Note on the cortex of some exponential Lie groups

Béchir Dali

Abstract. In this paper, we built a family of $4d$-dimensional two-step nilpotent Lie algebras $(\mathfrak{g}_d)_{d \geq 2}$ so that the cortex of the dual of each $\mathfrak{g}_d$ is a projective algebraic set. We also give a complete description of the cortex of the exponential connected and simply connected Lie group $G = \mathbb{R}^n \rtimes \mathbb{R}$.

Contents

1. Introduction 1247
2. Background material and notations 1248
3. The two-step nilpotent Lie algebras 1250
   3.1. Main example 1252
4. The Lie group $\mathbb{R}^n \rtimes \mathbb{R}$ 1256
References 1260

1. Introduction

The cortex of general locally compact group $G$ was defined in [9] as

$$\text{cor}(G) = \{ \pi \in \hat{G}, \pi \text{ is not Hausdorff-separated} \}$$

from the identity representation $1_G$, where $\hat{G}$ is the dual of $G$ (set of equivalence classes of unitary irreducible representations of $G$). Note that $\hat{G}$ is equipped with the topology of Fell which can be described in terms of weak containment (see [6]) and, in general, is not separated. However, if $G$ is abelian, then $\hat{G}$ is separated and hence $\text{cor}(G) = \{1_G\}$.

When $G$ is a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, the Kirillov theory says that $\mathfrak{g}^*/\text{Ad}^*(G)$ and $\hat{G}$ are homeomorphic,
where $\text{Ad}^*(G)$ denotes the coadjoint representation of $G$ on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. Hence, for this class of Lie groups, $\text{cor}(G)$ can be identified with a certain $\text{Ad}^*(G)$-invariant subset of $\mathfrak{g}^*$. From [2], one introduces the cortex of $\mathfrak{g}^*$ as

$$\text{Cor}(\mathfrak{g}^*) = \{\ell = \lim_{m \to \infty} \text{Ad}_{s_m}^*(\ell_m), \text{ where } \{s_m\} \subset G \text{ and } \{\ell_m\} \subset \mathfrak{g}^* \text{ such that } \lim_{m \to \infty} \ell_m = 0\}$$

and we have $\pi \ell \in \text{cor}(G)$ if and only if $\ell \in \text{Cor}(\mathfrak{g}^*)$. Note that in the case of general Lie groups, the two definitions are not so easily related. Motivated by this situation, the authors in [3] define the cortex $C_{V}(G)$ of a representation of a locally compact group $G$ on a finite-dimensional vector space $V$ as the set of all $v \in V$ for which $G.v$ and $\{0\}$ cannot be Hausdorff-separated in the orbit-space $V/G$. They give a precise description of $C_{V}(G)$ in the case $G = \mathbb{R}$. Moreover, they consider the subset $IC_{V}(G)$ of $V$ consisting of the common zeroes of all $G$-invariant polynomials $P$ on $V$ with $P(0) = 0$. Note that when $G$ is a nilpotent Lie group, one has $IC_{V}(G) \subset C_{V}(G)$ and they show that $IC_{V}(G) = C_{V}(G)$ when $G$ is a nilpotent Lie group of the form $G = \mathbb{R}^n \rtimes \mathbb{R}$ and $V = \mathfrak{g}^*$ the dual of the Lie algebra $\mathfrak{g}$. This fails for a general nilpotent Lie group, even in the case of two-step nilpotent Lie group (see [2]). In [7], the authors show that the cortex of a connected and simply connected nilpotent Lie group is a semi-algebraic set. In [5] one gives an explicit description of the cortex of certain class of exponential Lie algebras (using the results of parametrization in [1]).

Fixing the class of two-step nilpotent Lie algebras, we see that each coadjoint orbit is a flat (affine) symplectic manifold, however the cortex of that class of Lie algebras may not be flat and in this paper, we give a generalization of the example given in [2] p. 210. Our example consists of a family of $4d$-dimensional two-step nilpotent Lie algebras $(\mathfrak{g}_d)_{d \geq 2}$ such that the cortex of each $\mathfrak{g}_d^*$ is the zero set of a homogeneous polynomial of degree $d$ in the complement $\mathfrak{z}_d^*$ of the center $\mathfrak{z}_d$ of $\mathfrak{g}_d$. Finally we give some remarks on the cortex of $\mathbb{R}^n \rtimes \mathbb{R}$.

The paper is organized as follows: The next section is a review of the mathematics and basic tools used throughout the rest of the text. In the third section, we focus on the class of two-step nilpotent Lie algebras $\mathfrak{g}$, and we give a refinement of Theorem 4.5 ([1] p. 548) by which we give a description of the algebra of $G$-invariant polynomials on $\mathfrak{g}^*$ ($G$ is the corresponding Lie group of $\mathfrak{g}$). Next we give an interesting example of a family of two-step nilpotent Lie algebras $(\mathfrak{g}_d)_{d \geq 2}$ for which the cortex of the dual $\mathfrak{g}_d^*$ of each $\mathfrak{g}_d$ is the zero set of homogeneous polynomials of degree $d$. In the final section, we consider the exponential nonnilpotent Lie group $G = \mathbb{R}^n \rtimes \mathbb{R}$ and we give a complete and explicit description of the cortex of the dual of its Lie algebra.

2. Background material and notations

If $G$ is a locally compact group, Vershik and Karpushev [9] introduce the notion of cortex of $G$ as the set of all unitary irreducible representations of
Definition 2.1. Let \( \pi \) be a continuous representation of a locally compact Lie group \( G \) on a finite-dimensional (real) space \( V \) we define
\[
C_V(\pi) = \{ v = \lim_{m \to \infty} \pi(s_m)v_m, \lim_{m \to \infty} v_m = 0, \{ s_m \}_m \subset G \},
\]
and the cortex of invariants of \( \pi \) as
\[
IC_V(\pi) = \{ v \in V : p(v) = p(0) \text{ for all } G\text{-invariant polynomials on } V \}.
\]
In particular when \( G \) is a locally compact Lie group and \( \pi \) is the contragredient representation of \( G \) on the dual \( \mathfrak{g}^* \) of the Lie algebra \( \mathfrak{g} \) of \( G \), one has:

Definition 2.2. We define the cortex of \( \mathfrak{g}^* \) as
\[
\text{Cor}(\mathfrak{g}^*) = \{ \lim_{m \to \infty} \text{Ad}_{\mathfrak{s}_m}^*(\ell_m) \mid (s_m)_m \subset G, (\ell_m)_m \subset \mathfrak{g}^* \text{ with } \lim_{m \to \infty} \ell_m = 0 \},
\]
and the cortex of invariants
\[
\text{ICor}(\mathfrak{g}^*) = \{ \ell \in \mathfrak{g}^* : p(\ell) = p(0), \text{ for all } G\text{-invariant polynomial } p \text{ on } \mathfrak{g}^* \}.
\]
When \( G \) is a nilpotent connected and simply connected Lie group, Kirillov’s theory establishes a bijection between \( \mathfrak{g}^*/\text{Ad}^*(G) \) (the orbit space of the coadjoint representation of \( G \) on \( \mathfrak{g}^* \)) and \( \hat{G} \) (the unitary dual of \( G \)). More precisely, associated to \( \ell \in \mathfrak{g}^* \) is an irreducible representation \( \pi_\ell \) of \( G \), and \( \pi_f \) and \( \pi_\ell \) (\( f \in \mathfrak{g}^* \)) are equivalent if and only if \( f \in \text{Ad}^*(G)\ell \). The Kirillov correspondence is a homeomorphism provided that \( \mathfrak{g}^*/\text{Ad}^*(G) \) is endowed with the quotient topology \([4]\). In that case, the unitary dual \( \hat{G} \) of \( G \) can be parameterized via the orbit-method. More precisely, let \( \ell \in \mathfrak{g}^* \) and \( \mathfrak{p}_\ell \) be a Pukanszky polarization at \( \ell \), we define the representation \( \pi_{\ell,\mathfrak{p}_\ell} \) by
\[
\pi_{\ell,\mathfrak{p}_\ell} := \text{ind}^G_{\mathfrak{p}\ell} \chi_\ell,
\]
where \( \mathcal{P}_\ell = \exp \mathfrak{p}_\ell \) and \( \chi_\ell \) is the unitary character associated with \( \mathcal{P}_\ell \) given by
\[
\chi_\ell(\exp X) = e^{-i\langle \ell,X \rangle}, \quad X \in \mathfrak{p}_\ell.
\]
Then:

Theorem 2.1 (A. A. Kirillov). Let \( G \) be a simply connected nilpotent real Lie group with Lie algebra \( \mathfrak{g} \). If \( \ell \in \mathfrak{g}^* \), there exists a polarization \( \mathfrak{p}(\ell) \) of \( \mathfrak{g} \) for \( \ell \) such that the monomial representation \( \pi_{\ell,\mathfrak{p}(\ell)} := \text{ind}_{\mathfrak{e}\mathfrak{p}(\ell)} \chi_\ell \) is irreducible and of trace class. If \( \ell' \) is an element of \( \mathfrak{g}^* \) which belongs to the coadjoint orbit of \( \ell \) and \( \mathfrak{p}_{\ell'} \) is a polarization of \( \mathfrak{g} \) for \( \ell' \), then the monomial representations \( \pi_{\ell,\mathfrak{p}_\ell} \) and \( \pi_{\ell',\mathfrak{p}_{\ell'}} \) are unitarily equivalent. Conversely, if \( \mathfrak{h} \) and \( \mathfrak{h}' \) are polarizations of \( \mathfrak{g} \) for \( \ell \in \mathfrak{g}^* \) and \( \ell' \in \mathfrak{g}^* \) respectively such that the monomial representations \( \pi_{\ell,\mathfrak{h}} \) and \( \pi_{\ell',\mathfrak{h}'} \) of \( G \) are unitarily equivalent, then
and $\ell$ and $\ell'$ belong to the same coadjoint orbit of $G$ in $\mathfrak{g}^*$. Finally, for each irreducible unitary representation $\pi$ of $G$, there exists a unique coadjoint orbit $O$ of $G$ in $\mathfrak{g}^*$ such that for any linear from $\ell$ and each polarization $h$ of $\mathfrak{g}$ for $\ell$, the representations $\pi$ and $\text{ind}_{\exp h}^{G} \chi_{\ell}$ are unitarily equivalent. Any irreducible unitary representation of $G$ is strongly trace class. Moreover the mapping

$$K : \mathfrak{g}^*/\text{Ad}^*(G) \rightarrow \hat{G}$$

$$O_{\ell} \mapsto [\pi_{\ell, p(\ell)}]$$

is a homeomorphism (the Kirillov correspondence).

The above Kirillov’s result was generalized immediately to the class known as exponential solvable Lie groups, the Kirillov correspondence is still a bijection. For more details, see [8]. With this in mind, we see that if $G$ is an exponential Lie group, then $\pi := \pi_{\ell, p(\ell)} \in \text{cor}(G)$ (cortex of $G$) if and only if $\ell \in \text{Cor}(\mathfrak{g}^*)$. However if $G$ is exponential nonnilpotent, $\text{ICor}(\mathfrak{g}^*)$ may not be defined.

Throughout, $G$ will always denote a connected and simply connected Lie group with (real) Lie algebra $\mathfrak{g}$. We denote by $\mathfrak{z}$ the center of $\mathfrak{g}$ (if it exists) and $\mathfrak{g}^*$ denotes the dual of $\mathfrak{g}$. If $\ell \in \mathfrak{g}^*$, $O_{\ell}$ denotes the coadjoint orbit of $\ell$.

3. The two-step nilpotent Lie algebras

Let $G$ be a connected and simply connected two-step nilpotent Lie group with Lie algebra $\mathfrak{g}$, then if $O_{\ell} = \text{Ad}^*(G)\ell$, one has

$$O_{\ell} = \{\ell\} + T_{\ell}O_{\ell},$$

and

$$T_{\ell}O_{\ell} = \mathfrak{g}(\ell)^{\perp},$$

where $T_{\ell}O_{\ell}$ is the tangent space of $O_{\ell}$ at $\ell$, by which we see that the coadjoint orbits in two-step nilpotent Lie algebras are flat (and symplectic) manifolds. In [2], the authors show the following:

**Proposition 3.1.** Let $\mathfrak{g}$ be a nilpotent Lie algebra of class 2 (i.e, $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$), and let $G = \exp \mathfrak{g}$ be the associated Lie group. Denote by $\text{ad}^*$ the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$. Let $f \in \mathfrak{g}^*$. Then the corresponding representation $\pi_f$ of $G$ belongs to $\text{cor}(G)$ if and only if $f$ belongs to the closure of the subset $\{\text{ad}^*_X(\ell), X \in \mathfrak{g}, \ell \in \mathfrak{g}^*\}$ of $\mathfrak{g}^*$.

From this we can conclude the following:

**Corollary 3.2.** Let $\mathfrak{g}$ is a two-step nilpotent Lie algebra. If $T_{\ell}O_{\ell}$ denotes the tangent space to the coadjoint orbit $O_{\ell}$ at $\ell$, then the $\text{Cor}(\mathfrak{g}^*)$ is the closure in $\mathfrak{g}^*$ of the set

$$\bigcup_{\ell \in \mathfrak{g}^*} T_{\ell}O_{\ell} = \bigcup_{O_{\ell} \in \mathfrak{g}^*/\text{Ad}^*(G)} T_{O_{\ell}},$$

where $T_{O_{\ell}}$ is the fiber tangent of $O_{\ell}$ and $\mathfrak{g}^*/\text{Ad}^*(G)$ is the space of coadjoint orbits in $\mathfrak{g}^*$.
Proof. Indeed, for any $\ell \in \mathfrak{g}^*$, one has
\[ \{ ad_\ell^*(\ell); X \in \mathfrak{g} \} = T_\ell \mathcal{O}_\ell, \]
and hence with Proposition (3.1), the conclusion yields. $\square$

Proposition 3.3. Let $G$ be a two-step nilpotent Lie group with Lie algebra $\mathfrak{g}$, choose a real Jordan–Hölder basis $\{X_j\}$. Let $\mathcal{P}$ be the corresponding fine stratification of $\mathfrak{g}^*$, and let $\Omega$ be a layer belonging to $\mathcal{P}$. Then there is an explicit construction of an open set $U$ in $\mathfrak{g}^*$ and real-valued functions $p_1, p_2, \ldots, p_d, q_1, q_2, \ldots, q_d$ on $U$, such that $U$ contains $\Omega$, and such that for each coadjoint orbit $\mathcal{O}_\ell$ in $\Omega$, $p_1|_{\mathcal{O}_\ell}, p_2|_{\mathcal{O}_\ell}, \ldots, p_d|_{\mathcal{O}_\ell}, q_1|_{\mathcal{O}_\ell}, q_2|_{\mathcal{O}_\ell}, \ldots, q_d|_{\mathcal{O}_\ell}$ are real-valued, global canonical coordinates for $\mathcal{O}_\ell$. Moreover, for each $1 \leq j \leq n$, $0 \leq u \leq d$, there are rational functions $\alpha_{j,u}$ and $\beta_{j,u}$ such that for each $1 \leq j \leq n$ and $\ell \in \Omega$ one has
\[ \ell_j := \ell(X_j) = \sum_{u: j_u \leq j} \alpha_{j,u}(\ell)p_u + \sum_{r=1}^d \beta_{j,u}(\ell)q_u. \]

Proof. Recall that the construction of $p_r, q_r$ depends on the flag
\[ (\mathfrak{g}_j = \text{span}\{X_1, \ldots, X_j\})_{1 \leq j \leq n}. \]
More precisely if $j_t = \min\{j_r, 1 \leq r \leq d\}$, then:
\[ p_1^{(1)} = \ell_{j_t}, \quad q_1^{(1)} = \frac{\ell_{j_t}}{[X_{j_t}, X_{j_t}]} \]
Now suppose we have built $p_1^{(m)}, \ldots, p_k^{(m)}, \ldots, q_1^{(m)}, \ldots, q_k^{(m)}$, then for $\mathfrak{g}_{m+1}$ one has either $m+1 \not\in \mathfrak{e}$ and in this case $p_r^{(m+1)} = p_r^{(m)}$, $q_r^{(m+1)} = q_r^{(m)}$ or $m+1 = j_{k+1} \in \mathfrak{e}$ and in this case
\[ q_r^{(m+1)}(\ell) = q_r^{(m)}(\exp -qX_{m+1}\ell) = q_r^{(m)}(\ell) - q\{x_{m+1}, q_r^{(m)}\}, \]
and
\[ p_r^{(m+1)}(\ell) = p_r^{(m)}(\exp -qX_{m+1}\ell) = p_r^{(m)}(\ell) - q\{x_{m+1}, p_r^{(m)}\}, \]
with $q = \frac{y}{\ell(X_{m+1}, y)}$, where $y$ is a $G_m$-invariant and non-$G_{m+1}$-invariant polynomial function such that $\{x_{m+1}, y\}$ is nonvanishing on $\Omega$ (here $G_j = \exp \mathfrak{g}_j$). $\square$

Corollary 3.4. Let $\mathfrak{e} = \{e_1 < \cdots < e_{2d}\}$ be the set of jump indices corresponding to the minimal layer in $\mathfrak{g}^*$. Let $F$ be the cross-section mapping associated with the minimal layer $\Omega$ then $F(\ell) = (F_1(\ell), \ldots, F_n(\ell))$ and let $\mathfrak{e} = \{e_1 < \cdots < e_{2d}\}$ be the corresponding jump indices then
\[ F_k(\ell) = \begin{cases} \ell_k, & \text{if } k = 1, \ldots, e_1 - 1; \\ 0, & \text{if } k \in \mathfrak{e}; \\ \ell_k + \sum_{j:e_j \leq k-1} \alpha_j(\ell)\ell_{e_j}, & \text{if } k \not\in \mathfrak{e}, k \geq e_1, \end{cases} \]
where each of \( a_1(\ell), \ldots, a_{k-1}(\ell) \) is (nontrivial) a rational regular function on the minimal layer depending only upon \( \ell^0 = \ell_3 \) (\( \mathfrak{g} \) is the center of \( \mathfrak{g} \)).

**Proof.** For each layer in \( \mathfrak{g}^* \), the mapping \((\ell_{e_1}, \ldots, \ell_{e_{2d}}) \mapsto (p_i(\ell), q_i(\ell))_{1 \leq i \leq d}\) is a rational diffeomorphism whose inverse is also rational on any layer, then we consider the minimal layer and by Proposition 3.3, we can write

\[
p_i(\ell) = \sum_j u_j(\ell_1, \ldots, \ell_p)\ell_{e_j}, \quad q_i(\ell) = \sum_j v_j(\ell_1, \ldots, \ell_p)\ell_{e_j}, \quad i = 1, \ldots, d,
\]

where \( u_j \) and \( v_j \) are rational regular functions on the minimal layer. Then after substituting each of the functions \((p_i, q_i)\) by the above expressions in the coordinate functions \((\ell_k)_{k \notin \mathfrak{e}}\) we obtain the invariant functions of \( \mathfrak{g}^* \) and this ends the proof. \( \square \)

**Corollary 3.5.** If \( \mathfrak{g} \) is a two-step nilpotent Lie algebra and \( \mathfrak{g}^* \) denotes its dual, then

\[
\text{ICor}(\mathfrak{g}^*) = \{ \ell \in \mathfrak{g}^*: \ell(Z) = 0 \ \forall Z \in \mathfrak{z} \}.
\]

**Proof.** The nontrivial coordinates of the cross-section mapping \((F_k(\ell))_{k \geq p, k \notin \mathfrak{e}}\) associated with the minimal layer can be written as

\[
F_k(\ell) = \frac{B(\ell^0)\ell_k + A_k(\ell^0)}{B(\ell^0)}, \quad k \notin \mathfrak{e}, k > p := e_1,
\]

where each of \( B(\ell^0) \) and \( B(\ell^0)\ell_k + A_k(\ell^0), (k \geq p, k \notin \mathfrak{e}) \) is a nontrivial \( G \)-invariant polynomial on \( \mathfrak{g}^* \), with \( \ell^0 = \ell_3 \). Note that these polynomials are homogeneous and for each \( k \geq p, k \notin \mathfrak{e} \), one has

\[
\deg(B(\ell^0)\ell_k + A_k(\ell^0)) = \deg(B(\ell^0)) + 1.
\]

Finally the ring \( \text{Pol}(\mathfrak{g}^*)^G \) of \( G \)-invariant polynomials is spanned by the polynomials

\[
\ell_1, \ldots, \ell_p, B(\ell^0), (B(\ell^0)\ell_k + A_k(\ell^0))_{k \geq p, k \notin \mathfrak{e}};
\]

and this ends the proof. \( \square \)

**3.1. Main example.** In [2], one introduces an interesting example of 8-dimensional two-step nilpotent Lie algebra \( \mathfrak{g} \) so that the corresponding cor- tex in \( \mathfrak{g}^* \) is a projective algebraic set given by a quadric and such that \( \text{Cor}(\mathfrak{g}^*) \subsetneq \text{ICor}(\mathfrak{g}^*) \). Here we give a generalization of that example. Let \( d \in \mathbb{N} \) with \( d \geq 2 \) and let \( \mathfrak{g}_d \) be the Lie algebra with basis

\[
(Z_1, \ldots, Z_d, Y_1, Y_2, \ldots, Y_{2d-1}, Y_{2d}, X_1, \ldots, X_d),
\]

and nontrivial brackets

\[
[X_i, Y_{2i-1}] = Z_1, \quad \text{for } i = 1, \ldots, d,
\]

\[
[X_k, Y_{2k}] = Z_{k+1}, \quad \text{for } k = 1, \ldots, d-1,
\]

\[
[X_d, Y_{2d}] = Z_2 + \cdots + Z_d.
\]
Let’s denote the center of $g_d$ by $z_d = \text{span}\{Z_1, \ldots, Z_d\}$ and $G_d$ the corresponding connected and simply connected Lie group.

**Proposition 3.6.** For each Lie algebra $g_d$ ($d \geq 2$), one has:

(i) The minimal layer in $g_d^*$ is given by
\[
\Omega_d = \{\ell \in g_d^* : \ell(Z_1) \neq 0\}.
\]

(ii) The coadjoint orbits in $\Omega_d$ are $2d$-dimensional and if
\[
\ell = \sum_{k=1}^{d}(\lambda_k Z_k^* + \beta_k X_k^*) + \sum_{k=1}^{2d}\gamma_k Y_k^* \in \Omega_d,
\]
\[\xi = ((z_i)_{1 \leq i \leq d}, (y_j)_{1 \leq j \leq 2d}, (x_k)_{1 \leq k \leq d}) \in G\ell,
\]
then
\[
\xi = \begin{cases}
  z_k = \lambda_k, & \text{if } k = 1, \ldots, d; \\
  y_{2k-1} = \gamma_{2k-1} + s_j \lambda_1, & \text{if } k = 1, \ldots, d-1; \\
  y_{2k} = \gamma_{2k} + s_j \lambda_{k+1}, & \text{if } k = 1, \ldots, d-1; \\
  y_{2d} = \gamma_{2d-1} + s_d \lambda_1; \\
  y_{2d} = \gamma_{2d} + s_d (\lambda_2 + \cdots + \lambda_d); \\
  x_k = \beta_k + t_k & \text{if } k = 1, \ldots, d.
\end{cases}
\]

(iii) The algebra of $G$-invariant polynomials is
\[
\text{Pol}(g_d^*)^{G_d} = \mathbb{R}[z_1, \ldots, z_d, z_1 y_2 - z_2 y_1, \ldots, z_1 y_{2d-2} - z_{d-1} y_{2d-3}, z_1 y_{2d} - (z_2 + \cdots + z_d) y_{2d-1}].
\]

**Proof.** Let $B_d = (U_1, \ldots, U_{4d})$ be the Jordan–Hölder basis defined by
\[
U_i = \begin{cases}
  Z_i, & \text{if } 1 \leq i \leq d, \\
  Y_{i-d}, & \text{if } d + 1 \leq i \leq 3d; \\
  X_{i-3d}, & \text{if } 3d + 1 \leq i \leq 4d.
\end{cases}
\]

Using the methods of [1], we can see that the minimal layer in $g_d^*$ is
\[
\Omega_d = \{\ell \in g^* : \ell(U_1) = \ell(Z_1) \neq 0\},
\]
which corresponds to the set of jump indices $e_d = i_d \cup j_d$ with
\[
i_d = \{d + 1 < d + 3 < \cdots < 3d - 1\},
\]
\[
j_d = \{3d + 1, 3d + 2, \ldots, 4d\}.
\]

Then by using the methods of [1] (the parametrization of coadjoint orbits) we can deduce the results of (ii) and (iii). \qed

**Remark 3.1.**
(i) If $\Omega_d$ is the minimal layer given as above, then the canonical coordinates on $\Omega_d$ (see [1]) are given by

$$p_i(\ell) = x_i, \quad q_i(\ell) = \frac{y_{2i-1}}{z_1}, \quad i = 1, \ldots, d.$$ 

(ii) The cross-section $\Sigma_d$ is given by

$$\Sigma_d = \left( \sum_{k \in e} R U^*_k \right) \cap \Omega_d = \left( \sum_{k=1}^d R Z^*_k + R Y^*_{2k} \right) \cap \Omega_d.$$ 

(iii) The cross-section mapping $F_d : \Omega_d \to \Sigma_d$ is as follows

$$F_d(z_i, y_j, x_k) = \sum_{i=1}^d z_i Z^*_i + \sum_{j=1}^{d-1} \left( y_{2i} - \frac{z_i + 1}{z_i} y_{2i} - 1 \right) Y^*_{2i} + \left( y_{2d} - \frac{z_2 + \cdots + z_d}{z_1} \right) Y^*_{2d},$$

where $(Z^*_1, \ldots, Z^*_d, Y^*_1, \ldots, Y^*_{2d}, X^*_1, \ldots, X^*_d)$ is the dual basis of $B$.

**Proposition 3.7.** Let’s denote

$$\ell = \sum_{i=1}^d (z_i Z^*_i + x_i X^*_i) + \sum_{j=1}^{2d} y_j Y^*_j \in g^*$$

by $\ell = (z_i, y_j, x_k)$, where $(Z^*_1, \ldots, Z^*_d, Y^*_1, \ldots, Y^*_{2d}, X^*_1, \ldots, X^*_d)$ is the dual basis in $g^*_d$. Then the cortex of $g^*_d$ is the projective algebraic set given by

$$\text{Cor}(g^*_d) = \left\{ \ell = (z_i, y_j, x_k) : z_1 = \cdots = z_d = y_{2d-1} \left( \sum_{i=1}^{d-1} y_{2i} \prod_{j=1, j \neq i}^{d-1} y_{2j-1} \right) - y_{2d} \prod_{j=1}^{d-1} y_{2j-1} = 0 \right\}.$$ 

**Proof.** Note that since $\Omega_d$ is dense in $g^*_d$ (Zariski open subset in $g^*_d$) then

$$\text{Cor}(g^*_d) = \left\{ \lim_{m} \Ad_{\exp}^* x_m \ell_m, \quad (\ell_m)_{m} \in g^*_d, \quad (\ell_m)_{m} \in \Omega_d, \quad \text{and} \quad \lim_{m} \ell_m = 0 \right\}.$$ 

On the other hand if $O_\ell = G\ell$, then the tangent space $T_\ell O_\ell$ at $\ell$ is

$$T_\ell O_\ell = \{ ad^*_{X}(\ell), \quad \ell \in \Omega_d, \quad X \in \text{Vect}\{Y_{2k-1}, X_k, \quad 1 \leq k \leq d\} \}.$$ 

Now if $\ell = (\lambda_i, \gamma_j, \beta_k) \in \Omega_d$ and $\xi \in T_\ell O_\ell$, with

$$\xi = \sum_{i=1}^d (z_i Z^*_i + x_i X^*_i) + \sum_{j=1}^{2d} y_j Y^*_j,$$
then
\[
\begin{align*}
&z_i = 0, \quad \text{if } i = 1, \ldots, d; \\
y_{2j-1} = s_j \lambda_1, \quad \text{if } j = 1, \ldots, d-1; \\
y_{2j} = s_j \lambda_{j+1}, \quad \text{if } j = 1, \ldots, d-1; \\
y_{2d-1} = s_d \lambda_1; \\
y_{2d} = s_d (\lambda_2 + \cdots + \lambda_d); \\
x_k = t_k, \quad \text{if } k = 1, \ldots, d.
\end{align*}
\]

From which we can see that \(\xi = (z_i, y_j, x_k) \in T_\ell O\) if and only if
\[
\begin{align*}
z_i &= 0, \quad \text{if } i = 1, \ldots, d; \\
y_{2j} &= y_{2j-1} \frac{\lambda_{j+1}}{\lambda_1}, \quad \text{if } j = 1, \ldots, d-1; \\
y_{2d} &= y_{2d-1} \frac{\lambda_2 + \cdots + \lambda_d}{z_1}.
\end{align*}
\]
with \(y_{2j-1}, x_j\) are free variables in \(\mathbb{R}\) \((j = 1, \ldots, d)\). Then we see that a.e. \(\xi \in T_\ell O\) satisfies
\[
\frac{y_{2d}}{y_{2d-1}} = \sum_{j=1}^{d-1} \frac{y_{2j}}{y_{2j-1}},
\]
and hence
\[
\text{Cor}(g_d^*) = \left\{ \ell = (z_i, y_j, x_k) \in g_d^* : z_i = 0, \\
y_{2d-1} \left( \sum_{i=1}^{d-1} y_{2i} \prod_{j=1, j \neq i}^{d-1} y_{2j-1} \right) - y_{2d} \prod_{j=1}^{d-1} y_{2j-1} = 0 \right\}. \quad \square
\]

**Corollary 3.8.** For each integer \(d \geq 2\) if \(z_d\) denotes the center of the Lie algebra \(g_d\), then
\[
\text{Cor}(g_d^*) \subseteq \text{ICor}(g_d^*) = z_d^+.
\]

**Proof.** The ring of \(G\)-invariant polynomials on \(g^*\) is given by
\[
\text{Pol}(g_d^*)^{G_d} = \mathbb{R} \left[ z_1, \ldots, z_d, (z_1 y_{2i} - z_{i+1} y_{2i-1})_{1 \leq i \leq d-1} ; z_1 y_{2d} - (z_2 + \cdots + z_d) y_{2d-1} \right],
\]
where \(G_d\) is the connected and simply connected (nilpotent) Lie group corresponding to \(g_d\). Thus
\[
\{ \ell \in g_d^* : P(\ell) = P(0), \forall P \in \text{Pol}(g_d^*)^{G_d} \} = z_d^+,
\]
where
\[
z_d^+ = \{ \ell \in g_d^* : \ell(Z) = 0, \forall Z \in z_d \}.
\]
hence with Proposition 3.7, we conclude that
\[
\text{Cor}(g_d^*) \subseteq \text{ICor}(g_d^*) = z_d^+.
\] \(\square\)
4. The Lie group $\mathbb{R}^n \rtimes \mathbb{R}$

In [3], in there is a study of the cortex of the nilpotent Lie group
\[ G = \mathbb{R}^n \rtimes \mathbb{R}, \]
the authors show that
\[ \operatorname{Cor}(\mathfrak{g}^*) = \operatorname{ICor}(\mathfrak{g}^*) \]
\[ = \{ \ell \in \mathfrak{g}^* : P(\ell) = P(0), \ P \text{ is } G \text{ invariant polynomial on } \mathfrak{g}^* \}. \]
The definition of $\operatorname{ICor}(\mathfrak{g}^*)$ may not exist if $G$ is not nilpotent but we can define $G$-invariant (or semi-invariant) functions. Let’s consider the following example:

**Example 4.1.** Let $(X_1, X_2, A)$ be a basis in $\mathfrak{g}$ with
\[ [A, X_1] = X_1, [A, X_2] = -2X_2. \]
Let’s identify $\mathfrak{g}^*$ with $\mathbb{R}^3$ under the dual basis $(X_1^*, X_2^*, A^*)$, and denote $x = (x_1, x_2, a) \in \mathfrak{g}^*$, then the minimal layer is
\[ \Omega = \{ \ell = (\ell_1, \ell_2, a) \in \mathfrak{g}^* : \ell_1 \neq 0 \}. \]
If $\ell = (\ell_1, \ell_2, a) \in \mathfrak{g}^*$, then the coadjoint orbit of $\ell$ is given by
\[ \mathcal{O}_\ell = \{ x \in \mathfrak{g}^* : x = (\ell_1 e^t, \ell_2 e^{-2t}, a + s), t, s \in \mathbb{R} \}, \]
that is,
\[ \mathcal{O}_\ell = \{ x = (x_1, x_2, x_3) \in \mathfrak{g}^* : \text{sign}(x_1) = \text{sign}(\ell_1), x_1^2 x_2 = \ell_1^2 \ell_2, x_3 \in \mathbb{R} \}. \]
We can check that the cortex of $\mathfrak{g}^*$ is given by
\[ \operatorname{Cor}(\mathfrak{g}^*) = \{ \ell = (\ell_1, \ell_2, \ell_3) \in \mathfrak{g}^* : \ell_1 \ell_2 = 0 \}. \]
On other hand, the cross-section mapping is as follows
\[ F : \Omega \rightarrow \Omega, \quad \ell \mapsto \left( \text{sign}(\ell_1) = \frac{\ell_1}{|\ell_1|}, \frac{\ell_1^2 \ell_2}{|\ell_1|}, 0 \right), \]
from which we see the existence of $G$-invariant polynomial $p(x) = x_1^2 x_2$ and we see that
\[ \operatorname{Cor}(\mathfrak{g}^*) = \{ \ell \in \mathfrak{g}^* : p(\ell) = 0 \}. \]
In this example if we let $[A, X_1] = X_1, [A, X_2] = -\sqrt{2}X_2$ then there are no $G$-invariant polynomials on $\mathfrak{g}^*$, however the function $x_1^2 x_2$ is $G$-invariant and the cortex is still the same. This example can be generalized. To this end, if $\mathfrak{g} = \mathbb{R}^n \oplus \mathbb{R}A$, we denote $\text{sp}(adA) = \{ \lambda_1, \ldots, \lambda_n \}$ the set of eigenvalues of $adA$, and for $\lambda \in \text{sp}(adA)$, we set
\[ E_\lambda = \bigcup_{m \in \mathbb{N}} \ker(adA - \lambda)^m, \]
and
\[ E^+ = \bigcup_{\lambda \in \text{sp}(adA), \Re(\lambda) > 0} E_\lambda, \]
Then we have the following:

**Proposition 4.2.** Let $G = \mathbb{R}^n \times \mathbb{R}$ be the Lie group whose Lie algebra $\mathfrak{g} = \mathbb{R}^n \oplus \mathbb{R}A$. Suppose that $adA$ is diagonalizable, and let $sp(adA) = \{\lambda_1, \ldots, \lambda_n\}$ denote the set of eigenvalues of $adA$.

(a) If $\{\Re(\lambda_j)\}_{1 \leq j \leq n} \subset (0, \infty)$ or $\{\Re(\lambda_j)\}_{1 \leq j \leq n} \subset (-\infty, 0)$ then

$$
\text{Cor}(\mathfrak{g}^*) = \mathbb{R}^n.
$$

(b) If $\prod_{j=1}^n \Re(\lambda_j) < 0$. Then the cortex of $\mathfrak{g}^*$ is the union of two vector spaces. More precisely

$$
\text{Cor}(\mathfrak{g}^*) = (V^+ + \mathbb{R}A^+) \cup (V^- + \mathbb{R}A^*),
$$

where

$$
V^+ = (E^+)^*, \quad V^- = (E^-)^*.
$$

**Proof.** If $sp(adA) = \{\lambda_1, \ldots, \lambda_n\}$ denotes the set of eigenvalues of $adA$ (restricted to $\mathbb{R}^n$). Then identifying $\mathfrak{g}$ with $\mathbb{R}^{n+1}$ (respectively $\mathbb{C}^{n+1}$ if some of the eigenvalues of $adA$ are nonreal), the coadjoint orbit of any $\ell \in \mathfrak{g}^*$ is parameterized as follows

$$
\mathcal{O}_\ell = \{ (\ell_1 e^{\lambda_1 t}, \ldots, \ell_n e^{\lambda_n t}, \ell_{n+1} + s), \quad t, s \in \mathbb{R} \}.
$$

Since $\Re(\lambda_j) \neq 0, j = 1, \ldots, n$, then for any $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ the linear system

$$
\begin{pmatrix}
\begin{bmatrix}
e^{\lambda_1 t} & 0 & \cdots \\
0 & \ddots & 0 \\
\cdots & 0 & e^{\lambda_n t}
\end{bmatrix} & \ell_1 \\
& \vdots & \vdots \\
& \ell_n & \ell_n
\end{pmatrix}
= \begin{pmatrix}
\begin{bmatrix}
\alpha_1 \\
\vdots \\
\alpha_n
\end{bmatrix}
\end{pmatrix},
$$

has a unique solution $(\ell_1, \ldots, \ell_n)^\top$ with

$$
\| (\ell_1, \ldots, \ell_n)^\top \| = \| (e^{-\lambda_1 t} \alpha_1, \ldots, e^{-\lambda_n t} \alpha_n)^\top \|.
$$

(a) If $\{\Re(\lambda_j)\}_{1 \leq j \leq n} \subset (0, \infty)$ or $\{\Re(\lambda_j)\}_{1 \leq j \leq n} \subset (-\infty, 0)$ then for any $(\alpha_1, \ldots, \alpha_n, \beta) \in \mathfrak{g}^*$ it exists $\{x^{(m)} = x(t_m, \ell^{(m)})\}_m \in \mathfrak{g}^*$ with $\{\ell^{(m)}\}_m \subset \Omega$ and $\lim_{m:R(\lambda_1)\ell_m \to \infty} \ell^{(m)} = 0$ such that

$$
\lim_{m:R(\lambda_1)\ell_m \to \infty} x^{(m)} = (\alpha_1, \ldots, \alpha_n, \beta),
$$

and hence the cortex is all of $\mathfrak{g}^*$.

(b) In that case, let’s rearrange the basis $(X_1, \ldots, X_n)$ in $\mathbb{C}^n$ such that the matrix of $adA$ in this basis is $	ext{diag}(\lambda_1, \ldots, \lambda_{k_0}, \lambda_{k_0+1}, \ldots, \lambda_n)$ with

$$
\Re(\lambda_1) > 0, \ldots, \Re(\lambda_{k_0}) > 0, \Re(\lambda_{k_0+1}) < 0, \ldots, \Re(\lambda_n) < 0,
$$

then for any $x = (x_1, \ldots, x_n, x_{n+1}) \in \mathcal{O}_\ell$ one has

$$
\begin{align*}
|\ell_k|^\lambda_k |x_k|^\lambda_k &= |\ell_k|^\lambda_k |x_k|^\lambda_k, \quad k \leq k_0, \\
|x_k|^\lambda_k |x_1|^{-\lambda_k} &= |\ell_k|^\lambda_k |\ell_1|^{-\lambda_k}, \quad k \geq k_0 + 1.
\end{align*}
$$
Hence one has

\[ \text{Cor}(\mathfrak{g}^*) = \{ \ell \in \mathfrak{g}^* : \ell = (\ell_1, \ldots, \ell_{k_0}, 0, \ldots, 0, \ell_{n+1}) \} \]

\[ \cup \{ \ell \in \mathfrak{g}^* : \ell = (0, \ldots, 0, \ell_{k_0+1}, \ldots, \ell_n, \ell_{n+1}) \} \] \qed

**Remark 4.1.** Let \( \mathfrak{g} = \mathbb{R}^n \oplus \mathbb{R}A \) be a real Lie algebra. Suppose that there exists a basis \((X_1, \ldots, X_n)\) in \( \mathbb{R}^n \) such that

\[ [A, X_j] = m_j X_j, \quad j = 1, \ldots, n, \]

with \( \{m_1, \ldots, m_n\} \subset \mathbb{R}^\times \).

(a) If \( \{\frac{m_k}{m_1}\}_{1 \leq k \leq n} \in \mathbb{N} \), then any generic coadjoint orbit of \( \ell \,(\ell_1 \neq 0) \) is given by

\[ O_\ell = \left\{ x = (x_1, \ldots, x_n, x_{n+1}) \in \mathfrak{g}^* : x_1 \ell_1 > 0, x_k = \frac{x_{m_j}}{x_{m_1}^{m_j}}, \right\} \]

\[ k = 2, \ldots, n, x_{n+1} \in \mathbb{R} \}

and hence, it is an open semi-algebraic subset in \( \mathfrak{g}^* \).

(b) Now suppose that \( \{m_1, \ldots, m_n\} \subset \mathbb{Z}^\times \) with \( \prod_{j=1}^n m_j < 0 \). We can assume the existence of a basis \((X_1, \ldots, X_n)\) in \( \mathbb{R}^n \) so that with respect to this basis the matrix of \( \text{ad}A \) is

\[ \text{ad}A = \text{diag}(m_1, \ldots, m_{k_0}, m_{k_0+1}, \ldots, m_n) \]

with \( m_1 > 0, \ldots, m_{k_0} > 0, m_{k_0+1} < 0, \ldots, m_n < 0 \). Then for any \( x = (x_1, \ldots, x_n, x_{n+1}) \in O_\ell \) (with \( \ell_1 \neq 0 \)) one has

\[ \begin{cases} x_1 \ell_1 > 0, \\ \ell_1^{m_j} x_{m_j} = \ell_1^{m_1} x_{m_1}, \quad j = 2, \ldots, k_0, \\ x_{-m_j}^{m_j} x_{m_j} = \ell_1^{-m_j} \ell_1^{m_j}, \quad j = k_0 + 1, \ldots, n. \end{cases} \]

On other hand, the polynomials

\[ p_{i,j}(\ell) = \ell_i^{m_j} \ell_j^{m_j}, \quad i = 1, \ldots, k_0, \quad j = k_0 + 1, \ldots, n \]

are \( G \)-invariant on \( \mathfrak{g}^* \) and the cortex is the union of two vector spaces given by:

\[ \text{Cor}(\mathfrak{g}^*) = \{ \ell \in \mathfrak{g}^* : p_{i,j}(\ell) = 0 \quad \forall 1 \leq i \leq k_0, k_0 + 1 \leq j \leq n \}. \]

**Corollary 4.3.** Let \( \mathfrak{g} = \mathbb{R}^n \oplus \mathbb{R}A \) be a real Lie algebra. Let’s denote \( \text{sp}(\text{ad}A) = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C} \) the set of eigenvalues of \( \text{ad}A \). If

\[ \{\Re(\lambda_j)\}_{1 \leq j \leq n} \subset (0, \infty) \quad \text{or} \quad \{\Re(\lambda_j)\}_{1 \leq j \leq n} \subset (-\infty, 0), \]

then

\[ \text{Cor}(\mathfrak{g}^*) = \mathfrak{g}^*. \]
**Proof.** First let’s suppose that the real endomorphism $adA$ has a single eigenvalue $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. According to $\lambda$ is real or complex, we can suppose the existence of a basis in $\mathbb{R}^n$ (resp. in $\mathbb{C}^n$) such that the matrix of $adA$ is written in a Jordan block form:

$$adA = J_\lambda = \begin{pmatrix}
\lambda & 1 & 0 & \ldots \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 1 \\
0 & \ldots & 0 & \lambda
\end{pmatrix}$$

The coadjoint orbit of $\ell = (\ell_1, \ldots, \ell_n, \ell_{n+1})$ is given by

$$O_\ell = \left\{ \left( x_k = e^{\lambda t} \sum_{j \geq 1, j + k = 1}^t \frac{t^i}{i!} \ell_j, x_{n+1} = \ell_{n+1} + s \right), t, s \in \mathbb{R} \right\}.$$ 

Now let’s remark that for any $\alpha \in \mathbb{R}^n$ (resp. in $\mathbb{C}^n$), since $\Re(\lambda) \neq 0$, the linear system

$$e^{\lambda t} \begin{pmatrix}
1 & 0 & \ldots & 0 \\
t & 1 & \ddots & \\
\vdots & \ddots & \ddots & 0 \\
\frac{t^{n-1}}{(n-1)!} & \ldots & t & 1
\end{pmatrix} \begin{pmatrix}
\ell_1 \\
\ell_2 \\
\vdots \\
\ell_n
\end{pmatrix} = \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_n
\end{pmatrix}$$

has a unique solution and if we let

$$M(t) = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
t & 1 & \ddots & \\
\vdots & \ddots & \ddots & 0 \\
\frac{t^{n-1}}{(n-1)!} & \ldots & t & 1
\end{pmatrix},$$

then $M(t)$ is a unipotent matrix whose inverse $M^{-1}(t) = (p_{i,j}(t))_{1 \leq i, j \leq n}$ is also unipotent and all its entries $p_{i,j}(t)$ are polynomial functions in $t$, then

$$[\ell_1, \ldots, \ell_n]^\perp = e^{-\lambda t} M^{-1}(t)[\alpha_1, \ldots, \alpha_n]^\perp$$

and

$$||[\ell_1, \ldots, \ell_n]^\perp|| = e^{-\Re(\lambda)t} F(t),$$

where $F(t)^2$ is polynomial function in $t$ and then

$$\lim_{\Re(\lambda)t \to \infty} e^{-\Re(\lambda)t} F(t) = 0,$$

For instance for any $\alpha = [\alpha_1, \ldots, \alpha_n] \in \mathbb{R}^n$ if $\lambda$ is real (resp. $\alpha \in \mathbb{C}^n$ if $\lambda \in \mathbb{C} \setminus i\mathbb{R}$) and $\{t_m\}_m \subset \mathbb{R}$ such that $\lim_{m \to \infty} t_m \Re(\lambda) = \infty$ it exists $\{\ell_1^{(m)}, \ldots, \ell_n^{(m)}\}_m$ such that

$$\lim_{m \to \infty} e^{\lambda t_m} M(t_m)[\ell_1^{(m)}, \ldots, \ell_n^{(m)}]^\perp = [\alpha_1, \ldots, \alpha_n]^\perp.$$
\[ \lim_{m \to \infty} f_{1}^{(m)} = \cdots = \lim_{m \to \infty} f_{n}^{(m)} = 0. \]

This shows that the cortex of \( g^{*} \) coincides with \( g^{*} \). Finally if \( adA \) has more than one single eigenvalue, we can write \( adA = \text{diag}(J_{\lambda_{1}}, \ldots, J_{\lambda_{k}}) \) where each \( J_{\lambda} \) is a Jordan block matrix. \( \square \)

**Remark 4.2.** Let \( g = RX_{1} \oplus RX_{2} \oplus RA \) with
\[ [A, X_{1}] = X_{1}, [A, X_{2}] = X_{2}. \]

In this example the cortex of \( g^{*} \) is \( g^{*} \). The cross-section mapping of the minimal layer is given by
\[ F(\ell_{1}, \ell_{2}, \ell_{3}) = \left( \frac{\ell_{1}}{|\ell_{1}|}, \frac{\ell_{2}}{|\ell_{1}|}, 0 \right), \quad \ell_{1} \neq 0. \]

On other hand, we remark that the rational function \( r(\ell_{1}, \ell_{2}, \ell_{3}) = \frac{\ell_{2}}{\ell_{1}} \) is \( G \)-invariant on \( \Omega = \{ \ell = (\ell_{1}, \ell_{2}, \ell_{3}) \in g^{*} : \ell_{1} \neq 0 \} \) and
\[ \text{Cor}(g^{*}) \supseteq \{ \ell \in \Omega : r(\ell) = 0 \}. \]

**Corollary 4.4.** Let \( g = \mathbb{R}^{n} \oplus RA \), be a real Lie algebra. Let’s denote \( \text{sp}(adA) = \{ \lambda_{1}, \ldots, \lambda_{n} \} \subset \mathbb{C} \) the set of eigenvalues of \( adA \). If \( \prod_{j=1}^{n} \Re(\lambda_{j}) < 0 \), then the cortex of \( g^{*} \) is the union of two vector spaces. More precisely, with the notations of Proposition 4.2, one has
\[ \text{Cor}(g^{*}) = (V^{+} + RA^{*}) \cup (V^{-} + RA^{*}). \]

**Remark 4.3.** Let \( g = \mathbb{R}^{n} \oplus RA \) be a real Lie algebra, and assume that all the eigenvalues of \( adA \) are purely imaginary. Let’s denote \( h = \mathbb{R}^{n} \oplus RN \) where \( N \) is the nilpotent part in the Jordan decomposition of \( A \), then by [3], one has
\[ \text{Cor}(g^{*}) = \text{Cor}(h^{*}). \]

**References**


(Béchir Dali) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES OF BIZERTE, 7021 Zarzouna, Bizerte, Tunisia

Current address: King Saud University, college of science, Department of Mathematics, Riyadh, P.O Box 2455, Riyadh 11451, K.S.A.

bechir.dali@fss.rnu.tn

This paper is available via http://nyjm.albany.edu/j/2015/21-55.html.