Note on the cortex of some exponential Lie groups

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Abstract. In this paper, we built a family of $4d$-dimensional two-step nilpotent Lie algebras $(g_d)_{d \geq 2}$ so that the cortex of the dual of each $g_d$ is a projective algebraic set. We also give a complete description of the cortex of the exponential connected and simply connected Lie group $G = \mathbb{R}^n \rtimes \mathbb{R}$.

1. Introduction

The cortex of general locally compact group $G$ was defined in [9] as

$$\text{cor}(G) = \{ \pi \in \widehat{G}, \pi \text{ is not Hausdorff-separated} \}$$

from the identity representation $1_G$, where $\widehat{G}$ is the dual of $G$ (set of equivalence classes of unitary irreducible representations of $G$). Note that $\widehat{G}$ is equipped with the topology of Fell which can be described in terms of weak containment (see [6]) and, in general, is not separated. However, if $G$ is abelian, then $\widehat{G}$ is separated and hence $\text{cor}(G) = \{1_G\}$.

When $G$ is a connected and simply connected nilpotent Lie group with Lie algebra $g$, the Kirillov theory says that $g^*/\text{Ad}^*(G)$ and $\widehat{G}$ are homeomorphic,
where Ad\(^*\)(G) denotes the coadjoint representation of G on the dual g\(^*\) of g. Hence, for this class of Lie groups, cor(G) can be identified with a certain Ad\(^*\)(G)-invariant subset of g\(^*\). From [2], one introduces the cortex of g\(^*\) as

\[
\text{Cor}(g^\ast) = \{ \ell = \lim_{m \to \infty} \text{Ad}_{s_m}(\ell_m), \text{ where } \{s_m\} \subset G \\
\text{and } \{\ell_m\} \subset g^\ast \text{ such that } \lim_{m \to \infty} \ell_m = 0 \}
\]

and we have \(\pi \ell \in \text{cor}(G)\) if and only if \(\ell \in \text{Cor}(g^\ast)\). Note that in the case of general Lie groups, the two definitions are not so easily related. Motivated by this situation, the authors in [3] define the cortex \(C_V(G)\) of a representation of a locally compact group \(G\) on a finite-dimensional vector space \(V\) as the set of all \(v \in V\) for which \(G.v\) and \(\{0\}\) cannot be Hausdorff-separated in the orbit-space \(V/G\). They give a precise description of \(C_V(G)\) in the case \(G = \mathbb{R}\). Moreover, they consider the subset \(IC_V(G)\) of \(V\) consisting of the common zeroes of all \(G\)-invariant polynomials \(P\) on \(V\) with \(P(0) = 0\). Note that when \(G\) is a nilpotent Lie group, one has \(IC_V(G) \subset C_V(G)\) and they show that \(IC_V(G) = C_V(G)\) when \(G\) is a nilpotent Lie group of the form \(G = \mathbb{R}^n \rtimes \mathbb{R}\) and \(V = g^\ast\) the dual of the Lie algebra \(g\). This fails for a general nilpotent Lie group, even in the case of two-step nilpotent Lie group (see [2]). In [7], the authors show that the cortex of a connected and simply connected nilpotent Lie group is a semi-algebraic set. In [5] one gives an explicit description of the cortex of certain class of exponential Lie algebras (using the results of parametrization in [1]).

Fixing the class of two-step nilpotent Lie algebras, we see that each coadjoint orbit is a flat (affine) symplectic manifold, however the cortex of that class of Lie algebras may not be flat and in this paper, we give a generalization of the example given in [2] p. 210. Our example consists of a family of \(4d\)-dimensional two-step nilpotent Lie algebras \((g_d)_{d \geq 2}\) such that the cortex of each \(g_d^\ast\) is the zero set of a homogeneous polynomial of degree \(d\) in the complement \(z_d^\perp\) of the center \(z_d\) of \(g_d\). Finally we give some remarks on the cortex of \(\mathbb{R}^n \rtimes \mathbb{R}\).

The paper is organized as follows: The next section is a review of the mathematics and basic tools used throughout the rest of the text. In the third section, we focus on the class of two-step nilpotent Lie algebras \(g\), and we give a refinement of Theorem 4.5 ([1] p. 548) by which we give a description of the algebra of \(G\)-invariant polynomials on \(g^\ast\) (\(G\) is the corresponding Lie group of \(g\)). Next we give an interesting example of a family of two-step nilpotent Lie algebras \((g_d)_{d \geq 2}\) for which the cortex of the dual \(g_d^\ast\) of each \(g_d\) is the zero set of homogeneous polynomials of degree \(d\). In the final section, we consider the exponential nonnilpotent Lie group \(G = \mathbb{R}^n \rtimes \mathbb{R}\) and we give a complete and explicit description of the cortex of the dual of its Lie algebra.

2. Background material and notations

If \(G\) is a locally compact group, Vershik and Karpushev [9] introduce the notion of cortex of \(G\) as the set of all unitary irreducible representations of
G that cannot be Hausdorff separated from the trivial representation. If G is a Lie group with Lie algebra g, it’s known that G acts on g by the adjoint action denoted by Ad and on g* by the coadjoint action denoted by $\text{Ad}^*$. Following [3], we recall the following:

**Definition 2.1.** Let $\pi$ be a continuous representation of a locally compact Lie group $G$ on a finite-dimensional (real) space $V$ we define

$$C_V(\pi) = \{ v = \lim_{m \to \infty} \pi(s_m)v_m, \lim_{m \to \infty} v_m = 0, \{s_m\}_m \subset G \} ,$$

and the cortex of invariants of $\pi$ as

$$IC_V(\pi) = \{ v \in V : p(v) = p(0) \text{ for all } G\text{-invariant polynomials on } V \} .$$

In particular when $G$ is a locally compact Lie group and $\pi$ is the contra-redient representation of $G$ on the dual $g^*$ of the Lie algebra $g$ of $G$, one has:

**Definition 2.2.** We define the cortex of $g^*$ as

$$\text{Cor}(g^*) = \{ \lim_{m \to \infty} \text{Ad}_{s_m}(\ell)(m) \mid (s_m)_m \subset G, (\ell)_m \subset g^* \text{ with } \lim_{m \to \infty} \ell_m = 0 \} ,$$

and the cortex of invariants

$$\text{ICor}(g^*) = \{ \ell \in g^* : p(\ell) = p(0), \text{ for all } G\text{-invariant polynomial } p \text{ on } g^* \} .$$

When $G$ is a nilpotent connected and simply connected Lie group, Kirillov’s theory establishes a bijection between $g^*/\text{Ad}^*(G)$ (the orbit space of the coadjoint representation of $G$ on $g^*$) and $\hat{G}$ (the unitary dual of $G$). More precisely, associated to $\ell \in g^*$ is an irreducible representation $\pi_\ell$ of $G$, and $\pi_f$ and $\pi_\ell$ ($f \in g^*$) are equivalent if and only if $f \in \text{Ad}^*(G)\ell$. The Kirillov correspondence is a homeomorphism provided that $g^*/\text{Ad}^*(G)$ is endowed with the quotient topology [4]. In that case, the unitary dual $\hat{G}$ of $G$ can be parameterized via the orbit-method. More precisely, let $\ell \in g^*$ and $p_\ell$ be a Pukanszky polarization at $\ell$, we define the representation $\pi_{\ell,p_\ell}$ by

$$\pi_{\ell,p_\ell} := \text{ind}_{P_\ell}^G \chi_\ell ,$$

where $P_\ell = \exp p_\ell$ and $\chi_\ell$ is the unitary character associated with $P_\ell$ given by

$$\chi_\ell(\exp X) = e^{-i\langle \ell, X \rangle} , \quad X \in p_\ell .$$

Then:

**Theorem 2.1** (A. A. Kirillov). Let $G$ be a simply connected nilpotent real Lie group with Lie algebra $g$. If $\ell \in g^*$, there exists a polarization $p(\ell)$ of $g$ for $\ell$ such that the monomial representation $\pi_{\ell,p(\ell)} := \text{ind}_{\text{exp}p_\ell}^G \chi_\ell$ is irreducible and of trace class. If $\ell'$ is an element of $g^*$ which belongs to the coadjoint orbit of $\ell$ and $p_{\ell'}$ is a polarization of $g$ for $\ell'$, then the monomial representations $\pi_{\ell,p_\ell}$ and $\pi_{\ell',p_{\ell'}}$ are unitarily equivalent. Conversely, if $h$ and $h'$ are polarizations of $g$ for $\ell \in g^*$ and $\ell' \in g^*$ respectively such that the monomial representations $\pi_{\ell,h}$ and $\pi_{\ell',h'}$ of $G$ are unitarily equivalent, then
\( \ell \) and \( \ell' \) belong to the same coadjoint orbit of \( G \) in \( g^* \). Finally, for each irreducible unitary representation \( \pi \) of \( G \), there exists a unique coadjoint orbit \( O \) of \( G \) in \( g^* \) such that for any linear from \( \ell \) and each polarization \( h \) of \( g \) for \( \ell \), the representations \( \pi \) and \( \text{ind}^G_{\exp h} \chi_\ell \) are unitarily equivalent. Any irreducible unitary representation of \( G \) is strongly trace class. Moreover the mapping
\[
K : g^*/\text{Ad}^*(G) \rightarrow \hat{G} \\
O_\ell \mapsto [\pi_\ell, p(\ell)]
\]
is a homeomorphism (the Kirillov correspondence).

The above Kirillov's result was generalized immediately to the class known as exponential solvable Lie groups, the Kirillov correspondence is still a bijection. For more details, see [8]. With this in mind, we see that if \( G \) is an exponential Lie group, then \( \pi := \pi_{\ell, p(\ell)} \in \text{cor}(G) \) (cortex of \( G \)) if and only if \( \ell \in \text{Cor}(g^*) \). However if \( G \) is exponential nonnilpotent, \( \text{ICor}(g^*) \) may not be defined.

Throughout, \( G \) will always denote a connected and simply connected Lie group with (real) Lie algebra \( g \). We denote by \( z \) the center of \( g \) (if it exists) and \( g^* \) denotes the dual of \( g \). If \( \ell \in g^* \), \( O_\ell \) denotes the coadjoint orbit of \( \ell \).

3. The two-step nilpotent Lie algebras

Let \( G \) be a connected and simply connected two-step nilpotent Lie group with Lie algebra \( g \), then if \( O_\ell = \text{Ad}^*(G)\ell \), one has
\[
O_\ell = \{ \ell \} + T_\ell O_\ell,
\]
and
\[
T_\ell O_\ell = g(\ell)^\perp,
\]
where \( T_\ell O_\ell \) is the tangent space of \( O_\ell \) at \( \ell \), by which we see that the coadjoint orbits in two-step nilpotent Lie algebras are flat (and symplectic) manifolds. In [2], the authors show the following:

**Proposition 3.1.** Let \( g \) be a nilpotent Lie algebra of class 2 (i.e, \([g, [g, g]] = 0\)), and let \( G = \exp g \) be the associated Lie group. Denote by \( \text{ad}^* \) the coadjoint representation of \( g \) on \( g^* \). Let \( f \in g^* \). Then the corresponding representation \( \pi_f \) of \( G \) belongs to \( \text{cor}(G) \) if and only if \( f \) belongs to the closure of the subset \( \{ \text{ad}^*_{X}(\ell), X \in g, \ell \in g^* \} \) of \( g^* \).

From this we can conclude the following:

**Corollary 3.2.** Let \( g \) is a two-step nilpotent Lie algebra. If \( T_\ell O_\ell \) denotes the tangent space to the coadjoint orbit \( O_\ell \) at \( \ell \), then the \( \text{Cor}(g^*) \) is the closure in \( g^* \) of the set
\[
\bigcup_{\ell \in g^*} T_\ell O_\ell = \bigcup_{O_\ell \in g^*/\text{Ad}^*(G)} TO_\ell,
\]
where \( TO_\ell \) is the fiber tangent of \( O_\ell \) and \( g^*/\text{Ad}^*(G) \) is the space of coadjoint orbits in \( g^* \).
Proof. Indeed, for any $\ell \in \mathfrak{g}^*$, one has
\[
\{ad^*_{X}(\ell) ; X \in \mathfrak{g}\} = T_{\ell} \mathcal{O}_\ell,
\]
and hence with Proposition (3.1), the conclusion yields. \qed

Here we give a refinement of Theorem 4.5 ([1] p. 548).

**Proposition 3.3.** Let $G$ be a two-step nilpotent Lie group with Lie algebra $\mathfrak{g}$, choose a real Jordan–Hölder basis $\{X_j\}$. Let $\mathcal{P}$ be the corresponding fine stratification of $\mathfrak{g}^*$, and let $\Omega$ be a layer belonging to $\mathcal{P}$. Then there is an explicit construction of an open set $U$ in $\mathfrak{g}^*$ and real-valued functions $p_1, p_2, \ldots, p_d, q_1, q_2, \ldots, q_d$ on $U$, such that $U$ contains $\Omega$, and such that for each coadjoint orbit $\mathcal{O}_\ell$ in $\Omega$, $p_1|\mathcal{O}_\ell, p_2|\mathcal{O}_\ell, \ldots, p_d|\mathcal{O}_\ell, q_1|\mathcal{O}_\ell, q_2|\mathcal{O}_\ell, \ldots, q_d|\mathcal{O}_\ell$ are real-valued, global canonical coordinates for $\mathcal{O}_\ell$. Moreover, for each $1 \leq j \leq n$, $0 \leq u \leq d$, there are rational functions $\alpha_{j,u}$ and $\beta_{j,u}$ such that for each $1 \leq j < n$ and $\ell \in \Omega$ one has
\[
\ell_j := \ell(X_j) = \sum_{u: j_u \leq j} \alpha_{j,u}(\ell)p_u + \sum_{r=1}^{d} \beta_{j,u}(\ell)q_u.
\]

**Proof.** Recall that the construction of $p_r, q_r$ depends on the flag
\[
(\mathfrak{g}_j = \text{span}\{X_1, \ldots, X_j\})_{1 \leq j \leq n}.
\]
More precisely if $j_t = \min\{j_r, 1 \leq r \leq d\}$, then:
\[
p_1^{(1)} = \ell_{i_t}, \quad q_1^{(1)} = \ell_{[j_t]}^{\ell_{[j_t]}}.
\]

Now suppose we have built $p_1^{(m)}, \ldots, p_k^{(m)}, \ldots, q_1^{(m)}, \ldots, q_k^{(m)}$, then for $\mathfrak{g}_{m+1}$ one has either $m + 1 \notin e$ and in this case $p_r^{(m+1)} = p_r^{(m)}$, $q_r^{(m+1)} = q_r^{(m)}$ or $m + 1 = j_{k+1} \in e$ and in this case
\[
q_r^{(m+1)}(\ell) = q_r^{(m)}(\exp -q X_{m+1} \ell) = q_r^{(m)}(\ell) - q\{x_{m+1}, q_r^{(m)}\},
\]
and
\[
p_r^{(m+1)}(\ell) = p_r^{(m)}(\exp -q X_{m+1} \ell) = p_r^{(m)}(\ell) - q\{x_{m+1}, p_r^{(m)}\},
\]
with $q = \frac{y}{\ell(X_{m+1}, y)}$, where $y$ is a $G_m$-invariant and non-$G_{m+1}$-invariant polynomial function such that $\{x_{m+1}, y\}$ is nonvanishing on $\Omega$ (here $G_j = \exp \mathfrak{g}_j$). \qed

**Corollary 3.4.** Let $e = \{e_1 < \cdots < e_{2d}\}$ be the set of jump indices corresponding to the minimal layer in $\mathfrak{g}^*$. Let $F$ be the cross-section mapping associated with the minimal layer $\Omega$ then $F(\ell) = (F_1(\ell), \ldots, F_n(\ell))$ and let $e = \{e_1 < \cdots < e_{2d}\}$ be the corresponding jump indices then
\[
F_k(\ell) = \begin{cases} 
\ell_k, & \text{if } k = 1, \ldots, e_1 - 1; \\
0, & \text{if } k \in e; \\
\ell_k + \sum_{j : e_j \leq k-1} a_j(\ell)e_j, & \text{if } k \notin e, k \geq e_1,
\end{cases}
\]
where each of $a_1(\ell), \ldots, a_{k-1}(\ell)$ is (nontrivial) a rational regular function on the minimal layer depending only upon $\ell_0 = \ell_3$ ($z$ is the center of $\mathfrak{g}$).

**Proof.** For each layer in $\mathfrak{g}^*$, the mapping $(\ell_{e_1}, \ldots, \ell_{e_{2d}}) \mapsto (p_i(\ell), q_i(\ell))_{1 \leq i \leq d}$ is a rational diffeomorphism whose inverse is also rational on any layer, then we consider the minimal layer and by Proposition 3.3, we can write

$$p_i(\ell) = \sum_j u_j(\ell_1, \ldots, \ell_p)\ell_{e_j}, \quad q_i(\ell) = \sum_j v_j(\ell_1, \ldots, \ell_p)\ell_{e_j}, \quad i = 1, \ldots, d,$$

where $u_j$ and $v_j$ are rational regular functions on the minimal layer. Then after substituting each of the functions $(p_i, q_i)$ by the above expressions in the coordinate functions $(\ell_k)_{k \notin e}$ we obtain the invariant functions of $\mathfrak{g}^*$ and this ends the proof. \[\square\]

**Corollary 3.5.** If $\mathfrak{g}$ is a two-step nilpotent Lie algebra and $\mathfrak{g}^*$ denotes its dual, then

$$\text{ICor}(\mathfrak{g}^*) = \{\ell \in \mathfrak{g}^* : \ell(Z) = 0 \quad \forall Z \in \mathfrak{z}\}.$$

**Proof.** The nontrivial coordinates of the cross-section mapping $(F_k(\ell))_{k \geq p, k \notin e}$ associated with the minimal layer can be written as

$$F_k(\ell) = \frac{B(\ell^0)\ell_k + A_k(\ell^0)}{B(\ell^0)}, \quad k \notin e, k > p := e_1,$$

where each of $B(\ell^0)$ and $B(\ell^0)\ell_k + A_k(\ell^0), (k \geq p, k \notin e)$ is a nontrivial $G$-invariant polynomial on $\mathfrak{g}^*$, with $\ell^0 = \ell_3$. Note that these polynomials are homogeneous and for each $k \geq p, k \notin e$, one has

$$\deg(B(\ell^0)\ell_k + A_k(\ell^0)) = \deg(B(\ell^0)) + 1.$$

Finally the ring $\text{Pol}(\mathfrak{g}^*)^G$ of $G$-invariant polynomials is spanned by the polynomials

$$\ell_1, \ldots, \ell_p, B(\ell^0), (B(\ell^0)\ell_k + A_k(\ell^0))_{k \geq p, k \notin e},$$

and this ends the proof. \[\square\]

**3.1. Main example.** In [2], one introduces an interesting example of 8-dimensional two-step nilpotent Lie algebra $\mathfrak{g}$ so that the corresponding core in $\mathfrak{g}^*$ is a projective algebraic set given by a quadric and such that $\text{Cor}(\mathfrak{g}^*) \subseteq \text{ICor}(\mathfrak{g}^*)$. Here we give a generalization of that example. Let $d \in \mathbb{N}$ with $d \geq 2$ and let $\mathfrak{g}_d$ be the Lie algebra with basis

$$(Z_1, \ldots, Z_d, Y_1, Y_2, \ldots, Y_{2d-1}, Y_{2d}, X_1, \ldots, X_d),$$

and nontrivial brackets

$$[X_i, Y_{2i-1}] = Z_i, \quad i = 1, \ldots, d,$$

$$[X_k, Y_{2k}] = Z_{k+1}, \quad k = 1, \ldots, d - 1,$$

$$[X_d, Y_{2d}] = Z_2 + \cdots + Z_d.$$
Let's denote the center of $g_d$ by $z_d = \text{span}\{Z_1, \ldots, Z_d\}$ and $G_d$ the corresponding connected and simply connected Lie group.

**Proposition 3.6.** For each Lie algebra $g_d$ ($d \geq 2$), one has:

(i) The minimal layer in $g_d^*$ is given by
$$\Omega_d = \{\ell \in g_d^* : \ell(Z_1) \neq 0\}.$$ 

(ii) The coadjoint orbits in $\Omega_d$ are $2d$-dimensional and if
$$\ell = \sum_{k=1}^{d}(\lambda_k Z_k^* + \beta_k X_k^*) + \sum_{k=1}^{2d}\gamma_k Y_k^* \in \Omega_d,$$
then
$$\xi = \left\{\begin{array}{ll}
z_k = \lambda_k, & \text{if } k = 1, \ldots, d; \\
y_{2k-1} = \gamma_{2k-1} + s_j \lambda_1, & \text{if } k = 1, \ldots, d-1; \\
y_{2k} = \gamma_{2k} + s_j \lambda_{k+1}, & \text{if } k = 1, \ldots, d-1; \\
y_{2d-1} = \gamma_{2d-1} + s_d \lambda_1; \\
y_{2d} = \gamma_{2d} + s_d(\lambda_2 + \cdots + \lambda_d); \\
x_k = \beta_k + t_k & \text{if } k = 1, \ldots, d. \\
\end{array}\right.$$

(iii) The algebra of $G$-invariant polynomials is
$$\text{Pol}(g_d^*)^{G_d} = \mathbb{R}[z_1, \ldots, z_d, z_1 y_2 - z_2 y_1, \ldots, z_1 y_{2d-2} - z_d y_{2d-3}, z_1 y_{2d} - (z_2 + \cdots + z_d) y_{2d-1}].$$

**Proof.** Let $B_d = (U_1, \ldots, U_{4d})$ be the Jordan–Hölder basis defined by
$$U_i = \left\{\begin{array}{ll}
Z_i, & \text{if } 1 \leq i \leq d; \\
Y_{i-d}, & \text{if } d+1 \leq i \leq 3d; \\
X_{i-3d}, & \text{if } 3d+1 \leq i \leq 4d.
\end{array}\right.$$ 
Using the methods of [1], we can see that the minimal layer in $g_d^*$ is
$$\Omega_d = \{\ell \in g^* : \ell(U_1) = \ell(Z_1) \neq 0\},$$
which corresponds to the set of jump indices $e_d = i_d \cup j_d$ with
$$i_d = \{d + 1 < d + 3 < \cdots < 3d - 1\},$$
$$j_d = \{3d + 1, 3d + 2, \ldots, 4d\}.$$ 
Then by using the methods of [1] (the parametrization of coadjoint orbits) we can deduce the results of (ii) and (iii).

**Remark 3.1.**
(i) If \( \Omega_d \) is the minimal layer given as above, then the canonical coordinates on \( \Omega_d \) (see [1]) are given by

\[
p_i(\ell) = x_i, q_i(\ell) = \frac{y_{2i-1}}{z_1}, \quad i = 1, \ldots, d.
\]

(ii) The cross-section \( \Sigma_d \) is given by

\[
\Sigma_d = \left( \sum_{k \in \mathcal{E}} \mathbb{R}U_k^* \right) \cap \Omega_d = \left( \sum_{k=1}^d \mathbb{R}Z_k^* + \mathbb{R}Y_{2k}^* \right) \cap \Omega_d.
\]

(iii) The cross-section mapping \( F_d : \Omega_d \to \Sigma_d \) is as follows

\[
F_d(z_i, y_j, x_k) = \sum_{i=1}^d z_i Z_i^* + \sum_{i=1}^{d-1} \left( y_{2i} - \frac{z_i + 1}{z_i} y_{2i-1} \right) Y_{2i}^* + \left( y_{2d} - \frac{z_2 + \cdots + z_d}{z_1} \right) Y_{2d}^*,
\]

where \((Z_1^*, \ldots, Z_d^*, Y_1^*, \ldots, Y_{2d}^*, X_1^*, \ldots, X_d^*)\) is the dual basis of \( B \).

**Proposition 3.7.** Let’s denote \( \ell = \sum_{i=1}^d (z_i Z_i^* + x_i X_i^*) + \sum_{j=1}^{2d} y_j Y_j^* \in g^* \) by \( \ell = (z_i, y_j, x_k) \), where \((Z_1^*, \ldots, Z_d^*, Y_1^*, \ldots, Y_{2d}^*, X_1^*, \ldots, X_d^*)\) is the dual basis in \( g_d^* \). Then the cortex of \( g_d^* \) is the projective algebraic set given by

\[
\text{Cor}(g_d^*) = \left\{ \ell = (z_i, y_j, x_k) : z_1 = \cdots = z_d = y_{2d-1} \left( \sum_{i=1}^{d-1} y_{2i} \prod_{j=1, j \neq i}^{d-1} y_{2j-1} \right) - y_{2d} \prod_{j=1}^{d-1} y_{2j-1} = 0 \right\}.
\]

**Proof.** Note that since \( \Omega_d \) is dense in \( g_d^* \) (Zariski open subset in \( g_d^* \)) then

\[
\text{Cor}(g_d^*) = \left\{ \lim_m \text{Ad}_{\exp}^* x_m, (\ell_m) \in g_d^*, (\ell_m) \in \Omega_d, \text{ and } \lim_m \ell_m = 0 \right\}.
\]

On the other hand if \( \mathcal{O}_\ell = G\ell \), then the tangent space \( T_{\ell}\mathcal{O}_\ell \) at \( \ell \) is

\[
T_{\ell}\mathcal{O}_\ell = \{ \text{ad}_X^* (\ell), \ell \in \Omega_d, X \in \text{Vect}\{Y_{2k-1}, X_k, 1 \leq k \leq d\} \}.
\]

Now if \( \ell = (\lambda_i, \gamma_j, \beta_k) \in \Omega_d \) and \( \xi \in T_{\ell}\mathcal{O}_\ell \), with

\[
\xi = \sum_{i=1}^d (z_i Z_i^* + x_i X_i^*) + \sum_{j=1}^{2d} y_j Y_j^*,
\]
then

\begin{align*}
\begin{array}{l}
z_i = 0, & \text{if } i = 1, \ldots, d; \\
y_{2j-1} = s_j \lambda_1, & \text{if } j = 1, \ldots, d-1; \\
y_{2j} = s_j \lambda_{j+1}, & \text{if } j = 1, \ldots, d-1; \\
y_{2d-1} = s_d \lambda_1; \\
y_{2d} = s_d (\lambda_2 + \cdots + \lambda_d); \\
x_k = t_k, & \text{if } k = 1, \ldots, d.
\end{array}
\end{align*}

From which we can see that \( \xi = (z_i, y_j, x_k) \in T_e \mathcal{O} \) if and only if

\begin{align*}
\begin{array}{l}
z_i = 0, & \text{if } i = 1, \ldots, d; \\
y_{2j} = y_{2j-1} \frac{\lambda_{j+1}}{\lambda_1}, & \text{if } j = 1, \ldots, d-1; \\
y_{2d} = y_{2d-1} \frac{\lambda_2 + \cdots + \lambda_d}{z_1}.
\end{array}
\end{align*}

with \( y_{2j-1}, x_j \) are free variables in \( \mathbb{R} (j = 1, \ldots, d) \). Then we see that a.e. \( \xi \in T_e \mathcal{O} \) satisfies

\[
\frac{y_{2d}}{y_{2d-1}} = \sum_{j=1}^{d-1} \frac{y_{2j}}{y_{2j-1}},
\]

and hence

\[
\text{Cor}(g_d^*) = \left\{ \ell = (z_i, y_j, x_k) \in g_d^* : z_i = 0, \\
y_{2d-1} \left( \sum_{i=1}^{d-1} y_{2i} \prod_{j=1, j \neq i}^{d-1} y_{2j-1} \right) - y_{2d} \prod_{j=1}^{d-1} y_{2j-1} = 0 \right\}. \quad \square
\]

**Corollary 3.8.** For each integer \( d \geq 2 \) if \( z_d \) denotes the center of the Lie algebra \( g_d \), then

\[
\text{Cor}(g_d^*) \subset \mathbb{Z} \text{Cor}(g_d^*) = \frac{1}{3_d^+}.
\]

**Proof.** The ring of \( G \)-invariant polynomials on \( g^* \) is given by

\[
\text{Pol}(g_d^*)^{G_d} = \mathbb{R} \left[ z_1, \ldots, z_d, (z_1 y_{2i} - z_{i+1} y_{2i-1})_{1 \leq i \leq d-1}, \frac{z_1 y_{2d} - (z_2 + \cdots + z_d y_{2d-1})}{y_{2d-1}} \right],
\]

where \( G_d \) is the connected and simply connected (nilpotent) Lie group corresponding to \( g_d \). Thus

\[
\{ \ell \in g_d^* : P(\ell) = P(0), \forall P \in \text{Pol}(g_d^*)^{G_d} \} = \frac{1}{3_d^+},
\]

where

\[
\frac{1}{3_d^+} = \{ \ell \in g_d^* : \ell(Z) = 0, \forall Z \in 3_d \}.
\]

hence with Proposition 3.7, we conclude that

\[
\text{Cor}(g_d^*) \subset \mathbb{Z} \text{Cor}(g_d^*) = \frac{1}{3_d^+}. \quad \square
\]
4. The Lie group $\mathbb{R}^n \rtimes \mathbb{R}$

In [3], in there is a study of the cortex of the nilpotent Lie group
\[ G = \mathbb{R}^n \rtimes \mathbb{R}, \]
the authors show that
\[ \text{Cor}(g^*) = \text{ICor}(g^*) \]
\[ = \{ \ell \in g^* : P(\ell) = P(0), P \text{ is } G \text{ invariant polynomial on } g^* \}. \]
The definition of ICor($g^*$) may not exist if $G$ is not nilpotent but we can
define $G$-invariant (or semi-invariant) functions. Let’s consider the following example:

**Example 4.1.** Let $(X_1, X_2, A)$ be a basis in $g$ with
\[ [A, X_1] = X_1, [A, X_2] = -2X_2. \]
Let’s identify $g^*$ with $\mathbb{R}^3$ under the dual basis $(X_1^*, X_2^*, A^*)$, and denote
\[ x = (x_1, x_2, a) \in g^*, \]
then the minimal layer is
\[ \Omega = \{ \ell = (\ell_1, \ell_2, a) \in g^* : \ell_1 \neq 0 \}. \]
If $\ell = (\ell_1, \ell_2, a) \in g^*$, then the coadjoint orbit of $\ell$ is given by
\[ O_\ell = \{ x \in g^* : x = (\ell_1 e^t, \ell_2 e^{-2t}, a + s), t, s \in \mathbb{R} \}, \]
that is,
\[ O_\ell = \{ x = (x_1, x_2, x_3) \in g^* : \text{sign}(x_1) = \text{sign}(\ell_1), x_1^2 x_2 = \ell_1^2 \ell_2, x_3 \in \mathbb{R} \}. \]
We can check that the cortex of $g^*$ is given by
\[ \text{Cor}(g^*) = \{ \ell = (\ell_1, \ell_2, \ell_3) \in g^* : \ell_1 \ell_2 = 0 \}. \]
On other hand, the cross-section mapping is as follows
\[ F : \Omega \rightarrow \Omega, \quad \ell \mapsto \left( \text{sign}(\ell_1) = \frac{\ell_1}{|\ell_1|}, \ell_1^2 \ell_2, 0 \right), \]
from which we see the existence of $G$-invariant polynomial $p(x) = x_1^2 x_2$ and we see that
\[ \text{Cor}(g^*) = \{ \ell \in g^* : p(\ell) = 0 \}. \]
In this example if we let $[A, X_1] = X_1, [A, X_2] = -\sqrt{2}X_2$ then there are
no $G$-invariant polynomials on $g^*$, however the function $x_1^2 x_2$ is $G$-invariant
and the cortex is still the same. This example can be generalized. To this
end, if $g = \mathbb{R}^n \oplus RA$, we denote $\text{sp}(adA) = \{ \lambda_1, \ldots, \lambda_n \}$ the set of eigenvalues
of $adA$, and for $\lambda \in \text{sp}(adA)$, we set.
\[ E_\lambda = \bigcup_{m \in \mathbb{N}} \ker(adA - \lambda)^m, \]
and
\[ E^+ = \bigcup_{\lambda \in \text{sp}(adA), \Re(\lambda) > 0} E_\lambda, \]
\[ E^{-} = \bigcup_{\lambda \in \text{sp}(adA), \Re(\lambda) < 0} E_{\lambda}. \]

Then we have the following:

**Proposition 4.2.** Let \( G = \mathbb{R}^n \times \mathbb{R} \) be the Lie group whose Lie algebra \( g = \mathbb{R}^n \oplus \mathbb{R} A \). Suppose that \( adA \) is diagonalizable, and let \( \text{sp}(adA) = \{\lambda_1, \ldots, \lambda_n\} \) denote the set of eigenvalues of \( adA \).

(a) If \( \{\Re(\lambda_j)\}_{1 \leq j \leq n} \subset (0, \infty) \) or \( \{\Re(\lambda_j)\}_{1 \leq j \leq n} \subset (-\infty, 0) \) then
\[
\text{Cor}(g^*) = g^*.
\]

(b) If \( \prod_{j=1}^{n} \Re(\lambda_j) < 0 \). Then the cortex of \( g^* \) is the union of two vector spaces. More precisely
\[
\text{Cor}(g^*) = (V^+ + \mathbb{R}A^*) \cup (V^- + \mathbb{R}A^*),
\]
where
\[
V^+ = (E^+)^*, \quad V^- = (E^-)^*.
\]

**Proof.** If \( \text{sp}(adA) = \{\lambda_1, \ldots, \lambda_n\} \) denotes the set of eigenvalues of \( adA \) (restricted to \( \mathbb{R}^n \)). Then identifying \( g \) with \( \mathbb{R}^{n+1} \) (respectively \( \mathbb{C}^{n+1} \) if some of the eigenvalues of \( adA \) are nonreal), the coadjoint orbit of any \( \ell \in g^* \) is parameterized as follows
\[
\Omega_\ell = \{(\ell_1 e^{\lambda_1 t}, \ldots, \ell_n e^{\lambda_n t}, \ell_{n+1} + s), \quad t, s \in \mathbb{R}\}.
\]

Since \( \Re(\lambda_j) \neq 0, j = 1, \ldots, n \), then for any \((\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n\) the linear system
\[
\begin{pmatrix}
  e^{\lambda_1 t} & 0 & \ldots & 0 \\
  0 & \ddots & \ddots & 0 \\
  \ldots & \ldots & 0 & e^{\lambda_n t}
\end{pmatrix}
\begin{pmatrix}
  \ell_1 \\
  \vdots \\
  \ell_n
\end{pmatrix}
= \begin{pmatrix}
  \alpha_1 \\
  \vdots \\
  \alpha_n
\end{pmatrix},
\]
has a unique solution \((\ell_1, \ldots, \ell_n)^\top\) with
\[
\| (\ell_1, \ldots, \ell_n)^\top \| = \| (e^{-\lambda_1 t} \alpha_1, \ldots, e^{-\lambda_n t} \alpha_n)^\top \|.
\]

(a) If \( \{\Re(\lambda_j)\}_{1 \leq j \leq n} \subset (0, \infty) \) or \( \{\Re(\lambda_j)\}_{1 \leq j \leq n} \subset (-\infty, 0) \) then for any \((\alpha_1, \ldots, \alpha_n, \beta) \in g^* \) it exists \( \{x^{(m)} = x(t_m, \ell^{(m)})\}\}_{m} \in g^* \) with \( \ell^{(m)}\}_{m} \subset \Omega \) and \( \lim_{m: \Re(\lambda_1)t_m \to \infty} \ell^{(m)} = 0 \) such that
\[
\lim_{m: \Re(\lambda_1)t_m \to \infty} x^{(m)} = (\alpha_1, \ldots, \alpha_n, \beta),
\]
and hence the cortex is all of \( g^* \).

(b) In that case, let’s rearrange the basis \((X_1, \ldots, X_n)\) in \( \mathbb{C}^n \) such that the matrix of \( adA \) in this basis is \( \text{diag}(\lambda_1, \ldots, \lambda_{k_0}, \lambda_{k_0+1}, \ldots, \lambda_n) \) with
\[
\Re(\lambda_1) > 0, \ldots, \Re(\lambda_{k_0}) > 0, \Re(\lambda_{k_0+1}) < 0, \ldots, \Re(\lambda_n) < 0,
\]
then for any \( x = (x_1, \ldots, x_n, x_{n+1}) \in \Omega_\ell \) one has
\[
\begin{align*}
|\ell_1|^{\lambda_k} |x_k|^{\lambda_1} &= |\ell_1|^{\lambda_1} |x_1|^{\lambda_k}, \quad &k &\leq k_0, \\
|x_k|^{\lambda_1} |x_1|^{-\lambda_k} &= |\ell_1|^{\lambda_1} |\ell_1|^{-\lambda_k}, \quad &k &\geq k_0 + 1.
\end{align*}
\]
Hence one has
\[
\text{Cor}(\mathfrak{g}^*) = \{ \ell \in \mathfrak{g}^* : \ell = (\ell_1, \ldots, \ell_{k_0}, 0, \ldots, 0, \ell_{n+1}) \} \\
\cup \{ \ell \in \mathfrak{g}^* : \ell = (0, \ldots, 0, \ell_{k_0+1}, \ldots, \ell_n, \ell_{n+1}) \}.
\]

**Remark 4.1.** Let \( \mathfrak{g} = \mathbb{R}^n \oplus \mathbb{R}A \) be a real Lie algebra. Suppose that there exists a basis \((X_1, \ldots, X_n)\) in \( \mathbb{R}^n \) such that
\[
[A, X_j] = m_j X_j, \quad j = 1, \ldots, n,
\]
with \( \{m_1, \ldots, m_n\} \subset \mathbb{R}^\times \).

(a) If \( \{m_k\}_{1 \leq k \leq n} \subset \mathbb{N} \), then any generic coadjoint orbit of \( \ell \) \((\ell_1 \neq 0)\) is given by
\[
\mathcal{O}_\ell = \left\{ x = (x_1, \ldots, x_n, x_{n+1}) \in \mathfrak{g}^* : x_1\ell_1 > 0, x_k = \frac{\ell_k m_k}{\ell_1 m_1}, \quad k = 2, \ldots, n, x_{n+1} \in \mathbb{R} \right\},
\]
and hence, it is an open semi-algebraic subset in \( \mathfrak{g}^* \).

(b) Now suppose that \( \{m_1, \ldots, m_n\} \subset \mathbb{Z}^\times \) with \( \prod_{j=1}^n m_j < 0 \). We can assume the existence of a basis \((X_1, \ldots, X_n)\) in \( \mathbb{R}^n \) so that with respect to this basis the matrix of \( adA \) is
\[
adA = \text{diag}(m_1, \ldots, m_{k_0}, m_{k_0+1}, \ldots, m_n)
\]
with \( m_1 > 0, \ldots, m_{k_0} > 0, m_{k_0+1} < 0, \ldots, m_n < 0 \). Then for any \( x = (x_1, \ldots, x_n, x_{n+1}) \in \mathcal{O}_\ell \) (with \( \ell_1 \neq 0 \)) one has
\[
\begin{cases}
x_1\ell_1 > 0,
x_1^{-m_j} x_j^{-m_j} = \ell_1^{-m_j}, \quad j = 2, \ldots, k_0,
x_1^{-m_j} x_j^{m_j} = \ell_1^{-m_j}, \quad j = k_0 + 1, \ldots, n.
\end{cases}
\]

On other hand, the polynomials
\[
p_{i,j}(\ell) = \ell_i^{-m_j} p_{m_j}, \quad i = 1, \ldots, k_0, \quad j = k_0 + 1, \ldots, n
\]
are \( G \)-invariant on \( \mathfrak{g}^* \) and the cortex is the union of two vector spaces given by:
\[
\text{Cor}(\mathfrak{g}^*) = \{ \ell \in \mathfrak{g}^* : p_{i,j}(\ell) = 0 \quad \forall 1 \leq i \leq k_0, k_0 + 1 \leq j \leq n \}.
\]

**Corollary 4.3.** Let \( \mathfrak{g} = \mathbb{R}^n \oplus \mathbb{R}A \) be a real Lie algebra. Let’s denote \( sp(adA) = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C} \) the set of eigenvalues of \( adA \). If
\[
\{\Re(\lambda_j)\}_{1 \leq j \leq n} \subset (0, \infty) \quad \text{or} \quad \{\Re(\lambda_j)\}_{1 \leq j \leq n} \subset (\infty, 0),
\]
then
\[
\text{Cor}(\mathfrak{g}^*) = \mathfrak{g}^*.
\]
**Proof.** First let’s suppose that the real endomorphism $\text{ad}A$ has a single eigenvalue $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. According to $\lambda$ is real or complex, we can suppose the existence of a basis in $\mathbb{R}^n$ (resp. in $\mathbb{C}^n$) such that the matrix of $\text{ad}A$ is written in a Jordan block form:

$$\text{ad}A = J_\lambda = \begin{pmatrix}
\lambda & 1 & 0 & \cdots \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 1 \\
0 & \cdots & 0 & \lambda
\end{pmatrix}$$

The coadjoint orbit of $\ell = (\ell_1, \ldots, \ell_n, \ell_{n+1})$ is given by

$$O_\ell = \left\{ \left( \begin{array}{c}
x_k = e^{\lambda t} \sum_{j \geq 1, j+k \leq n} \frac{t^j}{j!} \ell_j \\
x_{n+1} = \ell_{n+1} + s
\end{array} \right), t, s \in \mathbb{R} \right\}.$$ 

Now let’s remark that for any $\alpha \in \mathbb{R}^n$ (resp. in $\mathbb{C}^n$), since $\Re(\lambda) \neq 0$, the linear system

$$e^{\lambda t} \begin{pmatrix}
1 & 0 & \cdots & 0 \\
t & 1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\frac{t^{n-1}}{(n-1)!} & \cdots & \cdots & t & 1
\end{pmatrix} \begin{pmatrix}
\ell_1 \\
\ell_2 \\
\vdots \\
\ell_n
\end{pmatrix} = \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_n
\end{pmatrix}$$

has a unique solution and if we let

$$M(t) = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
t & 1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\frac{t^{n-1}}{(n-1)!} & \cdots & \cdots & t & 1
\end{pmatrix},$$

then $M(t)$ is a unipotent matrix whose inverse $M^{-1}(t) = (p_{i,j}(t))_{1 \leq i,j \leq n}$ is also unipotent and all its entries $p_{i,j}(t)$ are polynomial functions in $t$, then

$$[\ell_1, \ldots, \ell_n] = e^{-\lambda t} M^{-1}(t)[\alpha_1, \ldots, \alpha_n]$$

and

$$\| [\ell_1, \ldots, \ell_n] \| = e^{-\Re(\lambda) t} F(t),$$

where $F(t)^2$ is polynomial function in $t$ and then

$$\lim_{\Re(\lambda) t \to \infty} e^{-\Re(\lambda) t} F(t) = 0,$$

For instance for any $\alpha = [\alpha_1, \ldots, \alpha_n] \in \mathbb{R}^n$ if $\lambda$ is real (resp. $\alpha \in \mathbb{C}^n$ if $\lambda \in \mathbb{C} \setminus i\mathbb{R}$) and \{t_m\}_m \subset \mathbb{R}$ such that $\lim_{m \to \infty} t_m \Re(\lambda) = \infty$ it exists \{t_{m}^{(1)}, \ldots, t_{m}^{(n)}\}_m such that

$$\lim_{m \to \infty} e^{\lambda m} M(t_m)[t_{m}^{(1)}, \ldots, t_{m}^{(n)}] = [\alpha_1, \ldots, \alpha_n]$$
and
\[ \lim_{m \to \infty} \ell_1^{(m)} = \cdots = \lim_{m \to \infty} \ell_n^{(m)} = 0. \]

This shows that the cortex of \(g^*\) coincides with \(g^*\). Finally if \(adA\) has more than one single eigenvalue, we can write \(adA = \text{diag}(J_{\lambda_1}, \ldots, J_{\lambda_k})\) where each \(J_{\lambda}\) is a Jordan block matrix. \(\Box\)

**Remark 4.2.** Let \(g = \mathbb{R}X_1 \oplus \mathbb{R}X_2 \oplus \mathbb{R}A\) with
\[ [A, X_1] = X_1, [A, X_2] = X_2. \]

In this example the cortex of \(g^*\) is \(g^*\). The cross-section mapping of the minimal layer is given by
\[ F(\ell_1, \ell_2, \ell_3) = \left( \frac{\ell_1}{|\ell_1|}, \frac{\ell_2}{|\ell_1|}, 0 \right), \quad \ell_1 \neq 0. \]

On other hand, we remark that the rational function \(r(\ell_1, \ell_2, \ell_3) = \frac{\ell_2}{\ell_1}\) is \(G\)-invariant on \(\Omega = \{ \ell = (\ell_1, \ell_2, \ell_3) \in g^*: \ell_1 \neq 0 \}\) and
\[ \text{Cor}(g^*) \supseteq \{ \ell \in \Omega : r(\ell) = 0 \}. \]

**Corollary 4.4.** Let \(g = \mathbb{R}^n \oplus \mathbb{R}A\), be a real Lie algebra. Let’s denote \(\text{sp}(adA) = \{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{C}\) the set of eigenvalues of \(adA\). If \(\prod_{j=1}^n \Re(\lambda_j) < 0\), then the cortex of \(g^*\) is the union of two vector spaces. More precisely, with the notations of Proposition 4.2, one has
\[ \text{Cor}(g^*) = (V^+ + RA^*) \cup (V^- + RA^*). \]

**Remark 4.3.** Let \(g = \mathbb{R}^n \oplus \mathbb{R}A\) be a real Lie algebra, and assume that all the eigenvalues of \(adA\) are purely imaginary. Let’s denote \(h = \mathbb{R}^n \oplus \mathbb{R}N\) where \(N\) is the nilpotent part in the Jordan decomposition of \(A\), then by [3], one has
\[ \text{Cor}(g^*) = \text{Cor}(h^*). \]

**References**


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