, j = 1, \ldots, m\}.$$ The following lemma is direct.
5. Condition (I) for $C^*$-textile dynamical systems

The notion of condition (I) for finite square matrices with entries in $\{0, 1\}$ has been introduced in [8]. The condition has been generalized by many authors to corresponding conditions for generalizations of the Cuntz–Krieger algebras (cf. [12], [15], [20], [41], etc.). The condition (I) for $C^*$-symbolic dynamical systems (including $\lambda$-graph systems) has been also defined in [29] (cf. [25], [26]). All of these conditions give rise to the uniqueness of the associated $C^*$-algebras subject to some operator relations among certain generating elements.

In this section, we will introduce the notion of condition (I) for $C^*$-textile dynamical systems to prove the uniqueness of the $C^*$-algebras $O^\kappa_{\rho,\eta}$ under the relation $(\rho, \eta; \kappa)$.

Let $(A, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ be a $C^*$-symbolic dynamical system over $\Sigma$ and $X^\kappa_{\rho,\eta}$ the associated two-dimensional subshift. Denote by $\Lambda_\rho, \Lambda_\eta$ the associated subshifts to the $C^*$-symbolic dynamical systems $(A, \rho, \Sigma^\rho), (A, \eta, \Sigma^\eta)$ respectively. For $\mu = (\mu_1, \ldots, \mu_j) \in B_j(\Lambda_\rho), \zeta = (\zeta_1, \ldots, \zeta_k) \in B_k(\Lambda_\eta)$, we put $\rho_\mu = \rho_{\mu_1} \circ \cdots \circ \rho_{\mu_j}, \eta_\kappa = \eta_{\kappa_1} \circ \cdots \circ \eta_{\kappa_k}$ respectively. Recall that $|\mu|, |\zeta|$ denotes the lengths $j, k$ respectively. In the algebra $O^\kappa_{\rho,\eta}$, we set the subalgebras

\[ F_{\rho,\eta} = C^*(S_\mu T_\zeta x T_\xi S_\nu^* : \mu, \nu \in B_s(\Lambda_\rho), \zeta, \xi \in B_s(\Lambda_\eta), |\mu| = |\nu|, |\zeta| = |\xi|, x \in A) \]

and for $j, k \in \mathbb{Z}_+$,

\[ F_{j,k} = C^*(S_\mu T_\zeta x T_\xi S_\nu^* : \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta), x \in A). \]

We notice that

\[ F_{j,k} = C^*(T_\xi S_\mu x S_\nu^* T_\zeta : \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta), x \in A). \]

The identities

\begin{align*}
(5.1) & \quad S_\mu T_\zeta x T_\xi S_\nu^* = \sum_{a \in \Sigma^\eta} S_\mu T_{\zeta a} \eta_a(x) T_{\xi a} S_\nu^*, \\
(5.2) & \quad T_\zeta S_\mu x S_\nu^* T_\xi = \sum_{a \in \Sigma^\rho} T_{\zeta a} S_{\mu a} \rho_a(x) S_{\nu a}^* T_{\xi}
\end{align*}

for $x \in A$ and $\mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta)$ yield the embeddings

\[ i_{*,+1} : F_{j,k} \hookrightarrow F_{j,k+1}, \quad i_{+,+} : F_{j,k} \hookrightarrow F_{j+1,k} \]

respectively, such that $\cup_{j,k \in \mathbb{Z}_+} F_{j,k}$ is dense in $F_{\rho,\eta}$.
By the universality of \( O_{\rho,\eta}^\kappa \) subject to the relations \((\rho, \eta; \kappa)\), we may define an action \( \theta : \mathbb{T}^2 \rightarrow \text{Aut}(O_{\rho,\eta}^\kappa) \) of the two-dimensional torus group

\[
\mathbb{T}^2 = \{(z, w) \in \mathbb{C}^2 \mid |z| = |w| = 1\}
\]
to \( O_{\rho,\eta}^\kappa \) by setting

\[
\theta_{z,w}(S_\alpha) = zS_\alpha, \quad \theta_{z,w}(T_a) = wT_a, \quad \theta_{z,w}(x) = x
\]
for \( \alpha \in \Sigma^\rho, a \in \Sigma^\eta, x \in A \) and \( z, w \in \mathbb{T} \). We call the action \( \theta : \mathbb{T}^2 \rightarrow \text{Aut}(O_{\rho,\eta}^\kappa) \) the gauge action of \( \mathbb{T}^2 \) on \( O_{\rho,\eta}^\kappa \). The fixed point algebra of \( O_{\rho,\eta}^\kappa \) under \( \theta \) is denoted by \( (O_{\rho,\eta}^\kappa)^\theta \). Let \( \mathcal{E}_{\rho,\eta} : O_{\rho,\eta}^\kappa \rightarrow (O_{\rho,\eta}^\kappa)^\theta \) be the conditional expectation defined by

\[
\mathcal{E}_{\rho,\eta}(X) = \int_{(z,w) \in \mathbb{T}^2} \theta_{z,w}(X) \, dzdw, \quad X \in O_{\rho,\eta}^\kappa
\]
where \( dzdw \) means the normalized Haar measure on \( \mathbb{T}^2 \). The following lemma is routine.

**Lemma 5.1.** \( (O_{\rho,\eta}^\kappa)^\theta = \mathcal{D}_{\rho,\eta} \).

Define homomorphisms \( \phi_\rho, \phi_\eta : \mathcal{D}_{\rho,\eta} \rightarrow \mathcal{D}_{\rho,\eta} \) by setting

\[
\phi_\rho(X) = \sum_{\alpha \in \Sigma^\rho} S_\alpha X S_\alpha^*, \quad \phi_\eta(X) = \sum_{a \in \Sigma^\eta} T_a X T_a^*, \quad X \in \mathcal{D}_{\rho,\eta}.
\]
It is easy to see that by (4.3)

\[
\phi_\rho \circ \phi_\eta = \phi_\eta \circ \phi_\rho \quad \text{on} \quad \mathcal{D}_{\rho,\eta}.
\]

**Definition 5.2.** A \( C^* \)-textile dynamical system \((A, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)\) is said to satisfy condition (I) if there exists a unital increasing sequence

\[
\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}
\]
of \( C^* \)-subalgebras of \( A \) such that:

1. \( \rho_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1}, \eta_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1} \) for all \( l \in \mathbb{Z}_+, \alpha \in \Sigma^\rho, a \in \Sigma^\eta \).
2. \( \cup_{l \in \mathbb{Z}_+} \mathcal{A}_l \) is dense in \( A \).
3. For \( \epsilon > 0, j, k, l \in \mathbb{N} \) with \( j + k \leq l \) and \( X_0 \in \mathcal{D}_{j,k} = C^*(S_\mu T_\zeta x T_\nu S_\nu^* : \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta), x \in \mathcal{A}_l) \),

there exists an element

\[
g \in \mathcal{D}_{\rho,\eta} \cap \mathcal{A}_l \setminus \{y \in \mathcal{D}_{\rho,\eta} \mid ya = ay \quad \text{for} \quad a \in \mathcal{A}_l\}
\]
with \( 0 \leq g \leq 1 \) such that:

(i) \( \|X_0 \phi_\rho^n \circ \phi_\eta^m(g)\| \geq \|X_0\| - \epsilon \),
(ii) \( \phi_\rho^n(g) \phi_\eta^m(g) = \phi_\rho^n(\phi_\eta^m(g))g = \phi_\rho^n(g)g = \phi_\eta^m(g)g = 0 \) for all \( n = 1, 2, \ldots, j, \quad m = 1, 2, \ldots, k \).

If in particular, one may take the above subalgebras \( \mathcal{A}_l \subset A, l = 0, 1, 2, \ldots \) to be of finite dimensional, then \((A, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)\) is said to satisfy \( AF \)-condition (I). In this case, \( A = \bigcup_{l=0}^{\infty} \mathcal{A}_l \) is an AF-algebra.
As the element $g$ above belongs to the diagonal subalgebra $D_{\rho, \eta}$ of $\mathcal{F}_{\rho, \eta}$, the condition (I) of $(\mathcal{A}, \rho, \eta, \Sigma^p, \Sigma^\eta, \kappa)$ is intrinsically determined by itself by virtue of Lemma 5.5 below.

We will also introduce the following condition called \textit{free}, which will be stronger than condition (I) but easier to confirm than condition (I).

**Definition 5.3.** A $C^*$-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^p, \Sigma^\eta, \kappa)$ is said to be \textit{free} if there exists a unital increasing sequence $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}$ of $C^*$-subalgebras of $\mathcal{A}$ such that:

1. $\rho_0(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$, $\eta_0(\mathcal{A}_l) \subset \mathcal{A}_{l+1}$ for all $l \in \mathbb{Z}_+$, $\alpha \in \Sigma^p$, $a \in \Sigma^\eta$.
2. $\bigcup_{l \in \mathbb{Z}_+} \mathcal{A}_l$ is dense in $\mathcal{A}$.
3. For $j, k, l \in \mathbb{N}$ with $j + k \leq l$ there exists a projection $q \in D_{\rho, \eta} \cap \mathcal{A}_l$ such that:
   
   (i) $qa \neq 0$ for $0 \neq a \in \mathcal{A}_l$.
   (ii) $\phi^\alpha(q) \phi^\eta(q) = \phi^\alpha(\phi^\eta(q))q = \phi^\eta(q)q = \phi^\eta(q) = 0$ for all $n = 1, 2, \ldots, j$, $m = 1, 2, \ldots, k$.

If in particular, one may take the above subalgebras $\mathcal{A}_l \subset \mathcal{A}$, $l = 0, 1, 2, \ldots$ to be of finite dimensional, then $(\mathcal{A}, \rho, \eta, \Sigma^p, \Sigma^\eta, \kappa)$ is said to be AF-free.

**Proposition 5.4.** If a $C^*$-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^p, \Sigma^\eta, \kappa)$ is free (resp. AF-free), then it satisfies condition (I) (resp. AF-condition (I)).

**Proof.** Assume that $(\mathcal{A}, \rho, \eta, \Sigma^p, \Sigma^\eta, \kappa)$ is free. Take an increasing sequence $\mathcal{A}_l, l \in \mathbb{N}$ of $C^*$-subalgebras of $\mathcal{A}$ satisfying the above conditions (1), (2), (3) of freeness. For $j, k, l \in \mathbb{N}$ with $j + k \leq l$ there exists a projection $q \in D_{\rho, \eta} \cap \mathcal{A}_l$ satisfying the above two conditions (3i) and (3ii). Put

$$Q_{j,k}^l = \phi^\alpha_j(\phi^\eta_k(q)).$$

For $x \in \mathcal{A}_l, \mu, \nu \in B_j(\Lambda_\rho), \xi, \zeta \in B_k(\Lambda_\eta)$, one has the equality

$$Q_{j,k}^l S_{\mu} T^* \xi^* S^*_\nu = S_{\mu} T^* \xi^* S^*_\nu$$

so that $Q_{j,k}^l$ commutes with all of elements of $\mathcal{F}_{j,k}^l$. By using the condition (3i) for $q$ one directly sees that $S_{\mu} T^* \xi^* S^*_\nu \neq 0$ if and only if

$$Q_{j,k}^l S_{\mu} T^* \xi^* S^*_\nu \neq 0.$$

Hence the map

$$X \in \mathcal{F}_{j,k}^l \mapsto XQ_{j,k}^l \in \mathcal{F}_{j,k}^l Q_{j,k}^l$$

defines a homomorphism, that is proved to be injective by a similar proof to the proof of [30, Proposition 3.7]. Hence we have $\|XQ_{j,k}^l\| = \|X\| \geq \|X\| - \epsilon$ for all $X \in \mathcal{F}_{j,k}^l$. \qed

Let $\mathcal{B}$ be a unital $C^*$-algebra. Suppose that there exist an injective $*$-homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{B}$ preserving their units and two families

$$s_\alpha \in \mathcal{B}, \alpha \in \Sigma^p \quad \text{and} \quad t_a \in \mathcal{B}, a \in \Sigma^\eta$$
of partial isometries satisfying
\[ \sum_{\beta \in \Sigma^p} s_\beta s_\beta^* = 1, \quad \pi(x)s_\alpha s_\alpha^* = s_\alpha s_\alpha^* \pi(x), \quad s_\alpha^* \pi(x)s_\alpha = \pi(\rho_\alpha(x)), \]
\[ \sum_{b \in \Sigma^q} t_b t_b^* = 1, \quad \pi(x)t_a t_a^* = t_a t_a^* \pi(x), \quad t_a^* \pi(x)t_a = \pi(\eta_a(x)), \]
\[ s_\alpha t_b = t_a s_\beta \quad \text{if} \quad \kappa(\alpha, b) = (a, \beta) \]
for all \( x \in A \) and \( \alpha \in \Sigma^p, a \in \Sigma^q \). Put \( \tilde{A} = \pi(A) \) and
\[ \tilde{\rho}_\alpha(\pi(x)) = \pi(\rho_\alpha(x)), \quad \tilde{\eta}_a(\pi(x)) = \pi(\eta_a(x)), \quad x \in A. \]

It is easy to see that \((\tilde{A}, \tilde{\rho}, \tilde{\eta}, \Sigma^p, \Sigma^q, \kappa)\) is a \( C^* \)-textile dynamical system such that the presented textile dynamical system \( X^\kappa_{\tilde{\rho}, \tilde{\eta}} \) is the same as the one \( X^\kappa_{\rho, \eta} \) presented by \((A, \rho, \eta, \Sigma^p, \Sigma^q, \kappa)\). Let \( O_{\pi,s,t} \) be the \( C^* \)-subalgebra of \( B \) generated by \( \pi(x) \) and \( s_a, t_a \) for \( x \in A, \alpha \in \Sigma^p, a \in \Sigma^q \). Let \( F_{\pi,s,t} \) be the \( C^* \)-subalgebra of \( O_{\pi,s,t} \) generated by \( s_\mu t_\zeta \pi(x)t_\xi^* \) for \( x \in A \) and \( \mu, \nu \in B_s(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta) \) with \( |\mu| = |\nu|, |\zeta| = |\xi| \). By the universality of the algebra \( O^\kappa_{\rho, \eta} \), the correspondence
\[ x \in A \rightarrow \pi(x) \in \tilde{A}, \quad S_a \rightarrow s_a, \quad \alpha \in \Sigma^p, \quad T_a \rightarrow t_a, \quad a \in \Sigma^q \]
extends to a surjective *-homomorphism \( \tilde{\pi} : O^\kappa_{\tilde{\rho}, \tilde{\eta}} \rightarrow O_{\pi,s,t} \).

**Lemma 5.5.** The restriction of \( \tilde{\pi} \) to the subalgebra \( F_{\tilde{\rho}, \tilde{\eta}} \) is a *-isomorphism from \( F_{\tilde{\rho}, \tilde{\eta}} \) to \( F_{\pi,s,t} \). Hence if \((A, \rho, \eta, \Sigma^p, \Sigma^q, \kappa)\) satisfies condition (I) (resp. is free), \((\tilde{A}, \tilde{\rho}, \tilde{\eta}, \Sigma^p, \Sigma^q, \kappa)\) satisfies condition (I) (resp. is free).

**Proof.** It suffices to show that \( \tilde{\pi} \) is injective on \( F_{j,k} \) for all \( j, k \in \mathbb{Z} \). Suppose
\[ \sum_{\mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta)} s_\mu t_\zeta \pi(x_{\mu, \zeta, \xi, \nu}) t_\xi^* s_\nu^* = 0 \]
with \( x_{\mu, \zeta, \xi, \nu} \in A \). For \( \mu', \nu' \in B_j(\Lambda_\rho), \zeta', \xi' \in B_k(\Lambda_\eta) \), one has
\[ \pi(\eta_{\zeta'}(\rho_{\nu'}(1)) x_{\mu', \zeta', \xi', \nu'}) \eta_{\xi'}(\rho_{\nu'}(1))) \]
\[ = t_{\zeta'}^* s_{\mu'}^* \left( \sum_{\mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta)} s_\mu t_\zeta \pi(x_{\mu, \zeta, \xi, \nu}) t_\xi^* s_\nu^* \right) s_{\nu'} t_{\xi'} = 0. \]
As \( \pi : A \rightarrow B \) is injective, one sees
\[ \eta_{\xi'}(\rho_{\nu'}(1)) x_{\mu', \zeta', \xi', \nu'} \eta_{\xi'}(\rho_{\nu'}(1)) = 0 \]
so that
\[ S_{\mu'} T_{\zeta'} x_{\mu', \zeta', \xi', \nu'} T_{\xi'}^* S_{\nu'}^* = 0. \]
Hence we have
\[ \sum_{\mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta)} S_{\mu} T_{\zeta} x_{\mu, \zeta, \xi, \nu} T_{\xi}^* S_{\nu}^* = 0. \]
Therefore \( \tilde{\pi} \) is injective on \( F_{j,k} \). \( \square \)
We henceforth assume that \((A, \rho, \eta, \Sigma^p, \Sigma^q, \kappa)\) satisfies condition (I) defined above. Take a unital increasing sequence \(\{A_t\}_{t \in \mathbb{Z}_+}\) of \(C^*-\)subalgebras of \(A\) as in the definition of condition (I). Recall that the algebra \(F_{j,k}^t\) for \(j, k \leq l\) is defined by

\[
F_{j,k}^t = C^*(S_{\mu}T_{\nu}x_{\xi}S_{\mu}^*: \mu, \nu \in B_j(A_{\rho}), \xi, \nu \in B_k(A_{\eta}), x \in A_t).
\]

There exists an inclusion relation \(F_{j,k}^t \subseteq F_{j',k'}^{t'}\) for \(j \leq j', k \leq k'\) and \(l \leq l'\) through the identities (5.1), (5.2). Let \(P_{\pi,s,t}\) be the \(*\)-subalgebra of \(O_{\pi,s,t}\) algebraically generated by \(\pi(x), s_{\alpha}, t_{\alpha}\) for \(x \in A_t, l \in \mathbb{Z}_+, \alpha \in \Sigma^p, \alpha \in \Sigma^q\).

**Lemma 5.6.** Any element \(x \in P_{\pi,s,t}\) can be expressed in a unique way as

\[
x = \sum_{|\mu|, |\xi| \geq 1} x_{-\xi,-\nu} t_{\nu} s_{\mu}^* + \sum_{|\xi|, |\nu| \geq 1} t_{\xi} x_{\xi,-\nu} s_{\mu}^* + \sum_{|\mu|, |\xi| \geq 1} s_{\mu} x_{\mu,-\xi} t_{\xi}^* + \sum_{|\mu| \geq 1} s_{\mu} x_{\mu,s} + \sum_{|\nu| \geq 1} t_{\nu} x_{\nu} + x_0
\]

where the above summations \(\Sigma\) are all finite sums and the elements

\[
x_{-\xi,-\nu}, x_{\mu,-\xi}, x_{\mu,\xi}, x_{-\xi,-\nu}, x_{\mu}, x_{\xi}, x_0
\]

for \(\mu, \nu \in B_s(A_{\rho}), \xi, \nu \in B_s(A_{\eta})\) all belong to the dense subalgebra \(P_{\pi,s,t} \cap F_{\pi,s,t}\)

which satisfy

\[
x_{-\xi,-\nu} = x_{-\xi,-\nu} \eta_{\nu}(\rho_{\nu}(1)), \quad x_{\xi,-\nu} = \eta_{\xi}(x_{\xi,-\nu} \rho_{\nu}(1)),
\]

\[
x_{\mu,-\xi} = \rho_{\mu}(1)x_{\mu,-\xi} \eta_{\xi}(1), \quad x_{\mu,\xi} = \eta_{\xi}(\rho_{\mu}(1))x_{\mu,\xi},
\]

\[
x_{-\xi} = x_{-\xi} \eta_{\xi}(1), \quad x_{-\nu} = x_{-\nu} \rho_{\nu}(1),
\]

\[
x_{\mu} = \rho_{\mu}(1)x_{\mu}, \quad x_{\xi} = \eta_{\xi}(1)x_{\xi}.
\]

**Proof.** Put

\[
x_{-\xi,-\nu} = \mathcal{E}_{\rho,\eta}(x_{\mu,s} t_{\xi}), \quad x_{\xi,-\nu} = \mathcal{E}_{\rho,\eta}(t_{\nu}^* x_{\mu,s}),
\]

\[
x_{\mu,-\xi} = \mathcal{E}_{\rho,\eta}(s_{\mu}^{*} t_{\xi}), \quad x_{\mu,\xi} = \mathcal{E}_{\rho,\eta}(t_{\xi}^* s_{\mu}^{*} x),
\]

\[
x_{-\xi} = \mathcal{E}_{\rho,\eta}(x_{\xi}), \quad x_{-\nu} = \mathcal{E}_{\rho,\eta}(x_{\nu}),
\]

\[
x_{\mu} = \mathcal{E}_{\rho,\eta}(s_{\mu}^{*} x), \quad x_{\xi} = \mathcal{E}_{\rho,\eta}(t_{\xi}^* x),
\]

\[
x_0 = \mathcal{E}_{\rho,\eta}(x).
\]

Then we have the desired expression of \(x\). The elements

\[
x_{-\xi,-\nu}, x_{\mu,-\xi}, x_{\mu,\xi}, x_{-\xi,-\nu}, x_{\mu}, x_{\xi}, x_0
\]

for \(\mu, \nu \in B_s(A_{\rho}), \xi, \nu \in B_s(A_{\eta})\) are automatically determined by the above formulae so that the expression is unique. \(\square\)
Lemma 5.7. For \( h \in D_{\rho,\eta} \cap A'_t \) and \( j, k \in \mathbb{Z} \) with \( j + k \leq l \), put
\[
h^{j,k} = \phi^j_{\rho} \circ \phi^k_{\eta}(h).
\]
Then we have
(i) \( h^{j,k}_{\mu} = s_{\mu} h^{j-|\mu|,k} \) for \( \mu \in B_n(\Lambda_\rho) \) with \( |\mu| \leq j \).
(ii) \( h^{j,k,\zeta} = t_\zeta h^{j,k,-|\zeta|} \) for \( \zeta \in B_n(\Lambda_\rho) \) with \( |\zeta| \leq k \).
(iii) \( h^{j,k} \) commutes with any element of \( F^j_{j,k} \).

Proof. (i) It follows that for \( \mu \in B_n(\Lambda_\rho) \) with \( |\mu| \leq j \)
\[
h^{j,k}_{\mu} = \sum_{|\mu'|=|\mu|} s_{\mu'} \phi^{j-|\mu|}_{\rho}(\phi^k_{\eta}(h)) s_{\mu'}^* s_{\mu} = s_{\mu} \phi^{j-|\mu|}_{\rho}(\phi^k_{\eta}(h)) s_{\mu}^* s_{\mu}.
\]
Since \( h \in A'_t \) and \( A_{j+k} \subset A_t \), one has
\[
\phi^{j-|\mu|}_{\rho}(\phi^k_{\eta}(h)) s_{\mu}^* s_{\mu} = \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_{\nu} t_\zeta h^\rho \xi \phi^k_{\eta}(h) s_{\mu}^* s_{\mu}
= \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_{\nu} t_\zeta h^\rho \xi \phi^k_{\eta}(h) s_{\mu}^* s_{\mu} t_\zeta h^\rho \xi s_{\nu}^*
= \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_{\nu} t_\zeta h^\rho \xi (\rho_{\mu\nu}(1)) h^\rho \xi s_{\nu}^*
= \sum_{\nu \in B_{j-|\mu|}(\Lambda_\rho)} \sum_{\xi \in B_k(\Lambda_\eta)} s_{\nu} \rho_{\mu\nu}(1) t_\zeta h^\rho \xi s_{\nu}^*
= s_{\mu} s_{\mu} \phi^{j-|\mu|}_{\rho}(\phi^k_{\eta}(h)) = s_{\mu} h^{j-|\mu|,k}.
\]
so that \( h^{j,k}_{\mu} = s_{\mu} h^{j-|\mu|,k} \).

(ii) Similarly we have \( h^{j,k,\zeta} = t_\zeta h^{j,k,-|\zeta|} \) for \( \zeta \in B_n(\Lambda_\rho) \) with \( |\zeta| \leq k \).

(iii) For \( x \in A_t, \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta) \), we have
\[
h^{j,k}_{\mu} t_{\zeta} = s_{\mu} h^{0,k,\zeta} = s_{\mu} t_{\zeta} h^{0,0} = s_{\mu} t_{\zeta} h.
\]
It follows that
\[
h^{j,k}_{\mu} t_{\zeta} x t_\nu^* s_{\nu}^* = s_{\mu} t_{\zeta} x h t_\nu^* s_{\nu}^* = s_{\mu} t_{\zeta} x h t_\nu^* s_{\nu}^* h^{j,k}.
\]
Hence \( h^{j,k} \) commutes with any element of \( F^j_{j,k} \).

Lemma 5.8. Assume that \( (A, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa) \) satisfies condition (I). For \( x \in \mathcal{P}_{\pi,s,t} \), let \( x_0 = E_{\rho,\eta}(x) \) as in Lemma 5.6. Then we have
\[
\|x_0\| \leq \|x\|.
\]
Proof. We may assume that the elements for \( x \in \mathcal{P}_{\pi,s,t} \)
\[
x-\xi, -\nu, x_\zeta -\nu, x_{\mu,-\xi}, x_{\mu,\zeta}, x-\xi, x-\nu, x_{\mu}, x_{\xi}, x_0
\]
in Lemma 5.6 belong to \( \bar{\pi}(F^j_{j,k}) \) for some \( j_1, k_1 \) and \( \mu, \nu \in \cup_{n=0}^{j_0} B_n(\Lambda_\rho) \),
\( \zeta, \xi \in \cup_{n=0}^{j_0} B_n(\Lambda_\eta) \) for some \( j_0, k_0 \). Take \( j, k, l \in \mathbb{Z}_+ \) such as
\[
j \geq j_0 + j_1, \quad k \geq k_0 + k_1, \quad l \geq \max\{j + k, l_1\}.
\]
By Lemma 5.5, \((\hat{A}, \hat{\rho}, \hat{\eta}, \Sigma^o, \Sigma^n, \kappa)\) satisfies condition (I). For any \(\epsilon > 0\), the numbers \(j, k, l\), and the element \(x_0 \in \hat{\pi}(\mathcal{F}_{j_1,k_1})\), one may find \(g \in \hat{\pi}(\mathcal{D}_{\rho,\eta}) \cap \pi(\mathcal{A}_t)\) with \(0 \leq g \leq 1\) such that:

(i) \(\|x_0\phi_p \circ \phi^k_{\eta}(g)\| \geq \|x_0\| - \epsilon\).

(ii) \(\phi_p^m(g)\phi^m_{\eta}(g) = \phi_p^m(\phi^m_{\eta}(g))g = \phi^m_{\eta}(g)g = 0\) for all \(n = 1, 2, \ldots, j, m = 1, 2, \ldots, k\).

Put \(h = g^{1/2}\) and \(h^{j,k} = \phi_p^j \circ \phi^k_{\eta}(h)\). It follows that \(\|x\| \geq \|h^{j,k}xh^{j,k}\|\) and

\[\|h^{j,k}xh^{j,k}\| = \|1 + (2) + (3) + (4) + (5) + (6)\|\]

where the summands are given by

\[(1) \sum_{|\nu|,|\xi| \geq 1} h^{j,k}x_{-\xi,-\nu}t^*_\xi s^*_\nu h^{j,k} \]

\[(2) \sum_{|\eta|,|\zeta| \geq 1} h^{j,k}t_\eta x_{\zeta,-\nu} s^*_\nu h^{j,k} \]

\[(3) \sum_{|\mu|,|\xi| \geq 1} h^{j,k}s_\mu x_{\mu,-\xi} t^*_\xi h^{j,k} \]

\[(4) \sum_{|\nu|,|\zeta| \geq 1} h^{j,k}s_\nu t_\zeta x_{\mu,-\zeta} h^{j,k} \]

\[(5) \sum_{|\xi| \geq 1} h^{j,k}x_{-\xi} t^*_\xi h^{j,k} + \sum_{|\nu| \geq 1} h^{j,k}x_{-\nu} s^*_\nu h^{j,k} + \sum_{|\mu| \geq 1} h^{j,k}s_\mu x_{\mu} h^{j,k} \]

\[\sum_{|\zeta| \geq 1} h^{j,k}t_\zeta x_{\xi} h^{j,k} \]

\[(6) h^{j,k}x_0 h^{j,k} \].

For (1), as \(x_{-\xi, -\nu} \in \hat{\pi}(\mathcal{F}_{j_1,k_1}) \subset \hat{\pi}(\mathcal{F}_{j,k})\), one sees that \(x_{-\xi, -\nu}\) commutes with \(h^{j,k}\). Hence we have

\[h^{j,k}x_{-\xi, -\nu} t^*_\xi s^*_\nu h^{j,k} = x_{-\xi, -\nu} h^{j,k} t^*_\xi s^*_\nu h^{j,k} = x_{-\xi, -\nu} h^{j,k} h^{j,\nu, j, k}_{-|\nu, |k-|\xi|} t^*_\xi s^*_\nu \]

and

\[h^{j,k} h^{j,\nu, j, k}_{-|\nu, |k-|\xi|} (h^{j,k} h^{j,\nu, j, k}_{-|\nu, |k-|\xi|})^* = \phi^j_{\rho}(\phi^j_{\eta}(g)) \cdot \phi^j_{\rho}(\phi^j_{\eta}(g)) = \phi^j_{\rho}(\phi^j_{\eta}(g)) \cdot \phi^j_{\rho}(\phi^j_{\eta}(g)) = 0 \]

so that

\[h^{j,k} x_{-\xi, -\nu} t^*_\xi s^*_\nu h^{j,k} = 0.\]

For (2), as \(x_{\xi, -\nu} \in \hat{\pi}(\mathcal{F}_{j_1,k_1}) \subset \hat{\pi}(\mathcal{F}_{j,k-|\zeta|})\), one sees that \(x_{\xi, -\nu}\) commutes with \(h^{j,k}_{-|\xi|}\). Hence we have

\[h^{j,k} t_\xi x_{\xi, -\nu} s^*_\nu h^{j,k} = t_\xi h^{j,k}_{-|\xi|} x_{\xi, -\nu} h^{j,\nu, j, k}_{-|\nu, |k-|\xi|} s^*_\nu = t_\xi x_{\xi, -\nu} h^{j,k}_{-|\xi|} h^{j,\nu, j, k}_{-|\nu, |k-|\xi|} s^*_\nu.\]
and
\[ h^{j,k-|\xi|}h^{j-|\nu|,k}(h^{j,k-|\xi|}h^{j-|\nu|,k})^* = \phi_p^j(\phi_\eta^k(\xi)(g)) \cdot \phi_p^{j-|\nu|}(\phi_\eta^k(g)) = 0 \]
so that
\[ h^{j,k}t_\xi x_\xi - \nu s_\nu h^{j,k} = 0. \]

For (3), as \( x_{\mu,-\zeta} \in \pi(\mathcal{F}_{j_1,k_1}) \subset \pi(\mathcal{F}_{j-|\mu|,k}), \) one sees that \( x_{\mu,-\zeta} \) commutes with \( h^{j-|\mu|,k}. \) Hence we have
\[ h^{j,k}s_\mu x_{\mu,-\zeta}t_\xi h^{j,k} = s_\mu h^{j-|\mu|,k}x_{\mu,-\zeta}h^{j,k-|\xi|}t_\xi^* = s_\mu x_{\mu,-\zeta}h^{j-|\mu|,k}h^{j,k-|\xi|}t_\xi^* \]
and
\[ h^{j-|\mu|,k}h^{j,k-|\xi|}(h^{j-|\mu|,k}h^{j,k-|\xi|})^* = \phi_p^{j-|\mu|}(\phi_\eta^k(\xi)(g)) \cdot \phi_p^{j-|\mu|}(\phi_\eta^k(\xi)(g)) = 0 \]
so that
\[ h^{j,k}s_\mu x_{\mu,-\zeta}t_\xi^* h^{j,k} = 0. \]

For (4), as \( x_{\mu,\zeta} \in \pi(\mathcal{F}_{j_1,k_1}) \subset \pi(\mathcal{F}_{j-|\mu|,k,-|\zeta|}), \) one sees that \( x_{\mu,\zeta} \) commutes with \( h^{j,\mu,k-|\zeta|}. \) Hence we have
\[ h^{j,k}s_\mu t_\zeta x_{\mu,\zeta}h^{j,k} = s_\mu t_\zeta h^{j,\mu,k-|\zeta|}x_{\mu,\zeta}h^{j,k} = s_\mu t_\zeta x_{\mu,\zeta}h^{j-|\mu|,k-|\zeta|}h^{j,k} \]
and
\[ h^{j-|\mu|,k-|\zeta|}h^{j,k}(h^{j-|\mu|,k-|\zeta|}h^{j,k})^* = \phi_p^{j-|\mu|}(\phi_\eta^k(\xi)(g)) \cdot \phi_p^{j-|\mu|}(\phi_\eta^k(\xi)(g)) = 0 \]
so that
\[ h^{j,k}s_\mu t_\zeta x_{\mu,\zeta}h^{j,k} = 0. \]

For (5), as \( x_{-\zeta} \) commutes with \( h^{j,k}, \) we have
\[ h^{j,k}x_{-\zeta}t_\xi^* h^{j,k} = x_{-\zeta}h^{j,k}h^{j,k-|\xi|}t_\xi^* \]
and
\[ h^{j,k}h^{j,k-|\zeta|}(h^{j,k}h^{j,k-|\zeta|})^* = \phi_p^{j}(\phi_\eta^k(\xi)(g)) \cdot \phi_p^{j}(\phi_\eta^k(\xi)(g)) = 0 \]
so that
\[ h^{j,k}x_{-\zeta}t_\xi^* h^{j,k} = 0. \]

We similarly see that
\[ h^{j,k}x_{-\nu}t_\zeta h^{j,k} = h^{j,k}t_\zeta x_\zeta h^{j,k} = h^{j,k}t_\zeta x_\zeta h^{j,k} = 0. \]

Therefore we have
\[ \|x\| \geq \|h^{j,k}x_0h^{j,k}\| = \|x_0(h^{j,k})^2\| = \|x_0\phi_p^j \circ \phi_\eta^k(g)\| \geq \|x_0\| - \epsilon. \]

By a similar argument to [8, 2.8 Proposition], one sees:
Corollary 5.9. Assume \((A, \rho, \eta, \Sigma^p, \Sigma^n, \kappa)\) satisfies condition (I). There exists a conditional expectation \(E_{\pi, s, t} : \mathcal{O}_{\pi, s, t} \rightarrow \mathcal{F}_{\pi, s, t}\) such that
\[
E_{\pi, s, t} \circ \tilde{\pi} = \tilde{\pi} \circ E_{\rho, \eta}.
\]
Therefore we have

Proposition 5.10. Assume that \((A, \rho, \eta, \Sigma^p, \Sigma^n, \kappa)\) satisfies condition (I). The *-homomorphism \(\tilde{\pi} : \mathcal{O}_{\rho, \eta}^\kappa \rightarrow \mathcal{O}_{\pi, s, t}\) defined by
\[
\tilde{\pi}(x) = \pi(x), \quad x \in A,
\]
\[
\tilde{\pi}(S_{\alpha}) = s_{\alpha}, \quad \alpha \in \Sigma^p,
\]
\[
\tilde{\pi}(T_{a}) = t_{a}, \quad a \in \Sigma^n
\]
becomes a surjective *-isomorphism, and hence the \(C^*\)-algebras \(\mathcal{O}_{\rho, \eta}^\kappa\) and \(\mathcal{O}_{\pi, s, t}\) are canonically *-isomorphic through \(\tilde{\pi}\).

Proof. The map \(\tilde{\pi} : \mathcal{F}_{\rho, \eta} \rightarrow \mathcal{F}_{\pi, s, t}\) is *-isomorphic and satisfies \(E_{\pi, s, t} \circ \tilde{\pi} = \tilde{\pi} \circ E_{\rho, \eta}\). Since \(E_{\rho, \eta} : \mathcal{O}_{\rho, \eta}^\kappa \rightarrow \mathcal{F}_{\rho, \eta}\) is faithful, a routine argument shows that the *-homomorphism \(\tilde{\pi} : \mathcal{O}_{\rho, \eta}^\kappa \rightarrow \mathcal{O}_{\pi, s, t}\) is actually a *-isomorphism.

Hence the following uniqueness of the \(C^*\)-algebra \(\mathcal{O}_{\rho, \eta}^\kappa\) holds.

Theorem 5.11. Assume that \((A, \rho, \eta, \Sigma^p, \Sigma^n, \kappa)\) satisfies condition (I). The \(C^*\)-algebra \(\mathcal{O}_{\rho, \eta}^\kappa\) is the unique \(C^*\)-algebra subject to the relation \((\rho, \eta; \kappa)\). This means that if there exist a unital \(C^*\)-algebra \(\mathcal{B}\), an injective *-homomorphism \(\pi : A \rightarrow B\) and two families of partial isometries \(s_{\alpha}, \alpha \in \Sigma^p, t_a, a \in \Sigma^n\) satisfying the following relations:
\[
\sum_{\beta \in \Sigma^p} s_{\alpha} \pi(x) s_{\beta}^* = s_{\alpha} s_{\beta}^* \pi(x), \quad s_{\alpha} \pi(x) s_{\alpha} = \pi(\rho_{\alpha}(x)),
\]
\[
\sum_{b \in \Sigma^n} t\pi(x) t_b^* = t_{\beta} t_{\alpha}^* \pi(x), \quad t_{\alpha} \pi(x) t_{\alpha} = \pi(\kappa_{\alpha}(x))
\]
then the correspondence
\[
x \in A \rightarrow \pi(x) \in B, \quad S_{\alpha} \rightarrow s_{\alpha} \in B, \quad T_{a} \rightarrow t_{a} \in B
\]
extends to a *-isomorphism \(\tilde{\pi}\) from \(\mathcal{O}_{\rho, \eta}^\kappa\) onto the \(C^*\)-subalgebra \(\mathcal{O}_{\pi, s, t}\) of \(\mathcal{B}\) generated by \(\pi(x), x \in A\) and \(s_{\alpha}, t_a, \alpha \in \Sigma^p, a \in \Sigma^n\).

For a \(C^*\)-textile dynamical system \((A, \rho, \eta, \Sigma^p, \Sigma^n, \kappa)\), let \(\lambda_{\rho, \eta} : A \rightarrow A\) be the positive map on \(A\) defined by
\[
\lambda_{\rho, \eta}(x) = \sum_{\alpha \in \Sigma^p, a \in \Sigma^n} \eta_{\alpha} \circ \rho_{\alpha}(x), \quad x \in A.
\]
Then \((A, \rho, \eta, \Sigma^p, \Sigma^n, \kappa)\) is said to be irreducible if there exists no nontrivial ideal of \(A\) invariant under \(\lambda_{\rho, \eta}\).

Corollary 5.12. If \((A, \rho, \eta, \Sigma^p, \Sigma^n, \kappa)\) satisfies condition (I) and is irreducible, the \(C^*\)-algebra \(\mathcal{O}_{\rho, \eta}^\kappa\) is simple.
Proof. Assume that there exists a nontrivial ideal \( \mathcal{I} \) of \( \mathcal{O}_{\rho, \eta}^\kappa \). Now suppose that \( \mathcal{I} \cap A = \{0\} \). As \( S_\alpha^* S_\alpha = \rho_\alpha(1), T_\alpha^* T_\alpha = \eta_\alpha(1) \in A \), one knows that \( S_\alpha, T_\alpha \notin \mathcal{I} \) for all \( \alpha \in \Sigma^p, \eta \in \Sigma^q \). By the above theorem, the quotient map \( q : \mathcal{O}_{\rho, \eta}^\kappa \rightarrow \mathcal{O}_{\rho, \eta}^\kappa / \mathcal{I} \) must be injective so that \( \mathcal{I} \) is trivial. Hence one sees that \( \mathcal{I} \cap A \neq \{0\} \) and it is invariant under \( \lambda_{\rho, \eta} \). \( \square \)

6. Concrete realization

In this section we will realize the \( C^* \)-algebra \( \mathcal{O}_{\rho, \eta}^\kappa \) for \( (A, \rho, \eta, \Sigma^p, \Sigma^q, \kappa) \) in a concrete way as a \( C^* \)-algebra constructed from a Hilbert \( C^* \)-bimodule. For \( \gamma_i \in \Sigma^p \cup \Sigma^q \), put

\[
\xi_{\gamma_i} = \begin{cases} 
\rho_{\gamma_i} & \text{if } \gamma_i \in \Sigma^p, \\
\eta_{\gamma_i} & \text{if } \gamma_i \in \Sigma^q.
\end{cases}
\]

A finite sequence of labels \( (\gamma_1, \gamma_2, \ldots, \gamma_k) \in (\Sigma^p \cup \Sigma^q)^k \) is said to be concatenated labeled path if \( \xi_{\gamma_k} \circ \cdots \circ \xi_{\gamma_2} \circ \xi_{\gamma_1}(1) \neq 0 \). For \( m, n \in \mathbb{Z}_+ \), let \( L_{(n,m)} \) be the set of concatenated labeled paths \( (\gamma_1, \gamma_2, \ldots, \gamma_{m+n}) \) such that symbols in \( \Sigma^p \) appear in \( (\gamma_1, \gamma_2, \ldots, \gamma_{m+n}) \) \( n \)-times and symbols in \( \Sigma^q \) appear in \( (\gamma_1, \gamma_2, \ldots, \gamma_{m+n}) \) \( m \)-times. We define a relation in \( L_{(n,m)} \) for \( i = 1, 2, \ldots, n + m - 1 \). We write

\[
(\gamma_1, \ldots, \gamma_{i-1}, \gamma_i, \gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_{m+n}) \\
\approx_i (\gamma_1, \ldots, \gamma_{i-1}, \gamma'_i, \gamma'_{i+1}, \gamma_{i+2}, \ldots, \gamma_{m+n})
\]

if one of the following two conditions holds:

1. \( (\gamma_i, \gamma_{i+1}) \in \Sigma^p, (\gamma'_i, \gamma'_{i+1}) \in \Sigma^q \) and \( \kappa(\gamma_i, \gamma_{i+1}) = (\gamma'_i, \gamma'_{i+1}) \),

2. \( (\gamma_i, \gamma_{i+1}) \in \Sigma^q, (\gamma'_i, \gamma'_{i+1}) \in \Sigma^p \) and \( \kappa(\gamma_i, \gamma_{i+1}) = (\gamma'_i, \gamma'_{i+1}) \).

Denote by \( \approx \) the equivalence relation in \( L_{(n,m)} \) generated by the relations \( \approx_i, i = 1, 2, \ldots, n + m - 1 \). Let \( \mathcal{I}_{(n,m)} = L_{(n,m)}/\approx \) be the set of equivalence classes of \( L_{(n,m)} \) under \( \approx \). Denote by \( \gamma \in \mathcal{I}_{(n,m)} \) the equivalence class of \( \gamma \in L_{(n,m)} \). Put the vectors \( e = (1, 0), f = (0, -1) \in \mathbb{R}^2 \). Consider the set of all paths consisting of sequences of vectors \( e, f \) starting at the point \( (-n, m) \in \mathbb{R}^2 \) for \( n, m \in \mathbb{Z}_+ \) and ending at the origin. Such a path consists of \( n \) \( e \)-vectors and \( m \) \( f \)-vectors. Let \( \mathcal{P}_{(n,m)} \) be the set of all such paths from \( (-n, m) \) to the origin. We consider the correspondence

\[
\rho_\alpha \rightarrow e \quad (\alpha \in \Sigma^p), \\
\eta_\alpha \rightarrow f \quad (\eta \in \Sigma^q),
\]

denoted by \( \pi \). It extends a surjective map from \( L_{(n,m)} \) to \( \mathcal{P}_{(n,m)} \) in a natural way. For a concatenated labeled path \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{n+m}) \in L_{(n,m)} \), put the projection in \( A \)

\[
P_\gamma = (\xi_{\gamma_{n+m}} \circ \cdots \circ \xi_{\gamma_2} \circ \xi_{\gamma_1})(1).
\]

We note that \( P_\gamma \neq 0 \) for all \( \gamma \in L_{(n,m)} \).
Lemma 6.1. For \( \gamma, \gamma' \in L_{(n,m)} \), if \( \gamma \approx \gamma' \), we have \( P_{\gamma} = P_{\gamma'} \). Hence the projection \( P_{[\gamma]} \) for \( [\gamma] \in \mathfrak{S}_{(n,m)} \) is well-defined.

Proof. If \( \kappa(\alpha, b) = (a, \beta) \), one has \( \eta_{b} \circ \rho_{\alpha}(1) = \rho_{\beta} \circ \eta_{a}(1) \neq 0 \). Hence the assertion is obvious. \( \square \)

Denote by \( |\mathfrak{S}_{(n,m)}| \) the cardinal number of the finite set \( \mathfrak{S}_{(n,m)} \). Let \( e_t, t \in \mathfrak{S}_{(n,m)} \) be the standard complete orthonormal basis of \( C^{\mathfrak{S}_{(n,m)}} \). Define

\[
H_{(n,m)} = \sum_{t \in \mathfrak{S}_{(n,m)}} \langle C e_t \otimes P_t A \rangle
\]

the direct sum of \( C e_t \otimes P_t A \) over \( t \in \mathfrak{S}_{(n,m)} \). \( H_{(n,m)} \) has a structure of \( C^{*} \)-bimodule over \( A \) by setting

\[
(e_t \otimes P_t x)y := e_t \otimes P_t xy, \\
\phi(y)(e_t \otimes P_t x) := e_t \otimes \xi_\gamma(y)x = e_t \otimes P_t \xi_\gamma(y)x \quad \text{for } x, y \in A
\]

where \( t = [\gamma] \) for \( \gamma = (\gamma_1, \ldots, \gamma_{n+m}) \) and \( \xi_\gamma(y) = (\xi_{\gamma_{n+m}} \circ \cdots \circ \xi_{\gamma_2} \circ \xi_{\gamma_1})(y) \).

Define an \( A \)-valued inner product on \( H_{(n,m)} \) by setting

\[
|e_t \otimes P_t x | e_s \otimes P_s y | := \begin{cases} x^* P_t y & \text{if } t = s, \\ 0 & \text{otherwise} \end{cases}
\]

for \( t, s \in \mathfrak{S}_{(n,m)} \) and \( x, y \in A \). Then \( H_{(n,m)} \) becomes a Hilbert \( C^{*} \)-bimodule over \( A \). Put \( H_{(0,0)} = A \). Denote by \( F_\kappa \) the Hilbert \( C^{*} \)-bimodule over \( A \) defined by the direct sum:

\[
F_\kappa = \sum_{(n,m) \in \mathbb{Z}_+^2} \langle H_{(n,m)} \rangle.
\]

For \( \alpha \in \Sigma^0, a \in \Sigma^n \), the creation operators \( s_\alpha, t_\alpha \) on \( F_\kappa \):

\[
s_\alpha : H_{(n,m)} \rightarrow H_{(n+1,m)}, \\
t_\alpha : H_{(n,m)} \rightarrow H_{(n,m+1)}
\]

are defined by

\[
s_\alpha x = e_{[\alpha]} \otimes P_{[\alpha]} x, \quad \text{for } x \in H_{(0,0)} (= A),
\]

\[
s_\alpha(e_{[\gamma]} \otimes P_{[\gamma]} x) = \begin{cases} e_{[\alpha\gamma]} \otimes P_{[\alpha\gamma]} x & \text{if } \alpha \gamma \in L_{(n+1,m)}, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
t_\alpha x = e_{[\alpha]} \otimes P_{[\alpha]} x, \quad \text{for } x \in H_{(0,0)} (= A),
\]

\[
t_\alpha(e_{[\gamma]} \otimes P_{[\gamma]} x) = \begin{cases} e_{[\alpha\gamma]} \otimes P_{[\alpha\gamma]} x & \text{if } a \gamma \in L_{(n,m+1)}, \\ 0 & \text{otherwise}. \end{cases}
\]

For \( y \in A \) an operator \( i_{F_\kappa}(y) \) on \( F_\kappa \):

\[
i_{F_\kappa}(y) : H_{(n,m)} \rightarrow H_{(n,m)}
\]
Lemma 6.2. Define the Cuntz–Toeplitz $C^*$-algebra for $(A, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ by

$$T_{\rho,\eta}^\kappa = C^*(s_\alpha, t_\alpha, i_{F_\kappa}(y) \mid \alpha \in \Sigma^\rho, a \in \Sigma^\eta, y \in A)$$

as the $C^*$-algebra on $F_\kappa$ generated by $s_\alpha, t_\alpha, i_{F_\kappa}(y)$ for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta, y \in A$.

Lemma 6.3. For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, we have

(i) $s_\alpha^* (e_{[\gamma]} \otimes P_{[\gamma]}x) = \begin{cases} \phi(\rho_\alpha(1))(e_{[\gamma]} \otimes P_{[\gamma]}x) & \text{if } \gamma \approx \gamma', \\ 0 & \text{otherwise.} \end{cases}$

(ii) $t_\alpha^* (e_{[\gamma]} \otimes P_{[\gamma]}x) = \begin{cases} \phi(\eta_\alpha(1))(e_{[\gamma]} \otimes P_{[\gamma]}x) & \text{if } \gamma \approx a\gamma', \\ 0 & \text{otherwise.} \end{cases}$

Proof. (i) For $\gamma \in L_{(n,m)}, \gamma' \in L_{(n-1,m)}$ and $\alpha \in \Sigma^\rho$, we have

$$\langle s_\alpha^* (e_{[\gamma]} \otimes P_{[\gamma]}x) \mid e_{[\gamma']} \otimes P_{[\gamma']}x' \rangle = \langle e_{[\gamma]} \otimes P_{[\gamma]}x \mid e_{[\alpha\gamma']} \otimes P_{[\alpha\gamma']}x' \rangle = \begin{cases} x^*P_{[\alpha\gamma']}x & \text{if } \gamma \approx \alpha\gamma', \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$\phi(\rho_\alpha(1))(e_{[\gamma]} \otimes P_{[\gamma]}x) = e_{[\gamma]} \otimes P_{[\alpha\gamma']}x = e_{[\gamma]} \otimes P_{[\alpha\gamma]}x$$

so that

$$\langle \phi(\rho_\alpha(1))(e_{[\gamma]} \otimes P_{[\gamma]}x) \mid e_{[\gamma']} \otimes P_{[\gamma']}x' \rangle = x^*P_{[\alpha\gamma']}x'.$$

Hence we obtain the desired equality. Similarly we see (ii). \hfill \Box

The following lemma is straightforward.

Lemma 6.4. For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and $\gamma \in L_{(n,m)}, x \in A$, we have:

(i) $s_\alpha s_\alpha^* (e_{[\gamma]} \otimes P_{[\gamma]}x) = \begin{cases} e_{[\gamma]} \otimes P_{[\gamma]}x & \text{if } \gamma \approx \gamma' \text{ for some } \gamma' \in L_{(n-1,m)}, \\ 0 & \text{otherwise.} \end{cases}$

(ii) $t_\alpha t_\alpha^* (e_{[\gamma]} \otimes P_{[\gamma]}x) = \begin{cases} e_{[\gamma]} \otimes P_{[\gamma]}x & \text{if } \gamma \approx a\gamma' \text{ for some } \gamma' \in L_{(n,m-1)}, \\ 0 & \text{otherwise.} \end{cases}$

Hence we see:

Lemma 6.4. (i) $1 - \sum_{\alpha \in \Sigma^\rho} s_\alpha s_\alpha^* = \text{the projection onto the subspace spanned by the vectors } e_{[\gamma]} \otimes P_{[\gamma]}x \text{ for } \gamma \in \bigcup_{m=0}^\infty L_{(0,m)}, x \in A.$
(ii) \( 1 - \sum_{a \in \Sigma^\rho} t_a t_a^* \) is the projection onto the subspace spanned by the vectors \( e_{[\gamma]} \otimes P_{[\gamma]} x \) for \( \gamma \in \bigcup_{n=0}^\infty L(n,0), x \in A \).

**Lemma 6.5.** For \( \alpha \in \Sigma^\rho, a \in \Sigma^n \) and \( x \in A \), we have:

(i) \( s_a^* x s_a = \phi(\rho_a(x)) \) and in particular \( s_a^* s_a = \phi(\rho_a(1)) \).

(ii) \( t_a^* x t_a = \phi(\eta_a(x)) \) and in particular \( t_a^* t_a = \phi(\eta_a(1)) \).

**Proof.** (i) It follows that for \( \gamma \in L(n, m) \) with \( \alpha \gamma \in L(n+1, m) \) and \( y \in A \),

\[
s_a^* x s_a (e_{[\gamma]} \otimes P_{[\gamma]} y) = s_a^* (e_{[\alpha\gamma]} \otimes P_{[\alpha\gamma]} y \xi_{\alpha\gamma}(x))
= e_{[\gamma]} \otimes P_{[\gamma]} y \xi_{\alpha\gamma}(\rho_a(x))
= \phi(\rho_a(x)) (e_{[\gamma]} \otimes P_{[\gamma]} y).
\]

If \( \alpha \gamma \notin L(n+1, m) \), we have

\[
s_a (e_{[\gamma]} \otimes P_{[\gamma]} y) = 0, \quad \phi(\rho_a(x)) (e_{[\gamma]} \otimes P_{[\gamma]} y) = 0.
\]

Hence we see that \( s_a^* x s_a = \phi(\rho_a(x)) \). Similarly we see (ii). \( \square \)

**Lemma 6.6.** For \( \alpha, \beta \in \Sigma^\rho, a, b \in \Sigma^n \) we have:

(6.1) \( s_a t_b = t_b s_a \) if \( \kappa(\alpha, b) = (a, \beta) \).

**Proof.** For \( \gamma \in L(n, m) \) with \( \alpha b \gamma, a \beta \gamma \in L(n+1, m+1) \) and \( x \in A \), we have

\[
s_a t_b (e_{[\gamma]} \otimes P_{[\gamma]} x) = e_{[a \beta \gamma]} \otimes P_{[a \beta \gamma]} y),
\]

\[
t_a s_b (e_{[\gamma]} \otimes P_{[\gamma]} x) = (e_{[a \beta \gamma]} \otimes P_{[a \beta \gamma]}) x.
\]

Since \( \kappa(\alpha, b) = (a, \beta) \), the condition \( \alpha b \gamma \in L(n+1, m+1) \) is equivalent to the condition \( a \beta \gamma \in L(n+1, m+1) \). We then have \( [a \beta \gamma] = [a \beta \gamma] \) and \( P_{[a \beta \gamma]} = P_{[a \beta \gamma]} \). \( \square \)

Let \( \mathcal{T}_{\rho, \eta}^\kappa \) be the ideal of \( \mathcal{T}_{\rho, \eta}^\kappa \) generated by the two projections:

\[
1 - \sum_{a \in \Sigma^\rho} s_a s_a^* \quad \text{and} \quad 1 - \sum_{a \in \Sigma^n} t_a t_a^*.
\]

Let \( \widehat{\mathcal{O}}_{\rho, \eta}^\kappa \) be the quotient \( C^* \)-algebra

\[
\widehat{\mathcal{O}}_{\rho, \eta}^\kappa = \mathcal{T}_{\rho, \eta}^\kappa / \mathcal{T}_{\rho, \eta}^\kappa.
\]

Let \( \pi_{\rho, \eta} : \mathcal{T}_{\rho, \eta}^\kappa \to \widehat{\mathcal{O}}_{\rho, \eta}^\kappa \) be the quotient map. Put

\[
\hat{S}_a = \pi_{\rho, \eta}(s_a), \quad \hat{T}_a = \pi_{\rho, \eta}(t_a), \quad \hat{i}(x) = \pi_{\rho, \eta}(i_{(F_a)})(x)
\]

for \( \alpha \in \Sigma^\rho, a \in \Sigma^n \) and \( x \in A \). By the above discussions, the following relations hold:

\[
\sum_{\beta \in \Sigma^n} \hat{S}_\beta \hat{S}_\beta^* = 1, \quad \hat{i}(x) \hat{S}_\alpha \hat{S}_\alpha^* = \hat{S}_\alpha \hat{S}_\alpha^* \hat{i}(x), \quad \hat{S}_\alpha \hat{i}(x) \hat{S}_\alpha = \hat{i}(\rho_a(x)),
\]

\[
\sum_{b \in \Sigma^n} \hat{T}_b \hat{T}_b^* = 1, \quad \hat{i}(x) \hat{T}_a \hat{T}_a^* = \hat{T}_a \hat{T}_a^* \hat{i}(x), \quad \hat{T}_a \hat{i}(x) \hat{T}_a = \hat{i}(\eta_a(x)),
\]

\[
\hat{S}_\alpha \hat{T}_b = \hat{T}_a \hat{S}_\beta \quad \text{if} \quad \kappa(\alpha, b) = (a, \beta)
\]
for all \( x \in \mathcal{A} \) and \( \alpha \in \Sigma^\rho, a \in \Sigma^\eta. \)

For \((z, w) \in \mathbb{T}^2\), the correspondence

\[
e_{[\gamma]} \otimes P_{[\gamma]} x \in H_{(n,m)} \longrightarrow z^n w^m e_{[\gamma]} \otimes P_{[\gamma]} x \in H_{(n,m)}
\]

yields a unitary representation of \( \mathbb{T}^2 \) on \( H_{(n,m)} \), which extends to \( F_\kappa \), denoted by \( u(z, w) \). Since

\[
u(z, w) \mathcal{T}_{\rho, \eta}^\kappa u^*(z, w) = \mathcal{T}_{\rho, \eta}^\kappa, \quad \nu(z, w) \tilde{\mathcal{T}}_{\rho, \eta}^\kappa u^*(z, w) = \tilde{\mathcal{T}}_{\rho, \eta}^\kappa,
\]

The map

\[
u(z, w) \mathcal{T}_{\rho, \eta}^\kappa \longrightarrow u(z, w) X u^*(z, w) \in \mathcal{T}_{\rho, \eta}^\kappa
\]

yields an action of \( \mathbb{T}^2 \) on the \( C^* \)-algebra \( \hat{\mathcal{O}}^\kappa_{\rho, \eta} \), which we denote by \( \hat{\theta} \). Similarly to the action \( \theta \) on \( \mathcal{O}^\kappa_{\rho, \eta} \), we may define the conditional expectation \( \hat{\mathcal{E}}_{\rho, \eta} \) from \( \hat{\mathcal{O}}^\kappa_{\rho, \eta} \) to the fixed point algebra \( \hat{\mathcal{O}}^\kappa_{\rho, \eta} \) by taking the integration of the function \( \hat{\theta}(z, w)(X) \) over \((z, w) \in \mathbb{T}^2 \) for \( X \in \hat{\mathcal{O}}^\kappa_{\rho, \eta} \). Then as in the proof of Proposition 5.10, one may prove the following theorem.

**Theorem 6.7.** The algebra \( \hat{\mathcal{O}}^\kappa_{\rho, \eta} \) is canonically \( \ast \)-isomorphic to the \( C^* \)-algebra \( \mathcal{O}^\kappa_{\rho, \eta} \) through the correspondences:

\[S_\alpha \longrightarrow \hat{S}_\alpha, \quad T_a \longrightarrow \hat{T}_a, \quad x \longrightarrow \hat{i}(x)\]

for \( \alpha \in \Sigma^\rho, a \in \Sigma^\eta \) and \( x \in \mathcal{A} \).

### 7. K-Theory machinery

Let us denote by \( \mathcal{K} \) the \( C^* \)-algebra of compact operators on a separable infinite dimensional Hilbert space. For a \( C^* \)-algebra \( \mathcal{B} \), we denote by \( M(\mathcal{B}) \) its multiplier algebra. In this section, we will study K-theory groups \( K_*(\mathcal{O}^\kappa_{\rho, \eta}) \) for the \( C^* \)-algebra \( \mathcal{O}^\kappa_{\rho, \eta} \). We fix a \( C^* \)-textile dynamical system \((\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)\). We define two actions

\[
\hat{\rho} : \mathbb{T} \longrightarrow \text{Aut}(\mathcal{O}^\kappa_{\rho, \eta}), \quad \hat{\eta} : \mathbb{T} \longrightarrow \text{Aut}(\mathcal{O}^\kappa_{\rho, \eta})
\]

of the circle group \( \mathbb{T} = \{z \in \mathbb{C} \mid \|z\| = 1\} \) to \( \mathcal{O}^\kappa_{\rho, \eta} \) by setting

\[
\hat{\rho}_z = \theta(z, 1), \quad \hat{\eta}_w = \theta(1, w), \quad z, w \in \mathbb{T}.
\]

They satisfy

\[
\hat{\rho}_z \circ \hat{\eta}_w = \hat{\eta}_w \circ \hat{\rho}_z = \theta(z, w), \quad z, w \in \mathbb{T}.
\]

Set the fixed point algebras

\[
(\mathcal{O}^\kappa_{\rho, \eta})^\hat{\rho} = \{x \in \mathcal{O}^\kappa_{\rho, \eta} \mid \hat{\rho}_z(x) = x \text{ for all } z \in \mathbb{T}\},
\]

\[
(\mathcal{O}^\kappa_{\rho, \eta})^\hat{\eta} = \{x \in \mathcal{O}^\kappa_{\rho, \eta} \mid \hat{\eta}_w(x) = x \text{ for all } w \in \mathbb{T}\}.
\]

For \( x \in (\mathcal{O}^\kappa_{\rho, \eta})^\hat{\rho} \), define the \( \mathcal{O}^\kappa_{\rho, \eta} \)-valued constant function

\[
\hat{x} \in L^1(\mathbb{T}, \mathcal{O}^\kappa_{\rho, \eta}) \subset \mathcal{O}^\kappa_{\rho, \eta} \times \mathbb{T}
\]
from $\mathbb{T}$ by setting $\widehat{x}(z) = x, z \in \mathbb{T}$. Put $p_0 = \hat{1}$. By [45], the algebra $(\mathcal{O}_{\rho,\eta}^\kappa)^\hat{\beta}$ is canonically isomorphic to $p_0(\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T})p_0$ through the map
\[
j_\rho : x \in (\mathcal{O}_{\rho,\eta}^\kappa)^\hat{\beta} \rightarrow \widehat{x} \in p_0(\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T})p_0
\]
which induces an isomorphism
\[
j_\rho : K_i((\mathcal{O}_{\rho,\eta}^\kappa)^\hat{\beta}) \rightarrow K_i(p_0(\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T})p_0), \quad i = 0, 1
\]
on their $K$-groups. By a similar manner to the proofs given in [23, Section 4], one may prove the following lemma.

**Lemma 7.1.**

(i) There exists an isometry
\[
v \in M((\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T}) \otimes \mathcal{K})
\]
such that $vv^* = p_0 \otimes 1, v^*v = 1$.

(ii) $\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T}$ is stably isomorphic to $(\mathcal{O}_{\rho,\eta}^\kappa)^\hat{\beta}$, and similarly $\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T}$ is stably isomorphic to $(\mathcal{O}_{\rho,\eta}^\kappa)^\hat{\beta}$.

(iii) The inclusion $t_{\hat{\beta}} : p_0(\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T})p_0 \hookrightarrow \mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T}$ induces an isomorphism
\[
t_{\hat{\beta}} : K_i(p_0(\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T})p_0) \cong K_i(\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T}), \quad i = 0, 1
\]
on their $K$-groups.

Thanks to the lemma above, the isomorphism
\[
\text{Ad}(v^*) : x \in p_0(\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T})p_0 \otimes \mathcal{K} \rightarrow v^*xv \in (\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T}) \otimes \mathcal{K}
\]
induces isomorphisms
\[
\text{Ad}(v^*)_* : K_i(p_0(\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T})p_0) \rightarrow K_i(\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T}), \quad i = 0, 1.
\]

Let $\hat{\beta}$ be the automorphism on $\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T}$ for the positive generator of $Z$ for the dual action of $\hat{\beta}$. By (7.1) and (7.2), we may define an isomorphism
\[
\beta_{p,i} = j_{\rho,\iota}^{-1} \circ \text{Ad}(v^*)^{-1} \circ \hat{\beta} \circ \text{Ad}(v^*) \circ j_{\rho,\iota} : K_i((\mathcal{O}_{\rho,\eta}^\kappa)^\hat{\beta}) \rightarrow K_i((\mathcal{O}_{\rho,\eta}^\kappa)^\hat{\beta})
\]
for $i = 0, 1$, so that the diagram is commutative:
\[
\begin{array}{ccc}
K_i((\mathcal{O}_{\rho,\eta}^\kappa)^\hat{\beta}) & \xrightarrow{\beta_{p,i}} & K_i((\mathcal{O}_{\rho,\eta}^\kappa)^\hat{\beta}) \\
\uparrow {\text{Ad}(v^*)}_* & & \uparrow {\text{Ad}(v^*)}_* \\
K_i(p_0(\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T})p_0) & \xrightarrow{\gamma_{p,i}} & K_i(p_0(\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T})p_0) \\
\uparrow j_{\rho,*} & & \uparrow j_{\rho,*} \\
K_i(\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T}) & \xrightarrow{\hat{\beta}_{p,\iota}} & K_i(\mathcal{O}_{\rho,\eta}^\kappa \times_{\hat{\beta}} \mathbb{T}).
\end{array}
\]
By [39] (cf. [15]), one has the six term exact sequence of K-theory:

\[
\begin{array}{cccc}
K_0(\mathcal{O}_{\rho,\eta}^\kappa \times_\hat{\rho} \mathbb{T}) & \xrightarrow{\text{id}-\hat{\beta}} & K_0(\mathcal{O}_{\rho,\eta}^\kappa \times_\hat{\rho} \mathbb{T}) & \xrightarrow{\iota^*} & K_0((\mathcal{O}_{\rho,\eta}^\kappa \times_\hat{\rho} \mathbb{T}) \times_\hat{\rho} \mathbb{Z}) \\
\downarrow & & \downarrow_{\exp} & & \downarrow \\
K_1((\mathcal{O}_{\rho,\eta}^\kappa \times_\hat{\rho} \mathbb{T}) \times_\hat{\rho} \mathbb{Z}) & \xleftarrow{\iota^*} & K_1(\mathcal{O}_{\rho,\eta}^\kappa \times_\hat{\rho} \mathbb{T}) & \xleftarrow{\text{id}-\hat{\beta}} & K_1(\mathcal{O}_{\rho,\eta}^\kappa \times_\hat{\rho} \mathbb{T}).
\end{array}
\]

Since \((\mathcal{O}_{\rho,\eta}^\kappa \times_\hat{\rho} \mathbb{T}) \times_\hat{\rho} \mathbb{Z} \cong \mathcal{O}_{\rho,\eta}^\kappa \otimes \mathcal{K}\) and \(K_i(\mathcal{O}_{\rho,\eta}^\kappa \times_\hat{\rho} \mathbb{T}) \cong K_i((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho})\), one has:

**Lemma 7.2.** The following six term exact sequence of K-theory holds:

\[
\begin{array}{cccc}
K_0((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}) & \xrightarrow{\text{id}-\beta_{\rho,q}} & K_0((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}) & \xrightarrow{\iota^*} & K_0(\mathcal{O}_{\rho,\eta}^\kappa) \\
\downarrow & & \downarrow_{\exp} & & \downarrow \\
K_1(\mathcal{O}_{\rho,\eta}^\kappa) & \xleftarrow{\iota^*} & K_1((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}) & \xleftarrow{\text{id}-\beta_{\rho,1}} & K_1((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}).
\end{array}
\]

Hence there exist short exact sequences for \(i = 0, 1\):

\[
\begin{align*}
0 & \longrightarrow \text{Coker}(\text{id} - \beta_{\rho,i}) \text{ in } K_i((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}) \\
& \longrightarrow K_i(\mathcal{O}_{\rho,\eta}^\kappa) \\
& \longrightarrow \text{Ker}(\text{id} - \beta_{\rho,i+1}) \text{ in } K_{i+1}((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}) \\
& \longrightarrow 0.
\end{align*}
\]

In the rest of this section, we will study the groups

\[
\text{Coker}(\text{id} - \beta_{\rho,i}) \text{ in } K_i((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}), \quad \text{Ker}(\text{id} - \beta_{\rho,i+1}) \text{ in } K_{i+1}((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}).
\]

The action \(\hat{\eta}\) acts on the subalgebra \((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}\), which we still denote by \(\hat{\eta}\). Then the fixed point algebra \(((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho})^{\hat{\eta}}\) of \((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}\) under \(\hat{\eta}\) coincides with \(\mathcal{F}_{\rho,\eta}\). The above discussions for the action \(\hat{\rho}: \mathbb{T} \longrightarrow \mathcal{O}_{\rho,\eta}^\kappa\) works for the action \(\hat{\eta}: \mathbb{T} \longrightarrow (\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}\) as in the following way. For \(y \in ((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho})^{\hat{\eta}}\), define the constant function \(\hat{\gamma} \in L^1(\mathbb{T}, (\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}) \subset (\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho} \times_\hat{\eta} \mathbb{T}\) by setting \(\hat{\gamma}(w) = y, w \in \mathbb{T}\). Putting \(q_0 = 1\), the algebra \(((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho})^{\hat{\eta}}\) is canonically isomorphic to \(q_0((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho} \times_\hat{\eta} \mathbb{T})q_0\) through the map

\[
\begin{align*}
\hat{j}_\eta: y \in ((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho})^{\hat{\eta}} & \longrightarrow \hat{\gamma} \in q_0((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho} \times_\hat{\eta} \mathbb{T})q_0,
\end{align*}
\]

which induces an isomorphism

\[
\hat{j}_{\eta,q_i}: K_i(((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho})^{\hat{\eta}}) \longrightarrow K_i(q_0((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho} \times_\hat{\eta} \mathbb{T})q_0), \quad i = 0, 1
\]
on their K-groups. Similarly to Lemma 7.1, we have:

**Lemma 7.3.**

(i) There exists an isometry

\[
u \in M(((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho} \times_\hat{\eta} \mathbb{T}) \otimes \mathcal{K})
\]
such that \(\nu \nu^* = q_0 \otimes 1, \nu^* \nu = 1\).
(ii) \((O^\kappa_{\rho,\eta})^{\hat{\rho}} \times \hat{\eta} \mathbb{T}\) is stably isomorphic to \(((O^\kappa_{\rho,\eta})^{\hat{\rho}})^{\hat{\eta}}\).

(iii) The inclusion
\[
i^\hat{\rho}_0 : q_0((O^\kappa_{\rho,\eta})^{\hat{\rho}} \times \hat{\eta} \mathbb{T})q_0 = ((O^\kappa_{\rho,\eta})^{\hat{\rho}})^{\hat{\eta}} = \mathcal{F}_{\rho,\eta} \hookrightarrow (O^\kappa_{\rho,\eta})^{\hat{\rho}} \times \hat{\eta} \mathbb{T}
\]
induces an isomorphism
\[
i^\hat{\rho}_i : K_i((O^\kappa_{\rho,\eta})^{\hat{\rho}} \times \hat{\eta} \mathbb{T})q_0 \cong K_i((O^\kappa_{\rho,\eta})^{\hat{\rho}} \times \hat{\eta} \mathbb{T}), \quad i = 0, 1
\]
on their \(K\)-groups.

The isomorphism
\[
\text{Ad}(u^*) : y \in q_0((O^\kappa_{\rho,\eta})^{\hat{\rho}} \times \hat{\eta} \mathbb{T})q_0 \rightarrow u^*yu \in (O^\kappa_{\rho,\eta})^{\hat{\rho}} \times \hat{\eta} \mathbb{T}
\]
induces isomorphisms
\[
\text{Ad}(u^*)_i : K_i((O^\kappa_{\rho,\eta})^{\hat{\rho}} \times \hat{\eta} \mathbb{T})q_0 \cong K_i((O^\kappa_{\rho,\eta})^{\hat{\rho}} \times \hat{\eta} \mathbb{T}), \quad i = 0, 1.
\]

Let \(\hat{\eta}_0\) be the automorphism on \((O^\kappa_{\rho,\eta})^{\hat{\rho}} \times \hat{\eta} \mathbb{T}\) for the positive generator of \(Z\) for the dual action of \(\hat{\eta}\). Define an isomorphism
\[
\gamma_{i,\eta} = j_0^{-1} \circ \text{Ad}(u^*)^{-1} \circ \hat{\eta}_0 \circ \text{Ad}(u^*)_i \circ j_0^*: K_i(\mathcal{F}_{\rho,\eta}) \rightarrow K_i(\mathcal{F}_{\rho,\eta}), \quad i = 0, 1
\]
such that the diagram is commutative for \(i = 0, 1:\)

\[
\begin{array}{ccc}
K_i((O^\kappa_{\rho,\eta})^{\hat{\rho}} \times \hat{\eta} \mathbb{T}) & \xrightarrow{\text{Ad}(u^*)_i} & K_i((O^\kappa_{\rho,\eta})^{\hat{\rho}} \times \hat{\eta} \mathbb{T}) \\
\uparrow \text{Ad}(u^*)_i & & \uparrow \text{Ad}(u^*)_i \\
K_i(q_0((O^\kappa_{\rho,\eta})^{\hat{\rho}} \times \hat{\eta} \mathbb{T})q_0) & \xrightarrow{j_0^*} & K_i(q_0((O^\kappa_{\rho,\eta})^{\hat{\rho}} \times \hat{\eta} \mathbb{T})q_0) \\
\uparrow \gamma_{i,\eta} & & \uparrow \gamma_{i,\eta} \\
K_i((O^\kappa_{\rho,\eta})^{\hat{\rho}}^{\hat{\eta}}) & = & K_i((O^\kappa_{\rho,\eta})^{\hat{\rho}}^{\hat{\eta}}) \\
\uparrow & & \uparrow \\
K_i(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\gamma_{i,\eta}} & K_i(\mathcal{F}_{\rho,\eta}).
\end{array}
\]

We similarly define an endomorphism \(\gamma_{i,\rho} : K_i(\mathcal{F}_{\rho,\eta}) \rightarrow K_i(\mathcal{F}_{\rho,\eta})\) by exchanging the rôles of \(\rho\) and \(\eta\).

Under the equality \(((O^\kappa_{\rho,\eta})^{\hat{\rho}})^{\hat{\eta}} = \mathcal{F}_{\rho,\eta}\), we have the following lemma which is similar to Lemma 7.2

**Lemma 7.4.** The following six term exact sequence of \(K\)-theory holds:

\[
\begin{array}{ccccccc}
K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{id^{-\gamma_{0,\eta}}} & K_0(\mathcal{F}_{\rho,\eta}) & \xrightarrow{\iota^*} & K_0((O^\kappa_{\rho,\eta})^{\hat{\rho}}) \\
\delta \uparrow & & \uparrow & & \exp \downarrow \\
K_1((O^\kappa_{\rho,\eta})^{\hat{\rho}}) & \xleftarrow{\iota_*} & K_1(\mathcal{F}_{\rho,\eta}) & \xleftarrow{id^{-\gamma_{1,\eta}}} & K_1(\mathcal{F}_{\rho,\eta}).
\end{array}
\]
In particular, if $K_1(F_{\rho,\eta}) = 0$, we have

\[
K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) = \text{Coker}(\text{id} - \gamma_{\eta,0}) \quad \text{in } K_0(F_{\rho,\eta}),
\]

\[
K_1((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) = \text{Ker}(\text{id} - \gamma_{\eta,0}) \quad \text{in } K_0(F_{\rho,\eta}).
\]

Denote by $M_n(B)$ the $n \times n$ matrix algebra over a $C^*$-algebra $B$, which is identified with the tensor product $B \otimes M_n(\mathbb{C})$. The following lemmas hold.

**Lemma 7.5.** For a projection $q \in M_n((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}})$ and a partial isometry $S \in \mathcal{O}^{\kappa}_{\rho,\eta}$ such that

\[
\hat{\rho}_z(S) = zS \quad \text{for } z \in \mathbb{T}, \quad q(SS^* \otimes 1_n) = (SS^* \otimes 1_n)q,
\]

we have

\[
\beta_{\rho,0}^{-1}([(SS^* \otimes 1_n)q]) = [(S^* \otimes 1_n)q(S \otimes 1_n)] \quad \text{in } K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}).
\]

**Proof.** As $q$ commutes with $SS^* \otimes 1_n$, $p = (S^* \otimes 1_n)q(S \otimes 1_n)$ is a projection in $(\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}$. Since $p \leq S^*S \otimes 1_n$, By a similar argument to the proof of [23, Lemma 4.5], one sees that $\beta_{\rho,0}([p]) = [(S \otimes 1_n)p(S^* \otimes 1_n)]$ in $K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}})$. \hfill $\square$

**Lemma 7.6.**

(i) For a projection $q \in M_n(F_{\rho,\eta})$ and a partial isometry $T \in (\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}$ such that

\[
\hat{\eta}_w(T) = wT \quad \text{for } w \in \mathbb{T}, \quad q(TT^* \otimes 1_n) = (TT^* \otimes 1_n)q,
\]

we have

\[
\gamma_{\eta,0}^{-1}([(TT^* \otimes 1_n)q]) = [(T^* \otimes 1_n)q(T \otimes 1_n)] \quad \text{in } K_0(F_{\rho,\eta}).
\]

(ii) For a projection $q \in M_n(F_{\rho,\eta})$ and a partial isometry $S \in (\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\eta}}$ such that

\[
\hat{\rho}_z(S) = zS \quad \text{for } z \in \mathbb{T}, \quad q(SS^* \otimes 1_n) = (SS^* \otimes 1_n)q,
\]

we have

\[
\gamma_{\rho,0}^{-1}([(SS^* \otimes 1_n)q]) = [(S^* \otimes 1_n)q(S \otimes 1_n)] \quad \text{in } K_0(F_{\rho,\eta}).
\]

Hence we have

**Lemma 7.7.** The diagram

\[
\begin{array}{ccc}
K_0(F_{\rho,\eta}) & \xrightarrow{\text{id} - \gamma_{\rho,0}} & K_0(F_{\rho,\eta}) \\
\downarrow{\iota_*} & & \downarrow{\iota_*} \\
K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}}) & \xrightarrow{\text{id} - \beta_{\rho,0}} & K_0((\mathcal{O}^{\kappa}_{\rho,\eta})^{\hat{\rho}})
\end{array}
\]

is commutative.
Proof. By [35, Proposition 3.3], the map \( \iota_* : K_0(\mathcal{F}_{\rho,\eta}) \to K_0((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}) \) is induced by the natural inclusion \( \mathcal{F}_{\rho,\eta} = ((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}) \to (\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho} \). For an element \([q] \in K_0(\mathcal{F}_{\rho,\eta})\) one may assume that \( q \in M_n(\mathcal{F}_{\rho,\eta}) \) for some \( n \in \mathbb{N} \) so that one has

\[
\gamma_{\rho,0}^{-1}([q]) = \sum_{\alpha \in \Sigma^\rho} [(S_\alpha S_\alpha^* \otimes 1_n)q]
= \sum_{\alpha \in \Sigma^\rho} [(S_\alpha^* \otimes 1_n)q(S_\alpha \otimes 1_n)]
= \sum_{\alpha \in \Sigma^\rho} \beta_{\rho,0}^{-1}([q(S_\alpha S_\alpha^* \otimes 1_n)]) = \beta_{\rho,0}^{-1}([q])
\]

so that \( \beta_{\rho,0}|_{K_0(\mathcal{F}_{\rho,\eta})} = \gamma_{\rho,0} \). \( \square \)

In the rest of this section, we assume that \( K_1(\mathcal{F}_{\rho,\eta}) = 0 \). The following lemma is crucial in our further discussions.

Lemma 7.8. In the six term exact sequence in Lemma 7.4 with \( K_1(\mathcal{F}_{\rho,\eta}) = 0 \), we have the following commutative diagrams:

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
K_1((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}) & \rightarrow & K_1((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}) \\
\delta & \downarrow & \delta \\
K_0(\mathcal{F}_{\rho,\eta}) & \rightarrow & K_0(\mathcal{F}_{\rho,\eta}) \\
\delta & \downarrow & \delta \\
K_0(\mathcal{F}_{\rho,\eta}) & \rightarrow & K_0(\mathcal{F}_{\rho,\eta}) \\
\iota_* & \downarrow & \iota_* \\
K_0((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}) & \rightarrow & K_0((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}) \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

(7.3)

Proof. It is well-known that \( \delta \)-map is functorial (see [48, Theorem 7.2.5], [4, p.266 (LX)]). Hence the diagram of the upper square

\[
\begin{array}{ccc}
K_1((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}) & \rightarrow & K_1((\mathcal{O}_{\rho,\eta}^\kappa)\hat{\rho}) \\
\delta & \downarrow & \delta \\
K_0(\mathcal{F}_{\rho,\eta}) & \rightarrow & K_0(\mathcal{F}_{\rho,\eta})
\end{array}
\]
is commutative. Since $\gamma_{\rho,0} \circ \gamma_{\eta,0} = \gamma_{\eta,0} \circ \gamma_{\rho,0}$, the diagram of the middle square

\[
\begin{array}{ccc}
K_0(F_{\rho,\eta}) & \xrightarrow{id-\gamma_{\rho,0}} & K_0(F_{\rho,\eta}) \\
\downarrow{\text{id}-\gamma_{\eta,0}} & & \downarrow{\text{id}-\gamma_{\eta,0}} \\
K_0(F_{\rho,\eta}) & \xrightarrow{id-\gamma_{\rho,0}} & K_0(F_{\rho,\eta})
\end{array}
\]

is commutative. The commutativity of the lower square comes from the preceding lemma.

We will describe the K-groups $K_*(\mathcal{O}^\kappa_{\rho,\eta})$ in terms of the kernels and cokernels of the homomorphisms $id - \gamma_{\rho,0}$ and $id - \gamma_{\eta,0}$ on $K_0(F_{\rho,\eta})$. Recall that there exist two short exact sequences by Lemma 7.2:

\[
0 \rightarrow \text{Coker}(\text{id} - \beta_{\rho,0}) \text{ in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^\hat{\rho}) \rightarrow K_0(\mathcal{O}^\kappa_{\rho,\eta}) \rightarrow \text{Ker}(\text{id} - \beta_{\rho,1}) \text{ in } K_1((\mathcal{O}^\kappa_{\rho,\eta})^\hat{\rho}) \rightarrow 0
\]

and

\[
0 \rightarrow \text{Coker}(\text{id} - \beta_{\rho,1}) \text{ in } K_1((\mathcal{O}^\kappa_{\rho,\eta})^\hat{\rho}) \rightarrow K_1(\mathcal{O}^\kappa_{\rho,\eta}) \rightarrow \text{Ker}(\text{id} - \beta_{\rho,0}) \text{ in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^\hat{\rho}) \rightarrow 0.
\]

As $\gamma_{\eta,0} \circ \gamma_{\rho,0} = \gamma_{\rho,0} \circ \gamma_{\eta,0}$ on $K_0(F_{\rho,\eta})$, the homomorphisms $\gamma_{\rho,0}$ and $\gamma_{\eta,0}$ naturally act on $\text{Coker}(\text{id} - \gamma_{\rho,0}) = K_0(F_{\rho,\eta})/(id - \gamma_{\rho,0})K_0(F_{\rho,\eta})$ and $\text{Coker}(\text{id} - \gamma_{\rho,0}) = K_0(F_{\rho,\eta})/(id - \gamma_{\rho,0})K_0(F_{\rho,\eta})$ as endomorphisms respectively, which we denote by $\bar{\gamma}_{\rho,0}$ and $\bar{\gamma}_{\eta,0}$ respectively.

**Lemma 7.9.**

(i) For $K_0(\mathcal{O}^\kappa_{\rho,\eta})$, we have

$\text{Coker}(\text{id} - \beta_{\rho,0}) \text{ in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^\hat{\rho})$

$\cong \text{Coker}(\text{id} - \bar{\gamma}_{\rho,0}) \text{ in } K_0(F_{\rho,\eta})/(id - \gamma_{\eta,0})K_0(F_{\rho,\eta})$

$\cong K_0(F_{\rho,\eta})/(id - \gamma_{\rho,0})K_0(F_{\rho,\eta}) + (id - \gamma_{\eta,0})K_0(F_{\rho,\eta})$

and

$\text{Ker}(\text{id} - \beta_{\rho,1}) \text{ in } K_1((\mathcal{O}^\kappa_{\rho,\eta})^\hat{\rho})$

$\cong \text{Ker}(\text{id} - \gamma_{\rho,0}) \text{ in } (\text{Ker}(\text{id} - \gamma_{\eta,0}) \text{ in } K_0(F_{\rho,\eta}))$

$\cong \text{Ker}(\text{id} - \gamma_{\rho,0}) \cap \text{Ker}(\text{id} - \gamma_{\eta,0}) \text{ in } K_0(F_{\rho,\eta})$. 
(ii) For $K_1(\mathcal{O}^\kappa_{\rho,\eta})$, we have
\[ \text{Coker}(\text{id} - \beta_{\rho,1}) \text{ in } K_1((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\beta}}) \]
\[ \cong (\text{Ker}(\text{id} - \gamma_{\rho,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}))/((\text{id} - \gamma_{\rho,0})(\text{Ker}(\text{id} - \gamma_{\rho,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})) \]
and
\[ \text{Ker}(\text{id} - \beta_{\rho,0}) \text{ in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\beta}}) \]
\[ \cong \text{Ker}(\text{id} - \tilde{\gamma}_{\rho,0}) \text{ in } (K_0(\mathcal{F}_{\rho,\eta}))/((\text{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})). \]

**Proof.** (i) We will first prove the assertions for the group
\[ \text{Coker}(\text{id} - \beta_{\rho,0}) \text{ in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\beta}}). \]
In the diagram (7.3), the exactness of the vertical arrows implies that $\iota_*$ is surjective so that
\[ K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\beta}}) \cong \iota_*(K_0(\mathcal{F}_{\rho,\eta})) \cong K_0(\mathcal{F}_{\rho,\eta})/\text{Ker}(\text{id} - \gamma_{\rho,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}). \]
By the commutativity in the lower square in the diagram (7.3), one has
\[ \text{Coker}(\text{id} - \beta_{\rho,0}) \text{ in } K_0((\mathcal{O}^\kappa_{\rho,\eta})^{\hat{\beta}}) \]
\[ \cong \text{Coker}(\text{id} - \tilde{\gamma}_{\rho,0}) \text{ in } (K_0(\mathcal{F}_{\rho,\eta}))/((\text{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})). \]
The latter group will be proved to be isomorphic to the group
\[ K_0(\mathcal{F}_{\rho,\eta})/((\text{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})) + (\text{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta}). \]
Put $H_{\rho,\eta} = (\text{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta}) + (\text{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})$ the subgroup of $K_0(\mathcal{F}_{\rho,\eta})$ generated by $\text{id} - \gamma_{\rho,0}K_0(\mathcal{F}_{\rho,\eta})$ and $(\text{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})$. Set the quotient maps
\[ K_0(\mathcal{F}_{\rho,\eta}) \xrightarrow{q_{\eta}} K_0(\mathcal{F}_{\rho,\eta})/((\text{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})) \]
\[ q_{(\text{id} - \gamma_{\rho,0})} \text{Coker}(\text{id} - \tilde{\gamma}_{\rho,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})/((\text{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})) \]
and
\[ \Phi = q_{(\text{id} - \gamma_{\rho,0})} \circ q_{\eta} : K_0(\mathcal{F}_{\rho,\eta}) \]
\[ \longrightarrow \text{Coker}(\text{id} - \tilde{\gamma}_{\rho,0}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})/((\text{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})). \]
It suffices to show the equality $\text{Ker}(\Phi) = H_{\rho,\eta}$. As $\text{id} - \gamma_{\rho,0}$ commutes with $(\text{id} - \gamma_{\rho,0})$, one has
\[ (\text{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta}) \subset \text{Ker}(\Phi), \quad (\text{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta}) \subset \text{Ker}(\Phi). \]
Hence we have $H_{\rho,\eta} \subset \text{Ker}(\Phi)$. On the other hand, for $g \in \text{Ker}(\Phi)$, we have $g \in (\text{id} - \tilde{\gamma}_{\rho,0})(K_0(\mathcal{F}_{\rho,\eta})/((\text{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta})))$ so that $g = (\text{id} - \gamma_{\rho,0})[h]$ for some $[h] \in K_0(\mathcal{F}_{\rho,\eta})/((\text{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta}))$. Hence
\[ g = (\text{id} - \gamma_{\rho,0})h + (\text{id} - \gamma_{\rho,0})(\text{id} - \gamma_{\rho,0})K_0(\mathcal{F}_{\rho,\eta}) \]
so that $g \in H_{\rho,\eta}$. Hence we have $\text{Ker}(\Phi) \subset H_{\rho,\eta}$ and $\text{Ker}(\Phi) = H_{\rho,\eta}$. 

We will second prove the assertions for the group
\[ \text{Ker}(\text{id} - \beta_{\rho,1}) \text{ in } K_1((O^\kappa_{\rho,\eta})^{\hat{\rho}}). \]

In the diagram (7.3), the exactness of the vertical arrows implies that \( \delta \) is injective and \( \text{Im}(\delta) = \text{Ker}(\text{id} - \gamma_{\eta,0}) \) so that we have

\[ K_1((O^\kappa_{\rho,\eta})^{\hat{\rho}}) \cong \text{Ker}(\text{id} - \gamma_{\eta,0}) \text{ in } K_0(F_{\rho,\eta}). \]  

By the commutativity in the upper square in the diagram (7.3), one has
\[ \text{Ker}(\text{id} - \beta_{\rho,1}) \text{ in } K_1((O^\kappa_{\rho,\eta})^{\hat{\rho}}) \cong \text{Ker}(\text{id} - \gamma_{\rho,0}) \text{ in } (\text{Ker}(\text{id} - \gamma_{\eta,0}) \text{ in } K_0(F_{\rho,\eta})). \]

Since \( \gamma_{\eta,0} \) commutes with \( \gamma_{\rho,0} \) in \( K_0(F_{\rho,\eta}) \), we have
\[ \text{Ker}(\text{id} - \gamma_{\rho,0}) \text{ in } (\text{Ker}(\text{id} - \gamma_{\eta,0}) \text{ in } K_0(F_{\rho,\eta})) \cong \text{Ker}(\text{id} - \gamma_{\rho,0}) \cap \text{Ker}(\text{id} - \gamma_{\eta,0}) \text{ in } K_0(F_{\rho,\eta}). \]

(ii) The assertions are similarly shown as in (i). \( \square \)

Therefore we have:

**Theorem 7.10.** Assume that \( K_1(F_{\rho,\eta}) = 0 \). There exist short exact sequences:

\[ 0 \rightarrow K_0(F_{\rho,\eta})/(\text{id} - \gamma_{\rho,0})K_0(F_{\rho,\eta}) + (\text{id} - \gamma_{\eta,0})K_0(F_{\rho,\eta}) \rightarrow K_0(O^\kappa_{\rho,\eta}) \rightarrow \text{Ker}(\text{id} - \gamma_{\rho,0}) \cap \text{Ker}(\text{id} - \gamma_{\eta,0}) \text{ in } K_0(F_{\rho,\eta}) \rightarrow 0 \]

and

\[ 0 \rightarrow (\text{Ker}(\text{id} - \gamma_{\eta,0}) \text{ in } K_0(F_{\rho,\eta}))/\text{(id} - \gamma_{\rho,0})(\text{Ker}(\text{id} - \gamma_{\eta,0}) \text{ in } K_0(F_{\rho,\eta})) \rightarrow K_1(O^\kappa_{\rho,\eta}) \rightarrow \text{Ker}(\text{id} - \gamma_{\rho,0}) \text{ in } (\text{Ker}(\text{id} - \gamma_{\eta,0}) \text{ in } K_0(F_{\rho,\eta})) \rightarrow 0. \]

We may describe the above formulae as follows.

**Corollary 7.11.** Suppose \( K_1(F_{\rho,\eta}) = 0 \). There exist short exact sequences:

\[ 0 \rightarrow \text{Coker}(\text{id} - \gamma_{\rho,0}) \text{ in } \text{(Coker}(\text{id} - \gamma_{\eta,0}) \text{ in } K_0(F_{\rho,\eta})) \rightarrow K_0(O^\kappa_{\rho,\eta}) \rightarrow \text{Ker}(\text{id} - \gamma_{\rho,0}) \text{ in } (\text{Ker}(\text{id} - \gamma_{\eta,0}) \text{ in } K_0(F_{\rho,\eta})) \rightarrow 0 \]
and
\[
0 \rightarrow \text{Coker}(id - \gamma_{\rho,0}) \text{ in } ((\text{Ker}(id - \gamma_{\eta,0}) \text{ in } K_0(F_{\rho,\eta}))
\rightarrow K_1(O_{\rho,\eta})
\rightarrow \text{Ker}(id - \bar{\gamma}_{\rho,0}) \text{ in } (\text{Coker}(id - \gamma_{\eta,0}) \text{ in } K_0(F_{\rho,\eta}))
\rightarrow 0.
\]

8. K-Theory formulae

In this section, we will present more useful formulae to compute the K-groups \( K_1(O_{\rho,\eta}) \) under a certain additional assumption on \((A, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)\).

The additional condition on \((A, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)\) is the following:

**Definition 8.1.** A \( C^*\)-textile dynamical system \((A, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)\) is said to form square if the \( C^*\)-subalgebra \( C^*(\rho_\alpha(1) : \alpha \in \Sigma^\rho) \) of \( A \) generated by the projections \( \rho_\alpha(1), \alpha \in \Sigma^\rho \), coincides with the \( C^*\)-subalgebra \( C^*(\eta_\alpha(1) : a \in \Sigma^\eta) \) of \( A \) generated by the projections \( \eta_\alpha(1), a \in \Sigma^\eta \).

**Lemma 8.2.** Assume that \((A, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)\) forms square. Put for \( l \in \mathbb{Z}_+ \)

\[
A_l^\rho = C^*(\rho_\mu(1) : \mu \in B_l(\Lambda_\rho)), \quad A_l^\eta = C^*(\eta_\xi(1) : \xi \in B_l(\Lambda_\eta)).
\]

Then \( A_l^\rho = A_l^\eta \).

**Proof.** By the assumption, we have \( A_1^\rho = A_1^\eta \). Hence the desired equality for \( l = 1 \) holds. Suppose that the equalities hold for all \( l \leq k \) for some \( k \in \mathbb{N} \). For \( \mu = (\mu_1, \mu_2, \ldots, \mu_k, \mu_{k+1}) \in B_{k+1}(\Lambda_\rho) \) we have \( \rho_\mu(1) = \rho_{\mu_{k+1}}(\rho_{\mu_1}\cdots\rho_k(1)) \) so that \( \rho_\mu(1) \in \rho_{\mu_{k+1}}(A_{k+1}^\rho) \). By the commutation relation (3.1), one sees that

\[
\rho_{\mu_{k+1}}(A_{k+1}^\rho) \subset C^*(\eta_\xi(\rho_\alpha(1)) : \xi \in B_k(\Lambda_\eta), \alpha \in \Sigma^\rho).
\]

Since \( C^*(\rho_\alpha(1) : \alpha \in \Sigma^\rho) = C^*(\eta_\alpha(1) : a \in \Sigma^\eta) \), the algebra \( C^*(\eta_\xi(\rho_\alpha(1)) : \xi \in B_k(\Lambda_\eta), \alpha \in \Sigma^\rho) \) is contained in \( A_{k+1}^\eta \) so that \( \rho_{\mu_{k+1}}(A_{k+1}^\rho) \subset A_{k+1}^\eta \). This implies \( \rho_\mu(1) \in A_{k+1}^\eta \) so that \( A_k(\mu) \subset A_{k+1}^\eta \) and hence \( A_k^\rho ) \subset A_{k+1}^\eta \). Therefore we have

**Lemma 8.3.** Assume that \((A, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)\) forms square. Put for \( j, k \in \mathbb{Z}_+ \)

\[
A_{j,k} = C^*(\rho_\mu(\eta_\xi(1)) : \mu \in B_j(\Lambda_\rho), \xi \in B_k(\Lambda_\eta))
\]

\[
(= C^*(\eta_\xi(\rho_\nu(1)) : \xi \in B_k(\Lambda_\eta), \nu \in B_j(\Lambda_\rho))).
\]

Then \( A_{j,k} \) is commutative and of finite dimensional such that

\[
A_{j,k} = A_{j+k}^\rho \subset A_{j+k}^\eta \subset A_{j+k}^\rho.
\]

Therefore, we have

\[
A_{j,k} = A_{j+k}^\rho \subset A_{j+k}^\eta = A_{j+k}^\rho.
\]

Hence \( A_{j,k} = A_{j+k} \) if \( j + k = j' + k' \).

**Proof.** Since \( \eta_\xi(1) \in \mathbb{Z}_A \) and \( \rho_\mu(Z_A) \subset Z_A \), the algebra \( A_{j,k} \) belongs to the center \( Z_A \) of \( A \). By the preceding lemma, we have

\[
A_{j,k} = C^*(\rho_\mu(\rho_\nu(1)) : \mu \in B_j(\Lambda_\rho), \nu \in B_k(\Lambda_\rho)) = A_{j+k}^\rho.
\]
For \( j, k \in \mathbb{Z}_+ \), put \( l = j + k \). We denote by \( \mathcal{A}_l \) the commutative finite dimensional algebra \( \mathcal{A}_{j,k} \). Put \( m(l) = \dim \mathcal{A}_l \). Take the finite sequence of minimal projections \( E^l_i, i = 1, 2, \ldots, m(l) \) in \( \mathcal{A}_l \) such that \( \sum_{i=1}^{m(l)} E^l_i = 1 \) and hence \( \mathcal{A}_l = \bigoplus_{i=1}^{m(l)} CE^l_i \). Since \( \rho_\alpha(\mathcal{A}_l) \subset \mathcal{A}_{l+1} \), there exists \( A_{l,l+1}(i, \alpha, n) \), which takes 0 or 1, such that

\[
\rho_\alpha(E^l_i) = \sum_{n=1}^{m(l+1)} A^\rho_{l+1}(i, \alpha, n)E^l_{n+1}, \quad \alpha \in \Sigma^\rho, \ i = 1, \ldots, m(l).
\]

Similarly, there exists \( A^n_0(i, a, n) \), which takes 0 or 1, such that

\[
\eta_\alpha(E^l_i) = \sum_{n=1}^{m(l+1)} A^n_{l+1}(i, a, n)E^l_{n+1}, \quad a \in \Sigma^n, \ i = 1, \ldots, m(l).
\]

Set for \( i = 1, \ldots, m(l) \)

\[
\mathcal{F}_{j,k}(i) = C^*(S_\mu T_\zeta E^l_i x E^l_i T_\xi S_\nu^* | \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta), x \in \mathcal{A}),
\]

\[
= C^*(T_\xi S_\mu E^l_i x E^l_i T_\zeta S_\nu^* | \mu, \nu \in B_j(\Lambda_\rho), \zeta, \xi \in B_k(\Lambda_\eta), x \in \mathcal{A}).
\]

Let \( N_{j,k}(i) \) be the cardinal number of the finite set

\[
\{(\mu, \zeta) \in B_j(\Lambda_\rho) \times B_k(\Lambda_\eta) | \rho_\zeta(\eta(1)) \geq E^l_i, \}
\]

Since \( E^l_i \) is a central projection in \( \mathcal{A} \), we have

**Lemma 8.4.** For \( j, k \in \mathbb{Z}_+ \), put \( l = j + k \). Then we have:

(i) \( \mathcal{F}_{j,k}(i) \) is isomorphic to the matrix algebra

\[
M_{N_{j,k}(i)}(E^l_i \mathcal{A} E^l_i) = M_{N_{j,k}(i)}(\mathbb{C}) \otimes E^l_i \mathcal{A} E^l_i
\]

over \( E^l_i \mathcal{A} E^l_i \) for \( i = 1, \ldots, m(l) \).

(ii) \( \mathcal{F}_{j,k} = \mathcal{F}_{j,k}(1) \oplus \cdots \oplus \mathcal{F}_{j,k}(m(l)) \).

**Proof.** (i) For \((\mu, \zeta) \in B_j(\Lambda_\rho) \times B_k(\Lambda_\eta) \) with \( S_\mu T_\zeta E^l_i \neq 0 \), one has

\[
\eta_\zeta(\rho_\mu(1))E^l_i \neq 0
\]

so that \( \eta_\zeta(\rho_\mu(1)) \geq E^l_i \). Hence \( (S_\mu T_\zeta E^l_i)^* S_\mu T_\zeta E^l_i = E^l_i \). One sees that the set

\[
\{S_\mu T_\zeta E^l_i | (\mu, \zeta) \in B_j(\Lambda_\rho) \times B_k(\Lambda_\eta); S_\mu T_\zeta E^l_i \neq 0 \}
\]

consist of partial isometries which give rise to matrix units of \( \mathcal{F}_{j,k}(i) \) such that \( \mathcal{F}_{j,k}(i) \) is isomorphic to \( M_{N_{j,k}(i)}(E^l_i \mathcal{A} E^l_i) \).

(ii) Since \( \mathcal{A} = E^l_1 \mathcal{A} E^l_1 \oplus \cdots \oplus E^l_{m(l)} \mathcal{A} E^l_{m(l)} \), the assertion is easy. \( \square \)

Define homomorphisms \( \lambda_\rho, \lambda_\eta : K_0(\mathcal{A}) \rightarrow K_0(\mathcal{A}) \) by setting

\[
\lambda_\rho([p]) = \sum_{\alpha \in \Sigma^\rho} [(\rho_\alpha \otimes 1_n)(p)], \quad \lambda_\eta([p]) = \sum_{\alpha \in \Sigma^n} [(\eta_\alpha \otimes 1_n)(p)]
\]
for a projection $p \in M_n(A)$ for some $n \in \mathbb{N}$. Recall that the identities (5.1), (5.2) give rise to the embeddings (5.3), which induce homomorphisms

$$K_0(F_{j,k}) \rightarrow K_0(F_{j,k+1}), \quad K_0(F_{j,k}) \rightarrow K_0(F_{j+1,k}).$$

We still denote them by $\iota_{+,1}, \iota_{+,1,*}$ respectively.

**Lemma 8.5.** Assume that $(A, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ forms square. There exists an isomorphism

$$\Phi_{j,k} : K_0(F_{j,k}) \rightarrow K_0(A)$$

such that the following diagrams are commutative:

(i)

$$\begin{array}{c}
K_0(F_{j,k}) \xrightarrow{\iota_{+,1,*}} K_0(F_{j+1,k}) \\
\Phi_{j,k} \downarrow \quad \Phi_{j+1,k} \\
K_0(A) \xrightarrow{\lambda_{\rho}} K_0(A)
\end{array}$$

(ii)

$$\begin{array}{c}
K_0(F_{j,k}) \xrightarrow{\iota_{+,1}} K_0(F_{j,k+1}) \\
\Phi_{j,k} \downarrow \quad \Phi_{j,k+1} \\
K_0(A) \xrightarrow{\lambda_{\eta}} K_0(A)
\end{array}$$

**Proof.** Put for $i = 1, 2, \ldots, m(l)$

$$P_i = \sum_{\mu \in B_j(\Lambda_\rho), \zeta \in B_k(\Lambda_\eta)} S_\mu T_\zeta E_i T_\zeta^* S_\mu^*.$$

Then $P_i$ is a central projection in $F_{j,k}$ such that $\sum_{i=1}^{m(l)} P_i = 1$. For $X \in F_{j,k}$, one has $P_iXP_i \in F_{j,k}(i)$ such that

$$X = \sum_{i=1}^{m(l)} P_iXP_i \in \bigoplus_{i=1}^{m(l)} F_{j,k}(i).$$

Define an isomorphism

$$\varphi_{j,k} : X \in F_{j,k} \rightarrow \sum_{i=1}^{m(l)} P_iXP_i \in \bigoplus_{i=1}^{m(l)} F_{j,k}(i)$$

which induces an isomorphism on their K-groups

$$\varphi_{j,k,*} : K_0(F_{j,k}) \rightarrow \bigoplus_{i=1}^{m(l)} K_0(F_{j,k}(i)).$$

Take and fix $\nu(i), \mu(i) \in B_j(\Lambda_\rho)$ and $\zeta(i), \xi(i) \in B_k(\Lambda_\eta)$ such that

$$(8.1) \quad T_{\xi(i)}S_{\nu(i)} = S_{\mu(i)}T_{\zeta(i)} \quad \text{and} \quad T_{\xi(i)}S_{\nu(i)}E_i \neq 0.$$
Hence \( S^*_\nu(i)T_{\xi(i)}^*T_{\xi(i)}^*S_{\nu(i)} \geq E_i^l \). Since \( F_{j,k}(i) \) is isomorphic to \( M_{N_{j,k}(i)}(\mathbb{C}) \otimes E_i^l A E_i^l \), the embedding
\[
\iota_{j,k}(i) : x \in E_i^l A E_i^l \longrightarrow T_{\xi(i)}^* S_{\nu(i)} x S_{\nu(i)}^* T_{\xi(i)}^* \in F_{j,k}(i)
\]
induces an isomorphism on their K-groups
\[
\iota_{j,k}(i)^* : K_0(E_i^l A E_i^l) \longrightarrow K_0(F_{j,k}(i)).
\]
Put
\[
\psi_{j,k} = \bigoplus_{i=1}^{m(l)} \iota_{j,k}(i) : \bigoplus_{i=1}^{m(l)} E_i^l A E_i^l \longrightarrow \bigoplus_{i=1}^{m(l)} F_{j,k}(i)
\]
and hence we have an isomorphism
\[
\psi_{j,k}^* = \bigoplus_{i=1}^{m(l)} \iota_{j,k}(i)^* : \bigoplus_{i=1}^{m(l)} K_0(E_i^l A E_i^l) \longrightarrow \bigoplus_{i=1}^{m(l)} K_0(F_{j,k}(i)).
\]
Since \( K_0(A) = \bigoplus_{i=1}^{m(l)} K_0(E_i^l A E_i^l) \), we have an isomorphism
\[
\Phi_{j,k} = \psi_{j,k}^{-1} \circ \varphi_{j,k}^* : K_0(F_{j,k}) \xrightarrow{\varphi_{j,k}^*} \bigoplus_{i=1}^{m(l)} K_0(F_{j,k}(i)) \xrightarrow{\psi_{j,k}^{-1}} K_0(A).
\]
(i) It suffices to show the following diagram
\[
\begin{array}{ccc}
K_0(F_{j,k}) & \longrightarrow & K_0(F_{j+1,k}) \\
\varphi_{j,k}^* \downarrow & & \varphi_{j+1,k}^* \downarrow \\
\bigoplus_{i=1}^{m(l)} K_0(F_{j,k}(i)) & \xrightarrow{\psi_{j,k}^*} & \bigoplus_{i=1}^{m(l)} K_0(F_{j+1,k}(i)) \\
\psi_{j,k} \uparrow & & \psi_{j+1,k} \uparrow \\
K_0(A) & \longrightarrow & K_0(A)
\end{array}
\]
is commutative. For \( x = \sum_{i=1}^{m(l)} E_i^l x E_i^l \in A \), we have
\[
\psi_{j,k}(x) = \sum_{i=1}^{m(l)} T_{\xi(i)} S_{\nu(i)} E_i^l x E_i^l S_{\nu(i)}^* T_{\xi(i)}^* = \sum_{i=1}^{m(l)} S_{\mu(i)} T_{\xi(i)} E_i^l x E_i^l T_{\xi(i)}^* S_{\mu(i)}^*.
\]
Since \( P_i T_{\xi(i)} S_{\nu(i)} E_i^l x E_i^l S_{\nu(i)}^* T_{\xi(i)}^* P_i = T_{\xi(i)} S_{\nu(i)} E_i^l x E_i^l S_{\nu(i)}^* T_{\xi(i)}^* \), we have
\[
\varphi_{j,k}^{-1} \circ \psi_{j,k}(x) = \sum_{i=1}^{m(l)} T_{\xi(i)} S_{\nu(i)} E_i^l x E_i^l S_{\nu(i)}^* T_{\xi(i)}^* 
\]
so that
\[
\iota_{j+1,k} \circ \varphi_{j,k}^{-1} \circ \psi_{j,k}(x) = \sum_{\alpha \in \Sigma^p} \sum_{i=1}^{m(l)} T_{\xi(i)} S_{\nu(i)} \alpha \rho_\alpha (E_i^l x E_i^l) S_{\nu(i)}^* T_{\xi(i)}^*.
\]
Since
\[ S_{\nu(i)}a_\alpha(E_{n}^{l+1})S_{\nu(i)}^{*}\alpha = \sum_{n=1}^{m(l+1)} A_{l+1}^{\rho}(i,\alpha,n)S_{\nu(i)}a_{\alpha}E_{n+1}^{l+1}a_{\alpha}E_{n}^{l+1}S_{\nu(i)}^{*}\alpha \]
and \( A_{l+1}^{\rho}(i,\alpha,n)S_{\nu(i)}a_{\alpha}E_{n+1}^{l+1} = S_{\nu(i)}a_{\alpha}E_{n}^{l+1} \), we have
\[ \sum_{\alpha \in \Sigma^{\rho}} S_{\nu(i)}a_{\alpha}(E_{n}^{l+1}E_{n}^{l+1})S_{\nu(i)}^{*}\alpha = \sum_{n=1}^{m(l+1)} \sum_{\alpha \in \Sigma^{\rho}} S_{\nu(i)}a_{\alpha}E_{n+1}^{l+1}a_{\alpha}E_{n}^{l+1}S_{\nu(i)}^{*}\alpha \]
so that
\[ \ell_{+1,*} \circ \varphi_{j,k}^{-1} \circ \psi_{j,k}(x) = \sum_{\alpha \in \Sigma^{\rho}} \sum_{i=1}^{m(l)} \sum_{n=1}^{m(l+1)} T_{\xi(i)}S_{\nu(i)}a_{\alpha}E_{n+1}^{l+1}a_{\alpha}E_{n}^{l+1}S_{\nu(i)}^{*}\alpha T_{\xi(i)} \times \]
On the other hand,
\[ \psi_{j,k}(\lambda_{\rho}(x)) = \psi_{j,k} \left( \sum_{n=1}^{m(l+1)} \sum_{\alpha \in \Sigma^{\rho}} E_{n+1}^{l+1}a_{\alpha}E_{n}^{l+1} \right) = \sum_{\alpha \in \Sigma^{\rho}} \sum_{i=1}^{m(l)} \sum_{n=1}^{m(l+1)} T_{\xi(i)}S_{\nu(i)}a_{\alpha}E_{n+1}^{l+1}a_{\alpha}E_{n}^{l+1}S_{\nu(i)}^{*}\alpha T_{\xi(i)} \times \]
Therefore we have
\[ \ell_{+1,*} \circ \varphi_{j,k}^{-1} \circ \psi_{j,k}(x) = \psi_{j,k}(\lambda_{\rho}(x)) \]
(ii) is symmetric to (i).

Define the abelian groups of the inductive limits:
\[ G_{\rho} = \operatorname{lim}(\lambda_{\rho} : K_{0}(\mathcal{A}) \to K_{0}(\mathcal{A})) \], \quad \[ G_{\eta} = \operatorname{lim}(\lambda_{\eta} : K_{0}(\mathcal{A}) \to K_{0}(\mathcal{A})) \].
Put the subalgebras of \( \mathcal{F}_{\rho,\eta} \) for \( j, k \in \mathbb{Z}_{+} \)
\[ \mathcal{F}_{\rho,k} = C^{*}(T_{\zeta}S_{\mu}xS_{\nu}^{*}T_{\xi}^{*} \mid \mu, \nu, \zeta, \xi \in \mathcal{B}_{k}(\Lambda_{\rho}), x \in \mathcal{A}) = C^{*}(T_{\zeta}yT_{\xi}^{*} \mid \zeta, \xi \in \mathcal{B}_{k}(\Lambda_{\eta}), y \in \mathcal{F}_{\rho}) \],
\[ \mathcal{F}_{\eta,j} = C^{*}(S_{\mu}T_{\zeta}xT_{\xi}^{*}S_{\nu}^{*} \mid \mu, \nu, \zeta, \xi \in \mathcal{B}_{j}(\Lambda_{\rho}), |\zeta| = |\xi|, x \in \mathcal{A}) = C^{*}(S_{\mu}yS_{\nu}^{*} \mid \mu, \nu, \zeta, \xi \in \mathcal{B}_{j}(\Lambda_{\eta}), y \in \mathcal{F}_{\eta}) \].
By the preceding lemma, we have:

**Lemma 8.6.** For \( j, k \in \mathbb{Z}_{+} \), there exist isomorphisms
\[ \Phi_{\rho,k} : K_{0}(\mathcal{F}_{\rho,k}) \to G_{\rho}, \quad \Phi_{j,\eta} : K_{0}(\mathcal{F}_{j,\eta}) \to G_{\eta} \]
such that the following diagrams are commutative:
Lemma 8.8. For \( \rho \) in Lemma 7.6, respectively.

Proof. Since \( \rho \) we have

\[
\xi \rightarrow \bar{\gamma}_{\rho} \rightarrow \bar{\gamma}_{\rho} \rightarrow \bar{\gamma}_{\rho}.
\]

Lemma 8.7. If \( \xi = (\xi_1, \ldots, \xi_k) \in B_k(\Lambda_\eta), \nu = (\nu_1, \ldots, \nu_j) \in B_j(\Lambda_\rho) \) satisfy the condition \( \rho_\nu(\eta_\xi(1)) \geq E_i^1 \) for some \( i = 1, \ldots, m(l) \) with \( l = j + k \), then \( T_\xi^* T_\xi S_\nu E_i^1 = T_\xi S_\nu E_i^1 \) where \( \xi = (\xi_2, \ldots, \xi_k) \).

Proof. Since \( T_\xi^* T_\xi = T_\xi^* T_\xi T_\xi^* T_\xi = T_\xi T_\xi^* T_\xi T_\xi = T_\xi T_\xi^* T_\xi \), we have

\[
T_\xi^* T_\xi S_\nu E_i^1 = T_\xi S_\nu S_\nu^* T_\xi T_\xi S_\nu E_i^1 = T_\xi S_\nu(\eta_\xi(1)) E_i^1 = T_\xi S_\nu E_i^1.
\]

Let us denote by \( \gamma_{\rho,0}, \gamma_{\eta,0} \) the endomorphisms \( \gamma_{\rho,0}, \gamma_{\eta,0} \) on \( K_0(F_{\rho,\eta}) \) appeared in Lemma 7.6, respectively.

Lemma 8.8. For \( k, j \in \mathbb{Z}_+ \), we have:

(i) The restriction of \( \gamma_{\rho}^{-1} \) to \( K_0(F_{j, k}) \) makes the following diagram commutative:

\[
\begin{array}{ccc}
K_0(F_{j, k}) & \overset{\gamma_{\rho}^{-1}}{\longrightarrow} & K_0(F_{j, k-1}) \\
\Phi_{j, k} & \downarrow & \Phi_{j, k} \\
K_0(A) & \overset{\lambda_{\rho}}{\longrightarrow} & K_0(A)
\end{array}
\]

(ii) The restriction of \( \gamma_{\rho}^{-1} \) to \( K_0(F_{j, k}) \) makes the following diagram commutative:

\[
\begin{array}{ccc}
K_0(F_{j, k}) & \overset{\gamma_{\rho}^{-1}}{\longrightarrow} & K_0(F_{j-1, k}) \\
\Phi_{j, k} & \downarrow & \Phi_{j, k} \\
K_0(A) & \overset{\lambda_{\rho}}{\longrightarrow} & K_0(A)
\end{array}
\]

Proof. (i) Put \( l = j + k \). Take a projection \( p \in M_n(A) \) for some \( n \in \mathbb{N} \).

Since \( A \otimes M_n(C) = \sum_{i=1}^{m(l)} (E_i^1 \otimes 1)(A \otimes M_n)(E_i^1 \otimes 1) \), by putting

\[
p_i^l = (E_i^1 \otimes 1)p(E_i^1 \otimes 1) \in M_n(E_i^1 A E_i^1),
\]

we have \( p = \sum_{i=1}^{m(l)} p_i^l \). Take

\[
\xi(i) = (\xi_1(i), \ldots, \xi_k(i)) \in B_k(\Lambda_\eta), \quad \nu(i) = (\nu_1(i), \ldots, \nu_j(i)) \in B_j(\Lambda_\rho)
\]
as in (8.1) so that $\rho_{\nu(i)}(\eta_{\xi(i)}(1)) \geq E_i^l$ and put $\bar{\xi}(i) = (\xi_2(i), \ldots, \xi_k(i))$ so that $\xi(i) = \xi_1(i)\bar{\xi}(i)$. We have

$$\psi_{j,k*}([p]) = \sum_{i=1}^{m(l)} \{ [T_{\xi(i)} S_{\nu(i)} \otimes 1_n] p_i [S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n] \} \in \bigoplus_{i=1}^{m(l)} K_0(F_{j,k}(i)).$$

As

$$(T_{\xi(i)} S_{\nu(i)} \otimes 1_n) p_i (S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n) \leq T_{\xi(i)} T_{\xi(i)}^* \otimes 1_n,$$

by the preceding lemma we have

$$T_{\xi(i)}^* T_{\xi(i)} S_{\nu(i)} E_i^l = T_{\xi(i)}^* S_{\nu(i)} E_i^l$$

so that by Lemma 7.6

$$\gamma_{\eta}(\{ (T_{\xi(i)} S_{\nu(i)} \otimes 1_n) p_i (S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n) \}) = \{ (T_{\xi(i)} S_{\nu(i)} \otimes 1_n) p_i (S_{\nu(i)}^* T_{\xi(i)}^* \otimes 1_n) \}.$$

Hence $K_0(F_{j,k})$ goes to $K_0(F_{j,k-1})$ by the homomorphism $\gamma_{\eta}^{-1}$. Take $\mu(i) \in B_j(\Lambda_{\eta}), \bar{\zeta}(i) \in B_{k-1}(\Lambda_{\eta})$ such that $T_{\bar{\zeta}(i)} S_{\nu(i)} = S_{\mu(i)} T_{\bar{\zeta}(i)}$ for $i = 1, \ldots, m(l)$. The element

$$\sum_{i=1}^{m(l)} \{ (S_{\mu(i)} T_{\bar{\zeta}(i)} \otimes 1_n) p_i (T_{\bar{\zeta}(i)}^* S_{\mu(i)}^* \otimes 1_n) \} \in K_0(F_{j,k-1})$$

goes to

$$\sum_{i=1}^{m(l)} \sum_{a \in \Sigma^g} [(S_{\mu(i)} T_{\bar{\zeta}(i)} \otimes 1_n) (T_a^* \otimes 1_n) p_i (T_a \otimes 1_n) (T_{\bar{\zeta}(i)}^* S_{\mu(i)}^* \otimes 1_n)] \in K_0(F_{j,k})$$

by $\iota_{\ast, +1}$. The latter one is expressed as

$$\sum_{i=1}^{m(l)} \sum_{a \in \Sigma^g} \{ [(S_{\mu(i)} T_{\bar{\zeta}(i)} \otimes 1_n) (T_a^* \otimes 1_n) p_i (T_a \otimes 1_n) (T_{\bar{\zeta}(i)}^* S_{\mu(i)}^* \otimes 1_n)] \} \in K_0(F_{j,k})$$

in $\bigoplus_{h=1}^{m(l)} K_0(F_{j,k}(h))$. On the other hand, we have

$$\lambda_{\eta}([p]) = \sum_{a \in \Sigma^g} [(T_a^* \otimes 1_n) p(T_a \otimes 1_n)]$$

$$= \sum_{h=1}^{m(l)} \sum_{a \in \Sigma^g} [E_h^l (T_a^* \otimes 1_n) p(T_a \otimes 1_n) E_h^l] \in \bigoplus_{h=1}^{m(l)} K_0(E_h^l A E_h^l),$$
which is expressed as

\[
\sum_{h=1}^{m(l)} \sum_{a \in \Sigma^n} \left[ (T_{\xi(h)} S_{\nu(h)} E^i_h \otimes 1_n) (T^*_a \otimes 1_n) p(T_a \otimes 1_n) (E^i_h S^*_\nu(h) T^*_\xi(h) \otimes 1_n) \right]
\]

\[
= \sum_{h=1}^{m(l)} \sum_{a \in \Sigma^n} \sum_{i=1}^{m(l)} \left[ (T_{\xi(h)} S_{\nu(h)} E^i_h \otimes 1_n) (T^*_a \otimes 1_n) \right.
\]

\[
\cdot p^i(T_a \otimes 1_n) (E^i_h S^*_\nu(h) T^*_\xi(h) \otimes 1_n) \right]
\]

This takes place in \( \bigoplus_{h=1}^{m(l)} K_0(\mathcal{F}_{j,k}(h)) \). Take \( \mu'(h) \in B_j(\Lambda_{\rho}), \zeta'(h) \in B_k(\Lambda_{\eta}) \) such that

\[ T_{\xi(h)} S_{\nu(h)} = S_{\mu'(h)} T_{\zeta'(h)} \] so that the above element is

\[(8.3)\]

\[
\sum_{h=1}^{m(l)} \sum_{a \in \Sigma^n} \sum_{i=1}^{m(l)} \left[ (S_{\mu'(h)} T_{\zeta'(h)} E^i_h \otimes 1_n) (T^*_a \otimes 1_n) p^i(T_a \otimes 1_n) (E^i_h T^*_\zeta'(h) S^*_\mu'(h) \otimes 1_n) \right]
\]

This takes place in \( \bigoplus_{h=1}^{m(l)} K_0(\mathcal{F}_{j,k}(h)) \). Since for \( h, i = 1, \ldots, m(l), a \in \Sigma^n \) their classes of the K-groups coincide such as

\[
[S_{\mu(i)} T_{\zeta(i)} a \otimes 1_n) E^i_h (T^*_a \otimes 1_n)]
\]

\[
= [(S_{\mu'(h)} T_{\zeta'(h)} E^i_h \otimes 1_n) (T^*_a \otimes 1_n) p^i(T_a \otimes 1_n) (E^i_h T^*_\zeta'(h) S^*_\mu'(h) \otimes 1_n)]
\]

\[ \in K_0(\mathcal{F}_{j,k}(h)), \]

the element of (8.2) is equal to the element of (8.3) in \( K_0(\mathcal{F}_{j,k}) \). Thus (i) holds.

(ii) is similar to (i). \( \square \)

We note that for \( j, k \in \mathbb{Z}_+ \),

\[ K_0(\mathcal{F}_{\rho,k}) = \lim_j \{ \iota_{i+1,*} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j+1,k}) \}, \]

\[ K_0(\mathcal{F}_{j,\eta}) = \lim_k \{ \iota_{i,*+1} : K_0(\mathcal{F}_{j,k}) \longrightarrow K_0(\mathcal{F}_{j,k+1}) \}. \]

The following lemma is direct.

**Lemma 8.9.** For \( k, j \in \mathbb{Z}_+ \), the following diagrams are commutative:

(i)

\[
\begin{array}{ccc}
K_0(\mathcal{F}_{j,k}) & \xrightarrow{\gamma_{\eta}^{-1}} & K_0(\mathcal{F}_{j,k-1}) \\
\downarrow \iota_{i+1,*} & & \downarrow \iota_{i+1,*} \\
K_0(\mathcal{F}_{j+1,k}) & \xrightarrow{\gamma_{\eta}^{-1}} & K_0(\mathcal{F}_{j+1,k-1})
\end{array}
\]

Hence \( \gamma_{\eta}^{-1} \) yields a homomorphism from \( K_0(\mathcal{F}_{\rho,k}) \) to \( K_0(\mathcal{F}_{\rho,k-1}) \).
(ii) \[
\begin{array}{c}
K_0(F_{j,k}) \xrightarrow{\gamma_{\rho}^{-1}} K_0(F_{j-1,k}) \\
\downarrow \iota_{\rho,1} \\
K_0(F_{j,k+1}) \xrightarrow{\gamma_{\rho}^{-1}} K_0(F_{j-1,k+1}).
\end{array}
\]

Hence \(\gamma_{\rho}^{-1}\) yields a homomorphism from \(K_0(F_{j,\eta})\) to \(K_0(F_{j-1,\eta})\).

The homomorphisms \(\iota_{+1,*} : K_0(F_{j,k}) \rightarrow K_0(F_{j+1,k}), \ i_{+1} : K_0(F_{j,k}) \rightarrow K_0(F_{j,k+1})\) are naturally induce homomorphisms \(K_0(F_{j,\eta}) \rightarrow K_0(F_{j+1,\eta}), \ i_{+1} : K_0(F_{\rho,k}) \rightarrow K_0(F_{\rho,k+1})\) which we denote by \(i_{+1,\eta}, i_{\rho,1}\) respectively. They are also induced by the identities (5.1), (5.2) respectively.

**Lemma 8.10.** For \(k, j \in \mathbb{Z}_+\), the following diagrams are commutative:

(i) \[
\begin{array}{c}
K_0(F_{\rho,k}) \xrightarrow{\gamma_{\eta}^{-1}} K_0(F_{\rho,k-1}) \\
\downarrow \iota_{\rho,1} \\
K_0(F_{\rho,k+1}) \xrightarrow{\gamma_{\eta}^{-1}} K_0(F_{\rho,k}).
\end{array}
\]

(ii) \[
\begin{array}{c}
K_0(F_{j,\eta}) \xrightarrow{\gamma_{\eta}^{-1}} K_0(F_{j-1,\eta}) \\
\downarrow \iota_{+1,\eta} \\
K_0(F_{j+1,\eta}) \xrightarrow{\gamma_{\eta}^{-1}} K_0(F_{j,\eta}).
\end{array}
\]

**Proof.** (i) As in the proof of Lemma 8.9, one may take an element of \(K_0(F_{\rho,k})\) as in the following form:

\[
\sum_{i=1}^{m(l)} \oplus [(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i(S_{\nu(i)}^* T_{\xi(i)} \otimes 1_n)] \in \bigoplus_{i=1}^{m(l)} K_0(F_{j,k}(i))
\]

for some projection \(p \in M_n(A)\) and \(j, l\) with \(l = j + k\), where \(p_i = (E_i^l \otimes 1)p(E_i^l \otimes 1) \in M_n(E_i^l A E_i^l)\).

Let \(\xi(i) = \xi_1(i)\bar{\xi}(i)\) with \(\xi_1(i) \in \Sigma^n, \bar{\xi}(i) \in B_{k-1}(\Lambda_\eta)\). One may assume that \(T_{\xi(i)}S_{\nu(i)} \neq 0\) so that \(T_{\xi(i)}S_{\nu(i)} = S_{\nu(i)} T_{\xi(i)}\) for some \(\nu(i)' \in B_j(\Lambda_\rho), \xi(i)' \in \Sigma^n\) and \(\bar{\xi}(i)' \in B_{k-1}(\Lambda_\eta)\).
$B_{k-1}(\Lambda_n)$. As in the proof of Lemma 8.9, one has
\[
\gamma_n^{-1}([(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S^{*}_{\nu(i)}T^{*}_{\xi(i)} \otimes 1_n)]
= [(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S^{*}_{\nu(i)}T^{*}_{\xi(i)} \otimes 1_n)]
= [(S_{\nu(i)}T^{*}_{\xi(i)})_{(1_n)}p_i^l(S^{*}_{\nu(i)}T^{*}_{\xi(i)})_{(1_n)}].
\]
Hence we have
\[
\tau_{*,1} \circ \gamma_n^{-1}([(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S^{*}_{\nu(i)}T^{*}_{\xi(i)} \otimes 1_n)]
= \tau_{*,1}([(S_{\nu(i)}T^{*}_{\xi(i)})_{(1_n)}p_i^l(S^{*}_{\nu(i)}T^{*}_{\xi(i)})_{(1_n)}]
= \sum_{b \in \Sigma^n} [(S_{\nu(i)}T^{*}_{\xi(i)})_{(1_n)}(T_{\xi(i)})_{(1_n)}p_i^l(T_b \otimes 1_n)(T^{*}_{\xi(i)})_{(1_n)}]_{(1_n)}.
\]
On the other hand, the equality $T_{\xi(i)}S_{\nu(i)} = T_{\xi(i)}S_{\nu(i)}T^{*}_{\xi(i)}$ implies
\[
\tau_{*,1}([(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S^{*}_{\nu(i)}T^{*}_{\xi(i)} \otimes 1_n)]
= \sum_{b \in \Sigma^n} [(T_{\xi(i)})_{(1_n)}S_{\nu(i)}T^{*}_{\xi(i)})_{(1_n)}p_i^l(T_b \otimes 1_n)(T^{*}_{\xi(i)})_{(1_n)}]_{(1_n)}
\]
and hence
\[
\gamma_n^{-1} \circ \tau_{*,1}([(T_{\xi(i)}S_{\nu(i)} \otimes 1_n)p_i^l(S^{*}_{\nu(i)}T^{*}_{\xi(i)} \otimes 1_n)]
= \sum_{b \in \Sigma^n} \gamma_n^{-1}([(T_{\xi(i)})_{(1_n)}S_{\nu(i)}T^{*}_{\xi(i)})_{(1_n)}(T_{\xi(i)})_{(1_n)}p_i^l(T_b \otimes 1_n)(T^{*}_{\xi(i)})_{(1_n)}]_{(1_n)}
= \sum_{b \in \Sigma^n} [(S_{\nu(i)}T^{*}_{\xi(i)})_{(1_n)}(T_{\xi(i)})_{(1_n)}p_i^l(T_b \otimes 1_n)(T^{*}_{\xi(i)})_{(1_n)}]_{(1_n)}.
\]
(ii) The proof is completely symmetric to the above proof. \hfill \Box

Since the homomorphisms $\lambda_\rho, \lambda_\eta : K_0(A) \rightarrow K_0(A)$ are mutually commutative, the map $\lambda_\eta$ induces a homomorphism on the inductive limit $G_\rho = \lim\{\lambda_\rho : K_0(A) \rightarrow K_0(A)\}$ and similarly $\lambda_\rho$ does on the inductive limit $G_\eta$. They are still denoted by $\lambda_\rho, \lambda_\eta$ respectively.

**Lemma 8.11.** For $k, j \in \mathbb{Z}_+$, the following diagrams are commutative:

(i)
\[
\begin{array}{ccc}
K_0(F_{\rho,k}) & \xrightarrow{\gamma_n^{-1}} & K_0(F_{\rho,k-1}) \\
\Phi_{\rho,k} \downarrow & & \downarrow \Phi_{\rho,k} \\
G_\rho & \xrightarrow{\lambda_n} & G_\rho.
\end{array}
\]
Therefore we have \( \Phi \). The map \( \lambda \) homomorphism \( \lambda \) from \( G \).

Lemma 8.12. For \( k, j \in \mathbb{Z}_+ \), the following diagrams are commutative:
We denote the abelian group $K_0(F_{\rho,\eta})$ by $G_{\rho,\eta}$. Since

$$K_0(F_{\rho,\eta}) = \lim_k \{ \iota_{\rho,+1} : K_0(F_{\rho,k}) \to K_0(F_{\rho,k+1}) \}$$

$$= \lim_j \{ \iota_{+1,\eta} : K_0(F_{j,\eta}) \to K_0(F_{j+1,\eta}) \},$$

one has

$$G_{\rho,\eta} = \lim_k \{ \lambda_\eta : G_{\rho,k} \to G_{\rho,k+1} \} = \lim_j \{ \lambda_\rho : G_{j,\eta} \to G_{j+1,\eta} \}.$$

Define two endomorphisms

$$\sigma_\eta \text{ on } G_{\rho,\eta} = \lim_k \{ \lambda_\eta : G_{\rho,k} \to G_{\rho,k+1} \} \quad \text{and}$$

$$\sigma_\rho \text{ on } G_{\rho,\eta} = \lim_j \{ \lambda_\rho : G_{j,\eta} \to G_{j+1,\eta} \}$$

by setting

$$\sigma_\rho : [g, k] \in G_{\rho,k} \mapsto [g, k-1] \in G_{\rho,k-1} \text{ for } g \in G_\rho \text{ and}$$

$$\sigma_\eta : [h, j] \in G_{j,\eta} \mapsto [h, j-1] \in G_{j-1,\eta} \text{ for } h \in G_\eta.$$

Therefore we have:

**Lemma 8.13.**

(i) There exists an isomorphism $\Phi_{\rho,\infty} : K_0(F_{\rho,\eta}) \to G_{\rho,\eta}$ such that the following diagrams are commutative:

$$\begin{array}{ccc}
K_0(F_{\rho,\eta}) & \xrightarrow{\gamma^{-1}} & K_0(F_{\rho,\eta}) \\
\Phi_{\rho,\infty} \downarrow & & \downarrow \Phi_{\rho,\infty} \\
G_{\rho,\eta} & \xrightarrow{\sigma_\eta} & G_{\rho,\eta}
\end{array}$$

and hence

$$\begin{array}{ccc}
K_0(F_{\rho,\eta}) & \xrightarrow{id \cdot \gamma^{-1}} & K_0(F_{\rho,\eta}) \\
\Phi_{\rho,\infty} \downarrow & & \downarrow \Phi_{\rho,\infty} \\
G_{\rho,\eta} & \xrightarrow{id \cdot \sigma_\eta} & G_{\rho,\eta}
\end{array}$$
There exists an isomorphism $\Phi_{\infty, \eta} : K_0(F_{\rho, \eta}) \to G_{\rho, \eta}$ such that the following diagrams are commutative:

$$
\begin{array}{ccc}
K_0(F_{\rho, \eta}) & \xrightarrow{\gamma_{\rho}^{-1}} & K_0(F_{\rho, \eta}) \\
\Phi_{\infty, \eta} \downarrow && \Phi_{\infty, \eta} \downarrow \\
G_{\rho, \eta} & \xrightarrow{\sigma_{\rho}} & G_{\rho, \eta}
\end{array}
$$

and hence

$$
\begin{array}{ccc}
K_0(F_{\rho, \eta}) & \xrightarrow{id-\gamma_{\rho}^{-1}} & K_0(F_{\rho, \eta}) \\
\Phi_{\infty, \eta} \downarrow && \Phi_{\infty, \eta} \downarrow \\
G_{\rho, \eta} & \xrightarrow{id-\sigma_{\rho}} & G_{\rho, \eta}
\end{array}
$$

Let us denote by $J_A$ the natural embedding $A = F_{0,0} \hookrightarrow F_{\rho, \eta}$, which induces a homomorphism $J_{A*} : K_0(A) \to K_0(F_{\rho, \eta})$.

**Lemma 8.14.** The homomorphism $J_{A*} : K_0(A) \to K_0(F_{\rho, \eta})$ is injective such that 

$$
J_{A*} \circ \lambda_{\rho} = \gamma_{\rho}^{-1} \circ J_{A*} \quad \text{and} \quad J_{A*} \circ \lambda_{\eta} = \gamma_{\eta}^{-1} \circ J_{A*}.
$$

**Proof.** We will first show that the endomorphisms $\lambda_{\rho}, \lambda_{\eta}$ on $K_0(A)$ are both injective. Put a projection $Q_{\alpha} = S_{\alpha}S_{\alpha}^*$ and a subalgebra $A_{\alpha} = \rho_{\alpha}(A)$ of $A$ for $\alpha \in \Sigma^\rho$. Then the endomorphism $\rho_{\alpha}$ on $A$ extends to an isomorphism from $AQ_{\alpha}$ onto $A_{\alpha}$ by setting $\rho_{\alpha}(x) = S_{\alpha}yS_{\alpha}^*, x \in AQ_{\alpha}$ whose inverse is $\phi_{\alpha} : A_{\alpha} \to AQ_{\alpha}$ defined by $\phi_{\alpha}(y) = S_{\alpha}yS_{\alpha}^*, y \in A_{\alpha}$. Hence the induced homomorphism $\rho_{\alpha*} : K_0(AQ_{\alpha}) \to K_0(A_{\alpha})$ is an isomorphism. Since $A = \bigoplus_{\alpha \in \Sigma^\rho} Q_{\alpha}A$, the homomorphism

$$
\sum_{\alpha \in \Sigma^\rho} \phi_{\alpha*} \circ \rho_{\alpha*} : K_0(A) \to \bigoplus_{\alpha \in \Sigma^\rho} K_0(Q_{\alpha}A)
$$

is an isomorphism, one may identify $K_0(A) = \bigoplus_{\alpha \in \Sigma^\rho} K_0(Q_{\alpha}A)$. Let $g \in K_0(A)$ satisfy $\lambda_{\rho}(g) = 0$. Put $g_{\alpha} = \phi_{\alpha*} \circ \rho_{\alpha*}(g) \in K_0(Q_{\alpha}A)$ for $\alpha \in \Sigma^\rho$ so that $g = \sum_{\alpha \in \Sigma^\rho} g_{\alpha}$. As $\rho_{\beta*} \circ \phi_{\alpha*} = 0$ for $\beta \neq \alpha$, one sees $\rho_{\beta*}(g_{\alpha}) = 0$ for $\beta \neq \alpha$. Hence

$$
0 = \lambda_{\rho}(g) = \sum_{\beta \in \Sigma^\rho} \sum_{\alpha \in \Sigma^\rho} \rho_{\beta*}(g_{\alpha}) = \sum_{\alpha \in \Sigma^\rho} \rho_{\alpha*}(g_{\alpha}) \in \bigoplus_{\alpha \in \Sigma^\rho} K_0(A_{\alpha}).
$$

It follows that $\rho_{\alpha*}(g_{\alpha}) = 0$ in $K_0(A_{\alpha})$. Since $\rho_{\alpha*} : K_0(Q_{\alpha}A) \to K_0(A_{\alpha})$ is isomorphic, one sees that $g_{\alpha} = 0$ in $K_0(Q_{\alpha}A)$ for all $\alpha \in \Sigma^\rho$. This implies that $g = \sum_{\alpha \in \Sigma^\rho} g_{\alpha} = 0$ in $K_0(A)$. Therefore the endomorphism $\lambda_{\rho}$ on $K_0(A)$ is injective, and similarly so is $\lambda_{\eta}$.
By the previous lemma, there exists an isomorphism \( \Phi_{j,k} : K_0(F_{j,k}) \to K_0(A) \) such that the diagram
\[
\begin{array}{ccc}
K_0(F_{j,k}) & \xrightarrow{\iota_{j+1,k}} & K_0(F_{j+1,k}) \\
\Phi_{j,k} & & \Phi_{j+1,k} \\
K_0(A) & \xrightarrow{\lambda_{j,k}} & K_0(A)
\end{array}
\]
is commutative so that the embedding \( \iota_{j+1,k} : K_0(F_{j,k}) \to K_0(F_{j+1,k}) \) is injective, and similarly \( \iota_{j,k+1} : K_0(F_{j,k}) \to K_0(F_{j,k+1}) \) is injective. Hence for \( n, m \in \mathbb{N} \), the homomorphism
\[
\iota_{n,m} : K_0(A) = K_0(F_{0,0}) \to K_0(F_{n,m})
\]
defined by the compositions of \( \iota_{j+1,k} \) and \( \iota_{j,k+1} \) is injective. By [44, Theorem 6.3.2 (iii)], one knows \( \text{Ker}(J_{A^*}) = \bigcup_{n,m \in \mathbb{N}} \text{Ker}(\iota_{n,m}) \), so that \( \text{Ker}(J_{A^*}) = 0 \).

We henceforth identify the group \( K_0(A) \) with its image \( J_{A^*}(K_0(A)) \) in \( K_0(F_{\rho,\eta}) \). As in the above proof, not only \( K_0(A)(= K_0(F_{0,0})) \) but also the groups \( K_0(F_{j,k}) \) for \( j, k \) are identified with subgroups of \( K_0(F_{\rho,\eta}) \) via injective homomorphisms from \( K_0(F_{j,k}) \) to \( K_0(F_{\rho,\eta}) \) induced by the embeddings of \( F_{j,k} \) into \( F_{\rho,\eta} \). We note that
\[
(id - \gamma_\eta)K_0(F_{\rho,\eta}) = (id - \gamma_\eta^{-1})K_0(F_{\rho,\eta}),
\]
\[
(id - \gamma_\rho)K_0(F_{\rho,\eta}) = (id - \gamma_\rho^{-1})K_0(F_{\rho,\eta})
\]
and
\[
\text{Ker}(id - \gamma_\rho) \cap \text{Ker}(id - \gamma_\eta) \text{ in } K_0(F_{\rho,\eta}) = \text{Ker}(id - \gamma_\rho^{-1}) \cap \text{Ker}(id - \gamma_\eta^{-1}) \text{ in } K_0(F_{\rho,\eta}).
\]

Denote by \( (id - \gamma_\rho)K_0(F_{\rho,\eta}) \) and \( (id - \gamma_\eta)K_0(F_{\rho,\eta}) \) the subgroups of \( K_0(F_{\rho,\eta}) \) generated by \( (id - \gamma_\rho)K_0(F_{\rho,\eta}) \) and \( (id - \gamma_\eta)K_0(F_{\rho,\eta}) \).

**Lemma 8.15.** Any element in \( K_0(F_{\rho,\eta}) \) is equivalent to some element of \( K_0(A) \) modulo the subgroup \( (id - \gamma_\rho)K_0(F_{\rho,\eta}) + (id - \gamma_\eta)K_0(F_{\rho,\eta}) \).

**Proof.** For \( g \in K_0(F_{\rho,\eta}) \), we may assume that \( g \in K_0(F_{j,k}) \) for some \( j, k \in \mathbb{Z}_+ \). As \( \gamma_\rho^{-1} \) commutes with \( \gamma_\eta^{-1} \), one sees that \( (\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(g) \in K_0(A) \). Put \( g_1 = \gamma_\rho^{-1}(g) \) so that
\[
g - (\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(g) = g_1 - (\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(g_1).
\]
We inductively see that \( g - (\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(g) \) belongs to the subgroup
\[
(id - \gamma_\rho)K_0(F_{\rho,\eta}) + (id - \gamma_\eta)K_0(F_{\rho,\eta}).
\]

Denote by \( (id - \lambda_\rho)K_0(A) + (id - \lambda_\eta)K_0(A) \) the subgroup of \( K_0(A) \) generated by \( (id - \lambda_\rho)K_0(A) \) and \( (id - \lambda_\eta)K_0(A) \).
Lemma 8.16. If \( g \in K_0(A) \) belongs to 
\[
(id - \gamma_\rho^{-1})K_0(F_{\rho,\eta}) + (id - \gamma_\eta^{-1})K_0(F_{\rho,\eta}),
\]
then \( g \) belongs to \((id - \lambda_\rho)K_0(A) + (id - \lambda_\eta)K_0(A)\).

Proof. By the assumption that \( g \in (id - \gamma_\rho^{-1})K_0(F_{\rho,\eta}) + (id - \gamma_\eta^{-1})K_0(F_{\rho,\eta})\), 
there exist \( h_1, h_2 \in K_0(F_{\rho,\eta}) \) such that 
\[
g = (id - \gamma_\rho^{-1})(h_1) + (id - \gamma_\eta^{-1})(h_2).
\]
We may assume that \( h_1, h_2 \in K_0(F_{\rho,\eta}) \) for large enough \( j, k \in \mathbb{Z}_+ \). 
Put \( e_i = (\gamma_\rho^{-1})^j \circ (\gamma_\eta^{-1})^k(h_i) \) which belongs to \( K_0(F_{0,0}) = K_0(A) \) for \( i = 0, 1 \). 
It follows that 
\[
\lambda_\rho^j \circ \lambda_\eta^k(g) = (id - \lambda_\eta)(e_1) + (id - \lambda_\rho)(e_2).
\]
Since \( g \in K_0(A) \) and \( \lambda_\rho^j \circ \lambda_\eta^k(g) \in (id - \lambda_\eta)K_0(A) + (id - \lambda_\rho)K_0(A) \), 
as in the proof of Lemma 8.15, by putting \( g^{(n)} = \lambda_\rho^n(g), g^{(n,m)} = \lambda_\eta^m(g^{(n)}) \in K_0(A) \) 
we have 
\[
g - \lambda_\rho^j \circ \lambda_\eta^k(g) \\
= g - \lambda_\rho(g) + g^{(1)} - \lambda_\rho(g^{(1)}) + g^{(2)} - \lambda_\rho(g^{(2)}) + \cdots + g^{(j-1)} - \lambda_\rho(g^{(j-1)}) \\
+ g^{(j)} - \lambda_\eta(g^{(j)}) + g^{(j,1)} - \lambda_\eta(g^{(j,1)}) + g^{(j,2)} - \lambda_\eta(g^{(j,2)}) + \cdots \\
+ g^{(j,k-1)} - \lambda_\eta(g^{(j,k-1)})
\]
\[
= (id - \lambda_\rho)(g + g^{(1)} + \cdots + g^{(j-1)}) + (id - \lambda_\eta)(g^{(j)} + g^{(j,1)} + \cdots + g^{(j,k-1)})
\]
so that \( g \) belongs to the subgroup \((id - \lambda_\eta)K_0(A) + (id - \lambda_\rho)K_0(A)\). \( \square \)

Hence we obtain the following lemma for the cokernel.

Lemma 8.17. The quotient group 
\[
K_0(F_{\rho,\eta})/((id - \gamma_\rho^{-1})K_0(F_{\rho,\eta}) + (id - \gamma_\eta^{-1})K_0(F_{\rho,\eta}))
\]
is isomorphic to the quotient group 
\[
K_0(A)/((id - \lambda_\eta)K_0(A) + (id - \lambda_\rho)K_0(A)).
\]

Proof. Surjectivity of the quotient map 
\[
K_0(A) \longrightarrow K_0(F_{\rho,\eta})/((id - \gamma_\rho^{-1})K_0(F_{\rho,\eta}) + (id - \gamma_\eta^{-1})K_0(F_{\rho,\eta}))
\]
comes from Lemma 8.15. Its kernel coincides with 
\[
(id - \lambda_\eta)K_0(A) + (id - \lambda_\rho)K_0(A)
\]
by the preceding lemma. \( \square \)

For the kernel, we have:

Lemma 8.18. The subgroup 
\[
\text{Ker}(id - \gamma_\eta^{-1}) \cap \text{Ker}(id - \gamma_\rho^{-1}) \text{ in } K_0(F_{\rho,\eta})
\]
is isomorphic to the subgroup 
\[
\text{Ker}(id - \lambda_\eta) \cap \text{Ker}(id - \lambda_\rho) \text{ in } K_0(A)
\]
through $J_{A^*}$.

**Proof.** For $g \in \text{Ker}(\text{id} - \gamma^{-1}_\eta) \cap \text{Ker}(\text{id} - \gamma^{-1}_\rho)$ in $K_0(F_{\rho,\eta})$, one may assume that $g \in K_0(F_{j,k})$ for some $j, k \in \mathbb{Z}_+$, so that $g = (\gamma^{-1}_\rho)^i(\gamma^{-1}_\eta)^k(g) \in K_0(A)$. Since $\lambda_\eta = \gamma^{-1}_\eta$ and $\lambda_\rho = \gamma^{-1}_\rho$ on $K_0(A)$ under the identification between $J_{A^*}(K_0(A))$ and $K_0(A)$ via $J_{A^*}$, one has that $g \in \text{Ker}(\text{id} - \lambda_\eta) \cap \text{Ker}(\text{id} - \lambda_\rho)$ in $K_0(A)$. The converse inclusion relation

$$\text{Ker}(\text{id} - \lambda_\eta) \cap \text{Ker}(\text{id} - \lambda_\rho) \subset \text{Ker}(\text{id} - \gamma^{-1}_\eta) \cap \text{Ker}(\text{id} - \gamma^{-1}_\rho)$$

is clear through the above identification. \qed

Therefore the short exact sequence for $K_0(O^\kappa_{\rho,\eta})$ in Theorem 7.10 is restated as the following proposition.

**Proposition 8.19.** Assume that $(A, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ forms square and $K_1(F_{\rho,\eta}) = \{0\}$. Then there exists a short exact sequence:

$$0 \longrightarrow K_0(A)/(\text{id} - \lambda_\eta)K_0(A) + (\text{id} - \lambda_\rho)K_0(A) \longrightarrow K_0(O^\kappa_{\rho,\eta}) \longrightarrow \text{Ker}(\text{id} - \lambda_\eta) \cap \text{Ker}(\text{id} - \lambda_\rho) \subset K_0(A) \longrightarrow 0.$$

Let $F_\rho$ be the fixed point algebra $(O_\rho)^{\hat{\rho}}$ of the $C^*$-algebra $O_\rho$ by the gauge action $\hat{\rho}$ for the $C^*$-symbolic dynamical system $(A, \rho, \Sigma^\rho)$. The algebra $F_\rho$ is isomorphic to the subalgebra $F_{\rho,0}$ of $F_{\rho,\eta}$ in a natural way. As in the proof of Lemma 8.15, the group $K_0(F_{\rho,0})$ is regarded as a subgroup of $K_0(F_{\rho,\eta})$ and the restriction of $\gamma^{-1}_\eta$ to $K_0(F_{\rho,0})$ satisfies $\gamma^{-1}_\eta(K_0(F_{\rho,0})) \subset K_0(F_{\rho,0})$ so that $\gamma^{-1}_\eta$ yields an endomorphism on $K_0(F_\rho)$, which we still denote by $\gamma^{-1}_\eta$.

For the group $K_1(O^\kappa_{\rho,\eta})$, we provide several lemmas.

**Lemma 8.20.**

(i) Any element in $K_0(F_{\rho,\eta})$ is equivalent to some element of $K_0(F_{\rho,0})(= K_0(F_\rho))$ modulo the subgroup $(\text{id} - \gamma_\eta)K_0(F_{\rho,\eta})$.

(ii) If $g \in K_0(F_{\rho,0})(= K_0(F_\rho))$ belongs to $(\text{id} - \gamma_\eta)K_0(F_{\rho,\eta})$, then $g$ belongs to $(\text{id} - \gamma_\eta)K_0(F_\rho)$.

As $\gamma_\rho$ commutes with $\gamma_\eta$ on $K_0(F_{\rho,\eta})$, it naturally acts on the quotient group $K_0(F_{\rho,\eta})/(\text{id} - \gamma^{-1}_\eta)K_0(F_{\rho,\eta})$. We denote it by $\gamma_\rho$. Similarly $\lambda_\rho$ naturally induces an endomorphism on $K_0(A)/(\text{id} - \lambda_\eta)K_0(A)$. We denote it by $\lambda_\rho$.

**Lemma 8.21.**

(i) The quotient group $K_0(F_{\rho,\eta})/(\text{id} - \gamma^{-1}_\eta)K_0(F_{\rho,\eta})$ is isomorphic to the quotient group $K_0(F_\rho)/(\text{id} - \gamma^{-1}_\eta)K_0(F_\rho)$, that is also isomorphic to the quotient group $K_0(A)/(\text{id} - \lambda_\eta)K_0(A)$.
Proof. (i) The fact that the three quotient groups
\[ K_0(\mathcal{F}_{\rho,\eta})/(\text{id} - \gamma_{\eta}^{-1})K_0(\mathcal{F}_{\rho,\eta}), \]
\[ K_0(\mathcal{F}_{\rho})/(\text{id} - \gamma_{\eta}^{-1})K_0(\mathcal{F}_{\rho}), \]
\[ K_0(\mathcal{A})/(\text{id} - \lambda_{\eta})K_0(\mathcal{A}), \]
are naturally isomorphic is similarly proved to the previous discussions.

(ii) The kernel \( \text{Ker}(\text{id} - \gamma_{\rho}) \) in \( K_0(\mathcal{F}_{\rho,\eta})/(\text{id} - \gamma_{\eta}^{-1})K_0(\mathcal{F}_{\rho,\eta}) \) is isomorphic to the kernel \( \text{Ker}(\text{id} - \lambda_{\rho}) \) in \( K_0(\mathcal{A})/(\text{id} - \lambda_{\eta})K_0(\mathcal{A}) \).

\[ \Box \]

**Lemma 8.22.** The kernel of \( \text{id} - \gamma_{\rho} \) in \( K_0(\mathcal{F}_{\rho,\eta}) \) is isomorphic to the kernel of \( \text{id} - \gamma_{\rho} \) in \( K_0(\mathcal{F}_{\rho}) \) that is also isomorphic to the kernel of \( \text{id} - \lambda_{\eta} \) in \( K_0(\mathcal{A}) \) such that the quotient group
\[ (\text{Ker}(\text{id} - \gamma_{\eta}) \text{ in } K_0(\mathcal{F}_{\rho,\eta}))/\text{id} - \gamma_{\rho})(\text{Ker}(\text{id} - \gamma_{\eta}) \text{ in } K_0(\mathcal{F}_{\rho,\eta})) \]
is isomorphic to the quotient group
\[ (\text{Ker}(\text{id} - \lambda_{\eta}) \text{ in } K_0(\mathcal{A}))/\text{id} - \lambda_{\rho})(\text{Ker}(\text{id} - \lambda_{\eta}) \text{ in } K_0(\mathcal{A})). \]

**Proof.** The proofs are similar to the previous discussions. \[ \Box \]

Therefore the short exact sequence for \( K_1(\mathcal{O}^\kappa_{\rho,\eta}) \) in Theorem 7.10 is restated as the following proposition.

**Proposition 8.23.** Assume that \( (\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa) \) forms square and
\[ K_1(\mathcal{F}_{\rho,\eta}) = \{0\}. \]

Then there exists a short exact sequence:
\[ 0 \rightarrow (\text{Ker}(\text{id} - \lambda_{\eta}) \text{ in } K_0(\mathcal{A}))/\text{id} - \lambda_{\rho})(\text{Ker}(\text{id} - \lambda_{\eta}) \text{ in } K_0(\mathcal{A})) \]
\[ \rightarrow K_1(\mathcal{O}^\kappa_{\rho,\eta}) \]
\[ \rightarrow \text{Ker}(\text{id} - \lambda_{\rho}) \text{ in } (K_0(\mathcal{A})/(\text{id} - \lambda_{\eta})K_0(\mathcal{A})) \]
\[ \rightarrow 0. \]

We give a condition on \( (\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa) \) which makes \( K_1(\mathcal{F}_{\rho,\eta}) = \{0\} \).

**Lemma 8.24.** Suppose that a \( C^* \)-textile dynamical system
\[ (\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa) \]
forms square and satisfies \( K_1(\mathcal{A}) = \{0\} \). Then \( K_1(\mathcal{F}_{\rho,\eta}) = \{0\} \).

**Proof.** The algebra \( \mathcal{F}_{\rho,\eta} \) is an inductive limit \( C^* \)-algebra of subalgebras \( \mathcal{F}_{j,k} \) with inclusion maps (5.3). Let \( E^l_i, i = 1, \ldots, m(l) \) be the minimal projections.
in $\mathcal{A}_i$ as in Lemma 8.4, which are central in $\mathcal{A}$ such that $\sum_{i=1}^{m(l)} E_i^l = 1$. By Lemma 8.4, we have

$$K_1(\mathcal{F}_{j,k}) = \bigoplus_{i=1}^{m(l)} K_1(\mathcal{F}_{j,k}(i)) = \bigoplus_{i=1}^{m(l)} K_1(E_i^l \mathcal{A} E_i^l) = K_1(\mathcal{A})$$

so that the condition $K_1(\mathcal{A}) = \{0\}$ implies $K_1(\mathcal{F}_{\rho,\eta}) = \{0\}$. □

A C*-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is said to have trivial $K_1$ if $K_1(\mathcal{A}) = \{0\}$.

Consequently we reach the following K-theory formulae for the C*-algebra $\mathcal{O}_{\rho,\eta}^\kappa$ by Proposition 8.19 and Proposition 8.23.

**Theorem 8.25.** Suppose that a C*-textile dynamical system

$$(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$$

forms square having trivial $K_1$. Then there exist short exact sequences for their $K$-groups as in the following way:

$$0 \rightarrow K_0(\mathcal{A})/(\text{id} - \lambda_\rho)K_0(\mathcal{A}) + (\text{id} - \lambda_\eta)K_0(\mathcal{A})$$

$$\rightarrow K_0(\mathcal{O}_{\rho,\eta}^\kappa)$$

$$\rightarrow \text{Ker}(\text{id} - \lambda_\eta) \cap \text{Ker}(\text{id} - \lambda_\rho) \text{ in } K_0(\mathcal{A})$$

$$\rightarrow 0$$

and

$$0 \rightarrow (\text{Ker}(\text{id} - \lambda_\eta) \text{ in } K_0(\mathcal{A}))/\text{id} - \lambda_\rho)(\text{Ker}(\text{id} - \lambda_\eta) \text{ in } K_0(\mathcal{A}))$$

$$\rightarrow K_1(\mathcal{O}_{\rho,\eta}^\kappa)$$

$$\rightarrow \text{Ker}(\text{id} - \bar{\lambda}_\rho) \text{ in } (K_0(\mathcal{A})/(\text{id} - \lambda_\eta)K_0(\mathcal{A}))$$

$$\rightarrow 0$$

where the endomorphisms $\lambda_\rho, \lambda_\eta : K_0(\mathcal{A}) \rightarrow K_0(\mathcal{A})$ are defined by

$$\lambda_\rho([p]) = \sum_{\alpha \in \Sigma^\rho} [\rho_\alpha(p)] \in K_0(\mathcal{A}) \text{ for } [p] \in K_0(\mathcal{A}),$$

$$\lambda_\eta([p]) = \sum_{\alpha \in \Sigma^\eta} [\eta_\alpha(p)] \in K_0(\mathcal{A}) \text{ for } [p] \in K_0(\mathcal{A}).$$

9. Examples

9.1. LR-textile $\lambda$-graph systems. A symbolic matrix

$$\mathcal{M} = [\mathcal{M}(i,j)]_{i,j=1}^N$$

is a matrix whose components consist of formal sums of elements of an alphabet $\Sigma$, such as

$$\mathcal{M} = \begin{bmatrix} a & a + c \\ c & 0 \end{bmatrix} \quad \text{where } \Sigma = \{a, b, c\}.$$
\(\mathcal{M}\) is said to be essential if there is no zero column or zero row. \(\mathcal{M}\) is said to be left-resolving if for each column a symbol does not appear in two different rows. For example, \[
\begin{bmatrix}
  a & a + b \\
  c & 0
\end{bmatrix}
\]
is left-resolving, but \[
\begin{bmatrix}
  a & a + b \\
  c & b
\end{bmatrix}
\]is not left-resolving because of \(b\) at the second column. We assume that symbolic matrices are always essential and left-resolving. We denote by \(\Sigma^\mathcal{M}\) the alphabet \(\Sigma\) of the symbolic matrix \(\mathcal{M}\).

Let \(\mathcal{M} = [\mathcal{M}(i,j)]_{i,j=1}^{N}\) and \(\mathcal{M}' = [\mathcal{M}'(i,j)]_{i,j=1}^{N}\) be \(N \times N\) symbolic matrices over \(\Sigma^\mathcal{M}\) and \(\Sigma^{\mathcal{M}'}\) respectively. Suppose that there is a bijection \(\kappa : \Sigma^\mathcal{M} \rightarrow \Sigma^{\mathcal{M}'}\). Following Nasu’s terminology [34] we say that \(\mathcal{M}\) and \(\mathcal{M}'\) are equivalent under specification \(\kappa\), or simply, specified equivalent if \(\mathcal{M}'\) can be obtained from \(\mathcal{M}\) by replacing every symbol \(\alpha \in \Sigma^\mathcal{M}\) by \(\kappa(\alpha) \in \Sigma^{\mathcal{M}'}\). That is if \(\mathcal{M}(i,j) = \alpha_1 + \cdots + \alpha_n\), then \(\mathcal{M}'(i,j) = \kappa(\alpha_1) + \cdots + \kappa(\alpha_n)\). We write this situation as \(\mathcal{M} \overset{\kappa}{\sim} \mathcal{M}'\) (see [34]).

For a symbolic matrix \(\mathcal{M} = [\mathcal{M}(i,j)]_{i,j=1}^{N}\) over \(\Sigma^\mathcal{M}\), we set for \(\alpha \in \Sigma^\mathcal{M}, i,j = 1, \ldots, N\)
\[
A^\mathcal{M}(i,\alpha,j) = \begin{cases} 1 & \text{if } \alpha \text{ appears in } \mathcal{M}(i,j), \\ 0 & \text{otherwise.} \end{cases}
\]
Put an \(N \times N\) nonnegative matrix \(A^\mathcal{M} = [A^\mathcal{M}(i,j)]_{i,j=1}^{N}\) by setting
\[
A^\mathcal{M}(i,j) = \sum_{\alpha \in \Sigma^\mathcal{M}} A^\mathcal{M}(i,\alpha,j).
\]
Let \(\mathcal{A}\) be an \(N\)-dimensional commutative \(\mathbb{C}^\ast\)-algebra \(\mathbb{C}^N\) with minimal projections \(E_1, \ldots, E_N\) such that
\[
\mathcal{A} = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_N.
\]
We set for \(\alpha \in \Sigma^\mathcal{M}:
\[
\rho^\mathcal{M}_\alpha(E_i) = \sum_{j=1}^{N} A^\mathcal{M}(i,\alpha,j)E_j, \quad i = 1, \ldots, N.
\]
Then we have a \(\mathbb{C}^\ast\)-symbolic dynamical system \((\mathcal{A}, \rho^\mathcal{M}, \Sigma^\mathcal{M})\).

Let \(\mathcal{M} = [\mathcal{M}(i,j)]_{i,j=1}^{N}\) and \(\mathcal{N} = [\mathcal{N}(i,j)]_{i,j=1}^{N}\) be \(N \times N\) symbolic matrices over \(\Sigma^\mathcal{M}\) and \(\Sigma^\mathcal{N}\) respectively. We have two \(\mathbb{C}^\ast\)-symbolic dynamical systems \((\mathcal{A}, \rho^\mathcal{N}, \Sigma^\mathcal{N})\) and \((\mathcal{A}, \rho^\mathcal{M}, \Sigma^\mathcal{M})\). Put
\[
\Sigma^{\mathcal{M}\mathcal{N}} = \{(a, b) \in \Sigma^\mathcal{M} \times \Sigma^\mathcal{N} | \rho^\mathcal{N}_b \circ \rho^\mathcal{M}_a \neq 0\},
\]
\[
\Sigma^{\mathcal{N}\mathcal{M}} = \{(a, \beta) \in \Sigma^\mathcal{N} \times \Sigma^\mathcal{M} | \rho^\mathcal{M}_\beta \circ \rho^\mathcal{N}_a \neq 0\}.
\]
Suppose that there is a bijection \(\kappa\) from \(\Sigma^{\mathcal{M}\mathcal{N}}\) to \(\Sigma^{\mathcal{N}\mathcal{M}}\) such that \(\kappa\) yields a specified equivalence
\[
(9.1) \quad \mathcal{M}\mathcal{N} \overset{\kappa}{\sim} \mathcal{N}\mathcal{M}
\]
and fix it.
Proposition 9.1. Keep the above situations. The specified equivalence (9.1) induces a specification \( \kappa : \Sigma^\mathcal{MN} \to \Sigma^\mathcal{NM} \) such that

\[
(9.2) \quad \rho_b^N \circ \rho_\alpha^M = \rho_\beta^M \circ \rho_a^N \quad \text{if} \quad \kappa(\alpha, b) = (a, \beta).
\]

Hence \((\mathcal{A}, \rho^M, \rho^N, \Sigma^\mathcal{M}, \Sigma^\mathcal{N}, \kappa)\) gives rise to a \( C^* \)-textile dynamical system which forms square having trivial \( K_1 \).

Proof. Since \( \mathcal{MN} \cong \mathcal{NM} \), one sees that for \( i, j = 1, 2, \ldots, N \),

\[
\kappa(\mathcal{MN}(i, j)) = \mathcal{NM}(i, j).
\]

For \( (\alpha, b) \in \Sigma^\mathcal{MN} \), there exists \( i, k \) such that

\[
\rho_b^N \circ \rho_\alpha^M(E_i) \geq E_k.
\]

As \( \kappa(\alpha, b) \) appears in \( \mathcal{NM}(i, k) \), by putting \( (a, \beta) = \kappa(\alpha, b) \), we have

\[
\rho_\beta^M \circ \rho_a^N(E_i) \geq E_k.
\]

Hence \( \kappa(\alpha, b) \in \Sigma^\mathcal{NM} \). One indeed sees that \( \rho_b^N \circ \rho_\alpha^M = \rho_\beta^M \circ \rho_a^N \) by the relation \( \mathcal{MN} \cong \mathcal{NM} \). \( \square \)

Two symbolic matrices satisfying (9.1) give rise to an LR textile system that has been introduced by Nasu (see [34]). Textile systems introduced by Nasu give a strong tool to analyze automorphisms and endomorphisms of topological Markov shifts. The author has generalized LR-textile systems to LR-textile \( \lambda \)-graph systems which consist of two pairs of sequences \( (\mathcal{M}, I) = (\mathcal{M}_{l,l+1}, I_{l,l+1})_{l \in \mathbb{Z}_+} \) and \( (\mathcal{N}, I) = (\mathcal{N}_{l,l+1}, I_{l,l+1})_{l \in \mathbb{Z}_+} \) such that

\[
(9.3) \quad \mathcal{M}_{l,l+1} \mathcal{N}_{l+1,l+2} \cong \mathcal{N}_{l,l+1} \mathcal{M}_{l+1,l+2}, \quad l \in \mathbb{Z}_+
\]

through a specification \( \kappa \) ([28]). We denote the LR-textile \( \lambda \)-graph system by \( \mathcal{T}_{K}^{\mathcal{MN}} \). Denote by \( \mathcal{L}^\mathcal{M} \) and \( \mathcal{L}^\mathcal{N} \) the associated \( \lambda \)-graph systems respectively. Since \( \mathcal{L}^\mathcal{M} \) and \( \mathcal{L}^\mathcal{N} \) have common sequences \( V^\mathcal{M}_i = V^\mathcal{N}_i, l \in \mathbb{Z}_+ \) of vertices which denoted by \( V_\ell, \ell \in \mathbb{Z}_+ \), and its common inclusion matrices \( I_{l,l+1}, l \in \mathbb{Z}_+ \). Hence \( \mathcal{L}^\mathcal{M} \) and \( \mathcal{L}^\mathcal{N} \) form square in the sense of [28, p.170]. Let \( (\mathcal{A}_\mathcal{M}, \rho^\mathcal{M}, \Sigma^\mathcal{M}) \) and \( (\mathcal{A}_\mathcal{N}, \rho^\mathcal{N}, \Sigma^\mathcal{N}) \) be the associated \( C^* \)-symbolic dynamical systems with the \( \lambda \)-graph systems \( \mathcal{L}^\mathcal{M} \) and \( \mathcal{L}^\mathcal{N} \) respectively. Since both the algebras \( \mathcal{A}_\mathcal{M} \) and \( \mathcal{A}_\mathcal{N} \) are the \( C^* \)-algebras of inductive limit of the system \( I^\mathcal{M}_l : C(V^\mathcal{M}_l) \to C(V^\mathcal{M}_{l+1}), l \in \mathbb{Z}_+ \), they are identical, which is denoted by \( \mathcal{A} \). It is easy to see that the relation (9.3) implies

\[
(9.4) \quad \rho_\alpha^M \circ \rho_b^N = \rho_a^M \circ \rho_\beta^N \quad \text{if} \quad \kappa(\alpha, b) = (a, \beta).
\]

Proposition 9.2. An LR-textile \( \lambda \)-graph system \( \mathcal{T}_{K}^{\mathcal{MN}} \) yields a \( C^* \)-textile dynamical system \( (\mathcal{A}, \rho^\mathcal{M}, \rho^\mathcal{N}, \Sigma^\mathcal{M}, \Sigma^\mathcal{N}, \kappa) \) which forms square. Conversely, a \( C^* \)-textile dynamical system \( (\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa) \) which forms square yields
an LR-textile λ-graph system \( T_{K,M} \) such that the associated \( C^* \)-textile dynamical system written \((A_{p,\eta}, \rho^M, \rho^N, \Sigma^M, \Sigma^N, \kappa)\) is a subsystem of \((A, \rho, \eta, \Sigma^p, \Sigma^q, \kappa)\) in the sense that the relations:

\[
A_{p,\eta} \subset A, \quad \rho|_{A_{p,\eta}} = \rho^M, \quad \eta|_{A_{p,\eta}} = \rho^N
\]

hold.

**Proof.** Let \( T_{K,M} \) be an LR-textile λ-graph system. As in the above discussions, we have a \( C^* \)-textile dynamical system \((A, \rho^M, \rho^N, \Sigma^M, \Sigma^N, \kappa)\). Conversely, let \((A, \rho, \eta, \Sigma^p, \Sigma^q, \kappa)\) be a \( C^* \)-textile dynamical system which forms square. Put for \( l \in \mathbb{N} \)

\[
A_l^\rho = C^*(\rho_l(1) : \mu \in B_l(A_\rho)), \quad A_l^\eta = C^*(\eta_l(1) : \xi \in B_l(A_\eta)).
\]

Since \( A_l^\rho = A_l^\eta \) and they are commutative and of finite dimensional, the algebra

\[
A_{p,\eta} = \bigcup_{l \in \mathbb{Z}_+} A_l^\rho = \bigcup_{l \in \mathbb{Z}_+} A_l^\eta
\]

is a commutative AF-subalgebra of \( A \). It is easy to see that both \((A_{p,\eta}, \rho, \Sigma^p)\) and \((A_{p,\eta}, \eta, \Sigma^q)\) are \( C^* \)-symbolic dynamical systems such that

\[
\eta_{b} \circ \rho_{a} = \rho_{\beta} \circ \eta_{a} \quad \text{if} \quad \kappa(a, b) = (a, \beta)
\]

By [27], there exist λ-graph systems \( \mathcal{L}_\rho \) and \( \mathcal{L}_\eta \) whose \( C^* \)-symbolic dynamical systems are \((A_{p,\eta}, \rho, \Sigma^p)\) and \((A_{p,\eta}, \eta, \Sigma^q)\) respectively. Let \((\mathcal{M}_\rho, I_\rho)\) and \((\mathcal{M}_\eta, I_\eta)\) be the associated symbolic matrix systems. It is easy to see that the relation (9.5) implies

\[
\mathcal{M}_{l+1,l+2}^\rho \sim \mathcal{M}_{l+1,l+2}^\eta, \quad l \in \mathbb{Z}_+.
\]

Hence we have an LR-textile λ-graph system \( T_{K,M} \). It is direct to see that the associated \( C^* \)-textile dynamical system is \((A_{p,\eta}, \rho|_{A_{p,\eta}}, \eta|_{A_{p,\eta}}, \Sigma^p, \Sigma^q, \kappa)\). \( \Box \)

Let \( A \) be an \( N \times N \) matrix with entries in nonnegative integers. We may consider a directed graph \( G_A = (V_A, E_A) \) with vertex set \( V_A \) and edge set \( E_A \). The vertex set \( V_A \) consists of \( N \) vertices which we denote by \( \{v_1, \ldots, v_N\} \). We equip \( A(i,j) \) edges from the vertex \( v_i \) to the vertex \( v_j \). Denote by \( E_A \) the set of the edges. Let \( \Sigma_A = E_A \) and the labeling map \( \lambda_A : E_A \rightarrow \Sigma_A \) be defined as the identity map. Then we have a labeled directed graph denoted by \( G_A \) as well as a symbolic matrix \( M_A = [M_A(i,j)]_{i,j=1}^N \) by setting

\[
M_A(i,j) = \begin{cases} e_1 + \cdots + e_n & \text{if } e_1, \ldots, e_n \text{ are edges from } v_i \text{ to } v_j, \\ 0 & \text{if there is no edge from } v_i \text{ to } v_j. \end{cases}
\]

Let \( B \) be an \( N \times N \) matrix with entries in nonnegative integers such that

\[
AB = BA.
\]
The equality (9.6) implies that the cardinal numbers of the sets of the pairs of directed edges
\[ \Sigma^{AB}(i,j) = \left\{ (e,f) \in E_A \times E_B \mid s(e) = v_i, t(e) = s(f), t(f) = v_j \right\} \]
and
\[ \Sigma^{BA}(i,j) = \left\{ (f,e) \in E_B \times E_A \mid s(f) = v_i, t(f) = s(e), t(e) = v_j \right\} \]
coincide with each other for each \( v_i \) and \( v_j \). We put
\[ \Sigma^{AB} = \bigcup_{i,j=1}^{N} \Sigma^{AB}(i,j) \]
and
\[ \Sigma^{BA} = \bigcup_{i,j=1}^{N} \Sigma^{BA}(i,j) \]
so that one may take a bijection \( \kappa : \Sigma^{AB} \rightarrow \Sigma^{BA} \) which gives rise to a specified equivalence \( \mathcal{M}_A \mathcal{M}_B \cong \mathcal{M}_B \mathcal{M}_A \). We then have a \( C^* \)-textile dynamical system
\[(A, \rho^A, \rho^B, \Sigma^A, \Sigma^B, \kappa)\]
which we denote by
\[(A, \rho^A, \rho^B, \Sigma^A, \Sigma^B, \kappa).\]
The associated \( C^* \)-algebra is denoted by \( \mathcal{O}^\kappa_{A,B} \). The algebra \( \mathcal{O}^\kappa_{A,B} \) depends on the choice of a specification \( \kappa : \Sigma^{AB} \rightarrow \Sigma^{BA} \). The algebras are \( C^* \)-algebras associated to textile systems studied by V. Deaconu [9]. By Theorem 8.25, we have:

**Proposition 9.3.** Keep the above situations. There exist short exact sequences:
\[
0 \rightarrow \mathbb{Z}^N/( (1-A)\mathbb{Z}^N + (1-B)\mathbb{Z}^N) \rightarrow K_0(\mathcal{O}^\kappa_{A,B}) \rightarrow \text{Ker}(1-A) \cap \text{Ker}(1-B) \text{ in } \mathbb{Z}^N \rightarrow 0
\]

and
\[
0 \rightarrow (\text{Ker}(1-B) \text{ in } \mathbb{Z}^N)/(1-A)(\text{Ker}(1-B) \text{ in } \mathbb{Z}^N) \rightarrow K_1(\mathcal{O}^\kappa_{A,B}) \rightarrow \text{Ker}(1-A) \text{ in } \mathbb{Z}^N/(1-B)\mathbb{Z}^N \rightarrow 0.
\]

We consider \( 1 \times 1 \) matrices \([N]\) and \([M]\) with its entries \( N \) and \( M \) respectively for \( 1 < N, M \in \mathbb{N} \). Let \( G_N \) be a directed graph with one vertex and \( N \) directed self-loops. Similarly we consider a directed graph \( G_M \) with \( M \) directed self-loops at the vertex. The self-loops are denoted by \( \Sigma^N = \{e_1, \ldots, e_N\} \) and \( \Sigma^M = \{f_1, \ldots, f_M\} \) respectively. As a specification \( \kappa \), we take the exchanging map \( (e,f) \in \Sigma^N \times \Sigma^M \rightarrow (f,e) \in \Sigma^M \times \Sigma^N \)
which we will fix. Put
\[ \rho^N_{e_i}(1) = 1, \quad \rho^M_{f_j}(1) = 1 \quad \text{for } i = 1, \ldots, N, \ j = 1, \ldots, M. \]
Then we have a \( C^* \)-textile dynamical system
\[(\mathbb{C}, \rho^N, \rho^M, \Sigma^N, \Sigma^M, \kappa).\]
The associated \( C^* \)-algebra is denoted by \( \mathcal{O}^\kappa_{N,M} \).
Lemma 9.4. \( \mathcal{O}_{N,M}^\kappa = \mathcal{O}_N \otimes \mathcal{O}_M \).

**Proof.** Let \( s_i, i = 1, \ldots, N \) and \( t_j, i = 1, \ldots, M \) be the generating isometries of the Cuntz algebra \( \mathcal{O}_N \) and those of \( \mathcal{O}_M \) respectively which satisfy
\[
\sum_{i=1}^{N} s_i s_i^* = 1, \quad \sum_{j=1}^{M} t_j t_j^* = 1.
\]

Let \( S_i, i = 1, \ldots, N \) and \( T_j, i = 1, \ldots, M \) be the generating isometries of \( \mathcal{O}_{N,M}^\kappa \) satisfying
\[
\sum_{i=1}^{N} S_i S_i^* = 1, \quad \sum_{j=1}^{M} T_j T_j^* = 1
\]
and
\[
S_i T_j = T_j S_i, \quad i = 1, \ldots, N, \quad j = 1, \ldots, M.
\]

The universality of \( \mathcal{O}_{N,M}^\kappa \) subject to the relations and that of the tensor product \( \mathcal{O}_N \otimes \mathcal{O}_M \) ensure us that the correspondence \( \Phi : \mathcal{O}_{N,M} \rightarrow \mathcal{O}_N \otimes \mathcal{O}_M \) given by \( \Phi(S_i) = s_i \otimes 1, \Phi(T_j) = 1 \otimes t_j \) yields an isomorphism. \( \square \)

Although we may easily compute the \( K \)-groups \( K_i(\mathcal{O}_{M,N}^\kappa) \) by using the Künneth formula for \( K_i(\mathcal{O}_N \otimes \mathcal{O}_M) \) ([46]), we will compute them by Proposition 9.3 as in the following way.

**Proposition 9.5** (cf. [19]). For \( 1 < N, M \in \mathbb{N} \), the \( C^* \)-algebra \( \mathcal{O}_{N,M}^\kappa \) is simple, purely infinite, such that
\[
K_0(\mathcal{O}_{N,M}^\kappa) \cong K_1(\mathcal{O}_{N,M}^\kappa) \cong \mathbb{Z}/d\mathbb{Z}
\]
where \( d = \gcd(N-1, M-1) \) the greatest common divisor of \( N-1, M-1 \).

**Proof.** It is easy to see that the group \( \mathbb{Z}/((N-1)\mathbb{Z}+(N-1)\mathbb{Z}) \) is isomorphic to \( \mathbb{Z}/d\mathbb{Z} \). As \( \text{Ker}(N-1) = \text{Ker}(M-1) = 0 \) in \( \mathbb{Z} \), we see that
\[
K_0(\mathcal{O}_{N,M}^\kappa) \cong \mathbb{Z}/d\mathbb{Z}.
\]

It is elementary to see that the subgroup
\[
\{ [k] \in \mathbb{Z}/(M-1)\mathbb{Z} \mid (N-1)k \in (M-1)\mathbb{Z} \}
\]
of \( \mathbb{Z}/(M-1)\mathbb{Z} \) is isomorphic to \( \mathbb{Z}/d\mathbb{Z} \). Hence we have
\[
K_1(\mathcal{O}_{N,M}^\kappa) \cong \mathbb{Z}/d\mathbb{Z}. \quad \square
\]

We will generalize the above examples from the view point of tensor products.
9.2. Tensor products. Let \((A^\rho, \rho, \Sigma^\rho)\) and \((A^\eta, \eta, \Sigma^\eta)\) be \(C^*\)-symbolic dynamical systems. We will construct a \(C^*\)-textile dynamical system by taking tensor product. Put
\[
\tilde{A} = A^\rho \otimes A^\eta, \quad \tilde{\rho}_\alpha = \rho_\alpha \otimes \text{id}, \quad \tilde{\eta}_\alpha = \text{id} \otimes \eta_\alpha, \quad \Sigma^{\tilde{\rho}} = \Sigma^\rho, \quad \Sigma^{\tilde{\eta}} = \Sigma^\eta
\]
for \(\alpha \in \Sigma^\rho, \eta \in \Sigma^\eta\), where \(\otimes\) means the minimal \(C^*\)-tensor product \(\otimes_{\text{min}}\).

For \((\alpha, \eta) \in \Sigma^\rho \times \Sigma^\eta\), we see \(\eta_\eta \circ \rho_\alpha(1) \neq 0\) if and only if \(\eta_\eta(1) \neq 0, \rho_\alpha(1) \neq 0\), so that
\[
\Sigma^{\tilde{\rho}\tilde{\eta}} = \Sigma^\rho \times \Sigma^\eta \quad \text{and similarly} \quad \Sigma^{\tilde{\eta}\tilde{\rho}} = \Sigma^\eta \times \Sigma^\rho.
\]
Define \(\tilde{\kappa} : \Sigma^{\tilde{\rho}\tilde{\eta}} \rightarrow \Sigma^{\tilde{\eta}\tilde{\rho}}\) by setting \(\tilde{\kappa}(\alpha, \eta) = (\eta, \alpha)\).

Lemma 9.6. \((\tilde{A}, \tilde{\rho}, \tilde{\eta}, \Sigma^{\tilde{\rho}}, \Sigma^{\tilde{\eta}}, \tilde{\kappa})\) is a \(C^*\)-textile dynamical system.

Proof. By [2], we have \(Z_{\tilde{A}} = Z_{A^\rho} \otimes Z_{A^\eta}\) so that
\[
\tilde{\rho}_\alpha(Z_{\tilde{A}}) \subseteq Z_{\tilde{A}}, \quad \alpha \in \Sigma^{\tilde{\rho}} \quad \text{and} \quad \tilde{\eta}_\alpha(Z_{\tilde{A}}) \subseteq Z_{\tilde{A}}, \quad \alpha \in \Sigma^{\tilde{\eta}}.
\]
We also have \(\sum_{\alpha \in \Sigma^{\tilde{\rho}}} \tilde{\rho}_\alpha(1) = \sum_{\alpha \in \Sigma^\rho} \rho_\alpha(1) \otimes 1 \geq 1\), and similarly \(\sum_{\alpha \in \Sigma^{\tilde{\eta}}} \tilde{\eta}_\alpha(1) \geq 1\)
so that both families \(\{\tilde{\rho}_\alpha\}_{\alpha \in \Sigma^{\tilde{\rho}}}\) and \(\{\tilde{\eta}_\alpha\}_{\alpha \in \Sigma^{\tilde{\eta}}}\) of endomorphisms are essential.

Since \(\{\rho_\alpha\}_{\alpha \in \Sigma^\rho}\) is faithful on \(A^\rho\), the homomorphism
\[
x \in A^\rho \longrightarrow \sum_{\alpha \in \Sigma^\rho} \oplus \rho_\alpha(x) \in \sum_{\alpha \in \Sigma^\rho} \oplus A^\rho
\]
is injective so that the homomorphism
\[
x \otimes y \in A^\rho \otimes A^\eta \longrightarrow \sum_{\alpha \in \Sigma^{\tilde{\rho}}} \oplus \rho_\alpha(x) \otimes y \in \sum_{\alpha \in \Sigma^{\tilde{\rho}}} \oplus A^\rho \otimes A^\eta
\]
is injective. This implies that \(\{\tilde{\rho}_\alpha\}_{\alpha \in \Sigma^{\tilde{\rho}}}\) is faithful. Similarly, so is \(\{\tilde{\eta}_\alpha\}_{\alpha \in \Sigma^{\tilde{\eta}}}\).

Hence \((\tilde{A}, \tilde{\rho}, \Sigma^{\tilde{\rho}})\) and \((\tilde{A}, \tilde{\eta}, \Sigma^{\tilde{\eta}})\) are both \(C^*\)-symbolic dynamical systems. It is direct to see that \(\tilde{\eta}_\eta \circ \tilde{\rho}_\alpha = \tilde{\rho}_\alpha \circ \tilde{\eta}_\eta\) for \((\alpha, \eta) \in \Sigma^{\tilde{\rho}\tilde{\eta}}\). Therefore \((\tilde{A}, \tilde{\rho}, \tilde{\eta}, \Sigma^{\tilde{\rho}}, \Sigma^{\tilde{\eta}}, \tilde{\kappa})\) is a \(C^*\)-textile dynamical system. \(\square\)

We call \((\tilde{A}, \tilde{\rho}, \tilde{\eta}, \Sigma^{\tilde{\rho}}, \Sigma^{\tilde{\eta}}, \tilde{\kappa})\) the tensor product between \((A^\rho, \rho, \Sigma^\rho)\) and \((A^\eta, \eta, \Sigma^\eta)\).

Denote by \(S_\alpha, \alpha \in \Sigma^{\tilde{\rho}}, T_\eta, \eta \in \Sigma^{\tilde{\eta}}\) the generating partial isometries of the \(C^*\)-algebra \(O_{\tilde{\rho}, \tilde{\eta}}\) for the \(C^*\)-textile dynamical system
\[
(\tilde{A}, \tilde{\rho}, \tilde{\eta}, \Sigma^{\tilde{\rho}}, \Sigma^{\tilde{\eta}}, \tilde{\kappa})\).

By the universality for the algebra \(O_{\tilde{\rho}, \tilde{\eta}}\) subject to the relations \((\tilde{\rho}, \tilde{\eta}; \tilde{\kappa})\), the algebra \(D_{\tilde{\rho}, \tilde{\eta}}\) is isomorphic to the tensor product \(D_{\rho} \otimes D_{\eta}\) through the correspondence
\[
S_\mu T_\xi(x \otimes y)T_\xi^* S_\mu^* \leftrightarrow S_\mu xS_\mu^* \otimes T_\xi yT_\xi^*
\]
for \(\mu \in B_{(\Lambda_\rho)}, \xi \in B_{(\Lambda_\eta)}, \ x \in A^\rho, y \in A^\eta\).

Lemma 9.7. Suppose that \((A^\rho, \rho, \Sigma^\rho)\) and \((A^\eta, \eta, \Sigma^\eta)\) are both free (resp. AF-free). Then the tensor product \((\tilde{A}, \tilde{\rho}, \tilde{\eta}, \Sigma^{\tilde{\rho}}, \Sigma^{\tilde{\eta}}, \tilde{\kappa})\) is free (resp. AF-free).
Proof. Suppose that \((A^\rho, \rho, \Sigma^\rho)\) and \((A^\eta, \eta, \Sigma^\eta)\) are both free. There exist increasing sequences \(A^\rho_l, l \in \mathbb{Z}_+\) and \(A^\eta_l, l \in \mathbb{Z}_+\) of \(C^*\)-subalgebras of \(A^\rho\) and \(A^\eta\) satisfying the conditions of their freeness respectively. Put
\[
\tilde{A}_l = A^\rho_l \otimes A^\eta_l, \quad l \in \mathbb{Z}_+.
\]
It is clear that:
1. \(\tilde{\rho}_\alpha(\tilde{A}_l) \subset \tilde{A}_{l+1}, \alpha \in \Sigma^\rho\) and \(\tilde{\eta}_\alpha(\tilde{A}_l) \subset \tilde{A}_{l+1}, \alpha \in \Sigma^\eta\) for \(l \in \mathbb{Z}_+\).
2. \(\bigcup_{l \in \mathbb{Z}_+} \tilde{A}_l\) is dense in \(\tilde{A}\).

We will show that the condition (3) for \(\tilde{A}\) in Definition 5.3 holds. Take and fix arbitrary \(j, k, l \in \mathbb{N}\) with \(j + k \leq l\). For \(j \leq l\), one may take a projection \(q_\rho \in D_\rho \cap A^\rho_l\) satisfying the condition (3) of the freeness of \((A^\rho, \rho, \Sigma^\rho)\), and similarly for \(k \leq l\), one may take a projection \(q_\eta \in D_\eta \cap A^\eta_l\). Put \(q = q_\rho \otimes q_\eta \in D_\rho \otimes D_\eta\) so that \(q \in D_\rho \otimes D_\eta\). As the maps \(\Phi^\rho_l : x \in A^\rho_l \rightarrow q_\rho x q_\rho \in q_\rho A^\rho_l\) and \(\Phi^\eta_l : y \in A^\eta_l \rightarrow q_\eta y q_\eta \in q_\eta A^\eta_l\) are both isomorphisms, the tensor product
\[
\Phi^\rho_l \otimes \Phi^\eta_l : x \otimes y \in A^\rho_l \otimes A^\eta_l \rightarrow (q_\rho \otimes q_\eta)(x \otimes y) \in (q_\rho \otimes q_\eta)(A^\rho_l \otimes A^\eta_l)
\]
is isomorphic. Hence \(qa \neq 0\) for \(0 \neq a \in \tilde{A}_l\). It is straightforward to see that \(q\) satisfies the condition (3) (ii) of Definition 5.3. Therefore the tensor product \((\tilde{A}, \tilde{\rho}, \tilde{\eta}, \Sigma^\rho, \Sigma^\eta, \tilde{\kappa})\) is free. It is obvious to see that if both \((A^\rho, \rho, \Sigma^\rho)\) and \((A^\eta, \eta, \Sigma^\eta)\) are AF-free, then \((\tilde{A}, \tilde{\rho}, \tilde{\eta}, \Sigma^\rho, \Sigma^\eta, \tilde{\kappa})\) is AF-free. \(\square\)

Proposition 9.8. Suppose that \((A^\rho, \rho, \Sigma^\rho)\) and \((A^\eta, \eta, \Sigma^\eta)\) are both free. Then the \(C^*\)-algebra \(O^\rho_{\tilde{\rho}, \tilde{\eta}}\) for the tensor product \(C^*\)-textile dynamical system \((\tilde{A}, \tilde{\rho}, \tilde{\eta}, \Sigma^\rho, \Sigma^\eta, \tilde{\kappa})\) is isomorphic to the minimal tensor product \(O^\rho \otimes O^\eta\) of the \(C^*\)-algebras between \(O^\rho\) and \(O^\eta\). If in particular, \((A^\rho, \rho, \Sigma^\rho)\) and \((A^\eta, \eta, \Sigma^\eta)\) are both irreducible, the \(C^*\)-algebra \(O^\rho_{\tilde{\rho}, \tilde{\eta}}\) is simple.

Proof. Suppose that \((A^\rho, \rho, \Sigma^\rho)\) and \((A^\eta, \eta, \Sigma^\eta)\) are both free. By the preceding lemma, the tensor product \((\tilde{A}, \tilde{\rho}, \tilde{\eta}, \Sigma^\rho, \Sigma^\eta, \tilde{\kappa})\) is free and hence satisfies condition (1). Let \(s_\alpha, \alpha \in \Sigma^\rho\) and \(t_a, a \in \Sigma^\eta\) be the generating partial isometries of the \(C^*\)-algebras \(O^\rho\) and \(O^\eta\) respectively. Let \(S_\alpha, \alpha \in \Sigma^\rho\) and \(T_a, a \in \Sigma^\eta\) be the generating partial isometries of the \(C^*\)-algebra \(O^\rho_{\tilde{\rho}, \tilde{\eta}}\). By the uniqueness of the algebra \(O^\rho_{\tilde{\rho}, \tilde{\eta}}\) with respect to the relations \((\tilde{\rho}, \tilde{\eta}; \tilde{\kappa})\), the correspondence
\[
S_\alpha \rightarrow s_\alpha \otimes 1 \in O^\rho \otimes O^\eta, \quad T_a \rightarrow 1 \otimes t_a \in O^\rho \otimes O^\eta
\]
naturally gives rise to an isomorphism from \(O^\rho_{\tilde{\rho}, \tilde{\eta}}\) onto the tensor product \(O^\rho \otimes O^\eta\).

If in particular, \((A^\rho, \rho, \Sigma^\rho)\) and \((A^\eta, \eta, \Sigma^\eta)\) are both irreducible, the \(C^*\)-algebras \(O^\rho\) and \(O^\eta\) are both simple so that \(O^\rho_{\tilde{\rho}, \tilde{\eta}}\) is simple. \(\square\)

We remark that the tensor product \((\tilde{A}, \tilde{\rho}, \tilde{\eta}, \Sigma^\rho, \Sigma^\eta, \tilde{\kappa})\) does not necessarily form square. The \(K\)-theory groups \(K_*(O^\rho_{\tilde{\rho}, \tilde{\eta}})\) are computed from the Künneth formulae for \(K_*(O^\rho \otimes O^\eta)\) [46].
10. Concluding remark

In [31], a different construction of $C^*$-algebra written $O_{\mathcal{H}_\kappa}$ from $C^*$-textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ is studied by using a 2-dimensional analogue of Hilbert $C^*$-bimodule. The $C^*$-algebra $O_{\mathcal{H}_\kappa}$ is different from the $C^*$-algebra $O_{\rho, \eta}$ in the present paper (see also [33], [32]).

References


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