On Φ-Mori modules

Ahmad Yousefian Darani and Mahdi Rahmatinia

Abstract. In this paper we introduce the concept of Mori module. An $R$-module $M$ is said to be a Mori module if it satisfies the ascending chain condition on divisorial submodules. Then we introduce a new class of modules which is closely related to the class of Mori modules. Let $R$ be a commutative ring with identity and set

$$
\mathbb{H} = \{ M \mid M \text{ is an } R\text{-module and } \Nil(M) \text{ is a divided prime submodule of } M \}.
$$

For an $R$-module $M \in \mathbb{H}$, set

$$
T = (R \setminus Z(M)) \cap (R \setminus Z(R)),
$$

$$
T(M) = T^{-1}(M),
$$

$$
P := [\Nil(M) : R M].
$$

In this case the mapping $\Phi : T(M) \longrightarrow M_P$ given by $\Phi(x/s) = x/s$ is an $R$-module homomorphism. The restriction of $\Phi$ to $M$ is also an $R$-module homomorphism from $M$ in to $M_P$ given by $\Phi(m/1) = m/1$ for every $m \in M$. A nonnil submodule $N$ of $M$ is $\Phi$-divisorial if $\Phi(N)$ is divisorial submodule of $\Phi(M)$. An $R$-module $M \in \mathbb{H}$ is called $\Phi$-Mori module if it satisfies the ascending chain condition on $\Phi$-divisorial submodules. This paper is devoted to study the properties of $\Phi$-Mori modules.

Contents

1. Introduction \quad 1269
2. Mori modules \quad 1272
3. $\Phi$-Mori modules \quad 1274
References \quad 1280

1. Introduction

We assume throughout this paper all rings are commmutative with $1 \neq 0$ and all modules are unitary. Let $R$ be a ring with identity and $\Nil(R)$ be the set of nilpotent elements of $R$. Recall from [Dobb76] and [Bada99-b], that a prime ideal $P$ of $R$ is called a divided prime ideal if $P \subset (x)$ for

Received July 18, 2015.

2010 Mathematics Subject Classification. 16D10, 16D80.

Key words and phrases. Mori module; divisorial submodule; $\Phi$-Mori module, $\Phi$-divisorial submodule.
every \( x \in R \setminus P \); thus a divided prime ideal is comparable to every ideal of \( R \). Badawi in [Bada99-a], [Bada00], [Bada99-b], [Bada01], [Bada02] and [Bada03] investigated the class of rings

\[ \mathcal{H} = \{ R \mid R \text{ is a commutative ring with } 1 \neq 0 \text{ and } \] 
\[ \text{Nil}(R) \text{ is a divided prime ideal of } R \}. \]

Anderson and Badawi in [AB04] and [AB05] generalized the concept of Prüfer, Dedekind, Krull and Bezout domain to context of rings that are in the class \( \mathcal{H} \). Also, Lucas and Badawi in [BadaL06] generalized the concept of Mori domains to the context of rings that are in the class \( \mathcal{H} \). Let \( R \) be a ring, \( Z(R) \) the set of zero divisors of \( R \) and \( S = R \setminus Z(R) \). Then \( T(R) := S^{-1}R \) denoted the total quotient ring of \( R \). We start by recalling some background material. A nonzero divisor of a ring \( R \) is called a regular element and an ideal of \( R \) is said to be regular if it contains a regular element. An ideal \( I \) of a ring \( R \) is said to be a nonnil ideal if \( I \not\subseteq \text{Nil}(R) \). If \( I \) is a nonnil ideal of \( R \in \mathcal{H} \), then \( \text{Nil}(R) \subseteq I \). In particular, it holds if \( I \) is a regular ideal of a ring \( R \in \mathcal{H} \). Recall from [AB04] that for a ring \( R \in \mathcal{H} \), the map \( \phi : T(R) \rightarrow R_{\text{Nil}(R)} \) given by \( \phi(a/b) = a/b, \) for \( a \in R \) and \( b \in R \setminus Z(R) \), is a ring homomorphism from \( T(R) \) into \( R_{\text{Nil}(R)} \) and \( \phi \) restricted to \( R \) is also a ring homomorphism from \( R \) into \( R_{\text{Nil}(R)} \) given by \( \phi(x) = x/1 \) for every \( x \in R \).

For a nonzero ideal \( I \) of \( R \) let \( I^{-1} = \{ x \in T(R) : xI \subseteq R \} \) and let \( I_\nu = (I^{-1})^{-1} \). It is obvious that \( II^{-1} \subseteq R \). An ideal \( I \) of \( R \) is called invertible, if \( II^{-1} = R \) and also \( I \) is called divisorial ideal if \( I_\nu = I \). \( I \) is said to be a divisorial ideal of finite type if \( I = J_\nu \) for some finitely generated ideal \( J \) of \( R \). A Mori domain is an integral domain that satisfies the ascending chain condition on divisorial ideals. Lucas in [Luc02], generalized the concept of Mori domains to the context of commutative rings with zero divisors. According to [Luc02] a ring is called a Mori ring if it satisfies a.c.c on divisorial regular ideals. Let \( R \in \mathcal{H} \). Then a nonnil ideal \( I \) of \( R \) is called \( \phi \)-invertible if \( \phi(I) \) is an invertible ideal of \( \phi(R) \). A nonnil ideal \( I \) is \( \phi \)-divisorial if \( \phi(I) \) is a divisorial ideal of \( \phi(R) \) [BadaL06]. Recall from [BadaL06] that \( R \) is called \( \phi \)-Mori ring if it satisfies a.c.c on \( \phi \)-divisorial ideals.

Let \( R \) be a ring and \( M \) be an \( R \)-module. Then \( M \) is a multiplication \( R \)-module if every submodule \( N \) of \( M \) has the form \( IM \) for some ideal \( I \) of \( R \). If \( M \) is a multiplication \( R \)-module and \( N \) a submodule of \( M \), then \( N = IM \) for some ideal \( I \) of \( R \). Hence \( I \subseteq (N :_R M) \) and so \( N = IM \subseteq (N :_R M)M \subseteq N \). Therefore \( N = (N :_R M)M \) [Bar81]. Let \( M \) be a multiplication \( R \)-module, \( N = IM \) and \( L = JM \) be submodules of \( M \) for ideals \( I \) and \( J \) of \( R \). Then, the product of \( N \) and \( L \) is denoted by \( NL \) and is defined by \( IJM \) [Ame03]. An \( R \)-module \( M \) is called a cancellation module if \( IM = JM \) for two ideals \( I \) and \( J \) of \( R \) implies \( I = J \) [Ali08-a]. By [Smi88, Corollary 1 to Theorem 9], finitely generated faithful multiplication modules are cancellation modules. It follows that if \( M \) is a finitely generated
faithful multiplication $R$-module, then $(IN :_RM) = I(N :_RM)$ for all ideals $I$ of $R$ and all submodules $N$ of $M$. If $R$ is an integral domain and $M$ a faithful multiplication $R$-module, then $M$ is a finitely generated $R$-module [ES98]. Let $M$ be an $R$-module and set

$$T = \{ t \in S : \text{ for all } m \in M, tm = 0 \text{ implies } m = 0 \}$$

$$= (R \setminus Z(M)) \cap (R \setminus Z(R)).$$

Then $T$ is a multiplicatively closed subset of $R$ with $T \subseteq S$, and if $M$ is torsion-free then $T = S$. In particular, $T = S$ if $M$ is a faithful multiplication $R$-module [ES98, Lemma 4.1]. Let $N$ be a nonzero submodule of $M$. Then we write $N^{-1} = (M :_{RT} N) = \{ x \in RT : xN \subseteq M \}$ and $N_{\nu} = (N^{-1})^{-1}$. Then $N^{-1}$ is an $R$-submodule of $RT$, $R \subseteq N^{-1}$ and $NN^{-1} \subseteq M$. We say that $N$ is invertible in $M$ if $NN^{-1} = M$. Clearly $0 \neq M$ is invertible in $M$. Following [Ali08-a], a submodule $N$ of $M$ is called a divisorial submodule of $M$ in case $N = N_{\nu}M$. We say that $N$ is a divisorial submodule of finite type if $N = L_{\nu}M$ for some finitely generated submodule $L$ of $M$. Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module. Let $N$ be a submodule of $M$, then it is obviously that, $N$ is a divisorial submodule of finite type if and only if $[N :_RM]$ is a divisorial ideal of finite type. If $M$ is a finitely generated faithful multiplication $R$-module, then $N_{\nu} = (N :_RM)$. Consequently, $M_{\nu} = R$. Let $M$ be a finitely generated faithful multiplication $R$-module, $N$ a submodule of $M$ and $I$ an ideal of $R$. Then $N$ is a divisorial submodule of $M$ if and only if $(N :_RM)$ is a divisorial ideal of $R$. Also $I$ is divisorial ideal of $R$ if and only if $IM$ is a divisorial submodule of $M$ [Ali09-a]. If $N$ is an invertible submodule of a faithful multiplication module $M$ over an integral domain $R$, then $(N :_RM)$ is invertible and hence is a divisorial ideal of $R$. So $N$ is a divisorial submodule of $M$ [Ali09-a]. If $R$ is an integral domain, $M$ a faithful multiplication $R$-module and $N$ a nonzero submodule of $M$, then $N_{\nu} = (N :_RM)_{\nu}$ [Ali09-a, Lemma 1]. We say that a submodule $N$ of $M$ is a radical submodule of $M$ if $N = \sqrt{N}$, where $\sqrt{N} = \sqrt{(N :_RM)M}$.

Let $M$ be an $R$-module. An element $r \in R$ is said to be zero divisor on $M$ if $rm = 0$ for some $0 \neq m \in M$. The set of zero divisors of $M$ is denoted by $Z_R(M)$ (briefly, $Z(M)$). It is easy to see that $Z(M)$ is not necessarily an ideal of $R$, but it has the property that if $a, b \in R$ with $ab \in Z(M)$, then either $a \in Z(M)$ or $b \in Z(M)$. A submodule $N$ of $M$ is called a nilpotent submodule if $[N :_RM]^nN = 0$ for some positive integer $n$. An element $m \in M$ is said to be nilpotent if $Rm$ is a nilpotent submodule of $M$ [Ali08-b]. We let $\text{Nil}(M)$ to denote the set of all nilpotent elements of $M$; then $\text{Nil}(M)$ is a submodule of $M$ provided that $M$ is a faithful module, and if in addition $M$ is multiplication, then $\text{Nil}(M) = \text{Nil}(R)M = \bigcap P$, where the intersection runs over all prime submodules of $M$, [Ali08-b, Theorem 6]. If $M$ contains no nonzero nilpotent elements, then $M$ is called a reduced $R$-module. A submodule $N$ of $M$ is said to be a nonnil submodule if $N \not\subseteq \text{Nil}(M)$. Recall
that a submodule $N$ of $M$ is prime if whenever $rm \in N$ for some $r \in R$ and $m \in M$, then either $m \in N$ or $rM \subseteq N$. If $N$ is a prime submodule of $M$, then $p := [N :_R M]$ is a prime ideal of $R$. In this case we say that $N$ is a $p$-prime submodule of $M$. Let $N$ be a submodule of multiplication $R$-module $M$, then $N$ is a prime submodule of $M$ if and only if $[N :_R M]$ is a prime ideal of $R$ if and only if $N = pM$ for some prime ideal $p$ of $R$ with $[0 :_R M] \subseteq p$, [ES98, Corollary 2.11]. Recall from [Ali09-b] that a prime submodule $P$ of $M$ is called a divided prime submodule if $P \subset Rm$ for every $m \in M \setminus P$; thus a divided prime submodule is comparable to every submodule of $M$.

Now assume that $T^{-1}(M) = \mathfrak{T}(M)$. Set

$$\mathbb{H} = \{ M \mid M \text{ is an } R\text{-module and } \text{Nil}(M) \text{ is a divided prime submodule of } M \}.$$ 

For an $R$-module $M \in \mathbb{H}$, $\text{Nil}(M)$ is a prime submodule of $M$. So

$$P := [\text{Nil}(M) :_R M]$$

is a prime ideal of $R$. If $M$ is an $R$-module and $\text{Nil}(M)$ is a proper submodule of $M$, then $[\text{Nil}(M) :_R M] \subseteq Z(R)$. Consequently,

$$R \setminus Z(R) \subseteq R \setminus [\text{Nil}(M) :_R M].$$

In particular, $T \subseteq R \setminus [\text{Nil}(M) :_R M]$ [Yous]. Recall from [Yous] that we can define a mapping $\Phi : \mathfrak{T}(M) \rightarrow M_P$ given by $\Phi(x/s) = x/s$ which is clearly an $R$-module homomorphism. The restriction of $\Phi$ to $M$ is also an $R$-module homomorphism from $M$ in to $M_P$ given by $\Phi(m/1) = m/1$ for every $m \in M$. A nonnil submodule $N$ of $M$ is said to be $\Phi$-invertible if $\Phi(N)$ is an invertible submodule of $\Phi(M)$ [MY]. An $R$-module $M$ is called a Nonnil-Noetherian module if every nonnil submodule of $M$ is finitely generated [Yous]. In this paper, we define concept of a Mori module and obtain some properties of this module. Then we introduce a generalization of $\phi$-Mori rings.

2. Mori modules

**Definition 2.1.** Let $R$ be a ring and $M$ be an $R$-module. Then $M$ is said to be a Mori module if it satisfies on divisorial submodules.

It is clear that, if $M$ is a Noetherian $R$-module, then $M$ is a Mori $R$-module.

**Theorem 2.2.** Let $R$ be an integral domain and $M$ a faithful multiplication $R$-module. Then $M$ is a Mori module if and only if $R$ is a Mori domain.

**Proof.** Let $M$ be a Mori module and $\{I_m\}$ be an ascending chain of divisorial ideals of $R$. Then $\{(I_m)M\}$ is an ascending chain of divisorial submodules of $M$. Thus there exists an integer $n \geq 1$ such that $(I_n)M = (I_m)M$ for each $m \geq n$. Hence $[(I_n)M :_R M] = [(I_m)M :_R M]$ and so $I_n = I_m$ for each $m \geq n$. Therefore $R$ is a Mori domain.
Conversely, let $R$ be a Mori ring and $\{N_m\}$ be an ascending chain of divisorial submodules of $M$. Thus $\{[N_m : R M]\}$ is an ascending chain of divisorial ideals of $R$. Then there exists an integer $n \geq 1$ such that $[N_n : R M] = [N_m : R M]$ for each $m \geq n$. Hence $[N_n : R M]M = [N_m : R M]M$ and so $N_n = N_m$. Therefore $M$ is a Mori module.

**Theorem 2.3.** Let $R$ be an integral domain and $M$ a faithful multiplication $R$-module. Then $M$ is a Mori module if and only if for every strictly descending chain of divisorial submodule $\{N_m\}$ of $M$, $\bigcap N_m = (0)$.

**Proof.** Let $M$ be a Mori module and $\{N_m\}$ is a strictly descending chain of divisorial submodule of $M$. Then, by Theorem 2.2, $R$ is a Mori domain and $\{[N_m : R M]\}$ is a strictly descending chain of divisorial ideals of $R$. So, by [Raill75, Theorem A.0], $\bigcap [N_m : R M] = (0)$. Therefore

$$\bigcap N_m = \bigcap ([N_m : R M])M = (0).$$

Conversely, let $\{N_m\}$ be a strictly descending chain of divisorial submodule of $M$ such that $\bigcap N_m = (0)$. Then $\{[N_m : R M]\}$ is a strictly descending chain of divisorial ideals of $R$ such that $\bigcap [N_m : R M] = (0)$. Hence, by [Raill75, Theorem A.0], $R$ is a Mori domain and therefore by Theorem 2.2, $M$ is a Mori module.

**Corollary 2.4.** Let $R$ be an integral domain and $M$ a faithful multiplication $R$-module. If $M$ is a Mori module, then every divisorial submodule of $M$ is contained in only a finite number of maximal divisorial submodules.

**Proof.** Let $M$ be a Mori module and $N$ a divisorial submodule of $M$. Then by Theorem 2.2, $R$ is a Mori domain and $[N : R M]$ is a divisorial submodule of $R$. So, by [BG87], $[N : R M]$ is contained in only a finite number of maximal divisorial ideals. Since $M$ is faithful multiplication module, $N$ is contained in only a finite number of maximal divisorial submodules of $M$.

Note that if $N$ is a divisorial submodule of $R$-module $M$, then $N_S$ is a divisorial submodule of $R_S$-module $M_S$ for each multiplicatively closed subset of $R$, because $N = N_vM$ and therefore $N_S = (N_vM)_S = (N_v)_S M_S$.

**Theorem 2.5.** Let $M$ be an Mori $R$-module. Then $M_S$ is a Mori $R_S$-module for each multiplicatively closed subset of $R$.

**Proof.** Let $\{N_m\}$ be an ascending chain of divisorial submodules of $M_S$. Then $\{N_m^c\}$ is an ascending chain of divisorial submodules of $M$. Thus there exits an integer $n \geq 1$ such that $N_n^c = N_m^c$ for each $m \geq n$. Therefore $N_n = N_n^c = N_m^c = N_m$ for each $m \geq n$. So $M_S$ is a Mori module.

**Definition 2.6.** A submodule $N$ of $M$ is said to be strong if $NN^{-1} = N$. $N$ is strongly divisorial if it is both strong and divisorial.

**Lemma 2.7.** Let $R$ be an integral domain an $M$ be a faithful multiplication $R$-module. Let $I$ be an ideal of $R$ and $N$ be a submodule of $M$. Then:
(1) \( N \) is strong (strong divisorial) submodule if and only if \([N :_R M]\) is strong (strong divisorial) ideal.

(2) \( I \) is strong (strong divisorial) ideal if and only if \( IM \) is strong (strong divisorial) submodule.

Proof. It is obvious by [Ali09-a, Lemma 1]. □

Proposition 2.8. Let \( R \) be an integral domain and \( M \) a faithful multiplication \( R \)-module. Let \( M \) be a Mori module and \( P \) be a prime submodule of \( M \) with \( \text{ht}(P) = 1 \). Then \( P \) is a divisorial submodule of \( M \). If \( \text{ht}(P) \geq 2 \), then either \( P^{-1} = R \) or \( P_{\nu} \) is a strong divisorial submodule of \( M \).

Proof. Let \( M \) be a Mori module and \( P \) be a prime submodule of \( M \) with \( \text{ht}(P) = 1 \). Then, by Theorem 2.2, \( R \) is a Mori domain and \([P :_R M]\) is a prime ideal of \( R \) such that \( \text{ht}([P :_R M]) = 1 \). Therefore, by [Querr71, Proposition 1], \([P :_R M]\) is a divisorial ideal of \( R \) and so \( N \) is a divisorial submodule of \( M \). If \( \text{ht}(P) \geq 2 \), then \( \text{ht}([P :_R M]) \geq 2 \). So, by [BG87], 
\[ [P :_R M]^{-1} = R \text{ or } [P :_R M]_{\nu} \] is a strong divisorial ideal of \( R \). Therefore, by [Ali09-a, Lemma 1], \( P^{-1} = R \) or \( P_{\nu} \) is a strong divisorial submodule of \( M \). □

Theorem 2.9. Let \( R \) be an integral domain and \( M \) a faithful multiplication \( R \)-module. Then \( M \) is a Mori module if and only if for each nonzero submodule \( N \) of \( M \), there is a finitely generated submodule \( L \subset N \) such that \( N^{-1} = L^{-1} \), equivalently, \( N_{\nu} = L_{\nu} \).

Proof. Let \( M \) be a Mori module and \( N \) be a nonzero submodule of \( M \). Then, by Theorem 2.2, \( R \) is a Mori domain and \([N :_R M]\) is a nonzero ideal of \( R \). Thus, by [Querr71, Theorem 1], there is a finitely generated ideal \( J \subset [N :_R M] := I \) such that \( J^{-1} = I^{-1} \). Hence there is a finitely generated submodule \( L := JM \subset IM = N \) such that \( N^{-1} = L^{-1} \) by [Ali09-a, Lemma 1].

Conversely, if for each nonzero submodule \( N \) of \( M \), there is a finitely generated submodule \( L \subset N \) such that \( N^{-1} = L^{-1} \), then for each nonzero ideal \([N :_R M]\) of \( R \), there is a finitely generated ideal \([L :_R M]\subset [N :_R M]\) such that \([N :_R M]^{-1} = [L :_R M]^{-1} \) by [Ali09-a, Lemma 1]. Thus, by [Querr71, Theorem 1], \( R \) is a Mori domain and so by Theorem 2.2, \( M \) is a Mori module. □

Corollary 2.10. Let \( R \) be an integral domain and \( M \) a faithful multiplication \( R \)-module. If \( M \) is a Mori module, then every divisorial submodule of \( M \) is a divisorial submodule of finite type.

3. \( \phi \)-Mori modules

In this section, we define the concept of \( \Phi \)-Mori module and give some results of this class of modules.
Definition 3.1. Let $R$ be a ring and $M \in \mathbb{H}$ be an $R$-module. A nonnil submodule $N$ of $M$ is said to be a $\Phi$-divisorial if $\Phi(N)$ is divisorial submodule of $\Phi(M)$. Also, $N$ is called a $\Phi$-divisorial of finite type of $M$ if $\Phi(N)$ is a divisorial submodule of finite type of $\Phi(M)$.

Definition 3.2. Let $R$ be a ring and $M \in \mathbb{H}$ be an $R$-module. Then $M$ is said to be a $\Phi$-Mori module if it satisfies the ascending chain condition on $\Phi$-divisorial submodules.

Lemma 3.3. Let $M \in \mathbb{H}$ be an $R$-module and $N,L$ be nonnil submodules of $M$. Then $N = L$ if and only if $\Phi(N) = \Phi(L)$.

Proof. It is clear that $N = L$ follows $\Phi(N) = \Phi(L)$. Conversely, since $\text{Nil}(M)$ is a divided prime submodule of $M$ and neither $N$ nor $L$ is contained in $\text{Nil}(M)$, both properly contain $\text{Nil}(M)$. Thus both contain $\text{Ker}(\Phi)$, by [MY, Proposition 2.1]. The result follows from standard module theory.

Proposition 3.4 ([MY, Proposition 2.2]). Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module with $M \in \mathbb{H}$. Then:

1. $\text{Nil}(\Phi(M)) = \Phi(\text{Nil}(M)) = \text{Nil}(\Phi(M))$.
2. $\text{Nil}(\Phi(M)) = \text{Nil}(M)$.
3. $\Phi(M) \in \mathbb{H}$.

Theorem 3.5. Let $M \in \mathbb{H}$. Then $M$ is a $\Phi$-Mori module if and only if $\Phi(M)$ is a Mori module.

Proof. Each submodule of $\Phi(M)$ is the image of a unique nonnil submodule of $M$ and $\Phi(N)$ is a submodule of $\Phi(M)$ for each nonnil submodule $N$ of $M$. Moreover, by definition, if $L = \Phi(N)$, then $L$ is a divisorial submodule of $\Phi(M)$ if and only if $N$ is a $\Phi$-divisorial submodule of $M$. Thus a chain of $\Phi$-divisorial submodules of $M$ stabilizes if and only if the corresponding chain of divisorial submodules of $\Phi(M)$ stabilizes. It follows that $M$ is a $\Phi$-Mori module if and only if $\Phi(M)$ is a Mori module.

It is worthwhile to note that if $R$ is a commutative ring and $M \in \mathbb{H}$ is an $R$-module, then $\frac{N}{\text{Nil}(M)}$ is a divisorial submodule of $\frac{M}{\text{Nil}(M)}$ if and only if $\frac{\Phi(N)}{\text{Nil}(\Phi(M))}$ is a divisorial submodule of $\frac{\Phi(M)}{\text{Nil}(\Phi(M))}$. For if $\frac{\Phi(N)}{\text{Nil}(\Phi(M))}$ is not divisorial, then $\frac{\Phi(N)}{\text{Nil}(\Phi(M))} \neq \frac{\Phi(N)_\nu}{\text{Nil}(\Phi(M))} \frac{\Phi(M)}{\text{Nil}(\Phi(M))}$. So $\Phi(N) \neq \Phi(N)_\nu \Phi(M) = \Phi(N_\nu M)$. Thus, by Lemma 3.3, $N \neq N_\nu M$. Therefore,

$$\frac{N}{\text{Nil}(M)} \neq \frac{N_\nu M}{\text{Nil}(M)} = \left(\frac{N}{\text{Nil}(M)}\right)_\nu \frac{M}{\text{Nil}(M)},$$

which is a contradiction.

Lemma 3.6. Let $M \in \mathbb{H}$. For each nonnil submodule $N$ of $M$, $N$ is $\Phi$-divisorial if and only if $\frac{N}{\text{Nil}(M)}$ is a divisorial submodule of $\frac{M}{\text{Nil}(M)}$. Moreover, $\Phi(N)$ is invertible if and only if $\frac{N}{\text{Nil}(M)}$ is invertible.
Proof. Let $N$ be a divisorial submodule of $M$. Then $\Phi(N)$ is divisorial and so $\Phi(N) = \Phi(N)\nu\Phi(M)$. Thus $\frac{\Phi(N)}{\text{Nil}(\Phi(M))} = \frac{\Phi(N)\nu}{\text{Nil}(\Phi(M))}\frac{\Phi(M)}{\text{Nil}(\Phi(M))}$. Therefore $\frac{\Phi(N)}{\text{Nil}(\Phi(M))}$ is a divisorial submodule of $\frac{\Phi(M)}{\text{Nil}(\Phi(M))}$. Thus $\frac{N}{\text{Nil}(M)}$ is a divisorial submodule of $\frac{M}{\text{Nil}(M)}$. Conversely, is same. □

Theorem 3.7. Let $M \in \mathbb{H}$. Then $M$ is a $\Phi$-Mori module if and only if $\frac{M}{\text{Nil}(M)}$ is a Mori module.

Proof. Suppose that $M$ is a $\Phi$-Mori module. Let $\{N_m\}$ be an ascending chain of divisorial submodules of $M$. Hence $\{\Phi(N_m)\}$ is an ascending chain of divisorial submodules of $\Phi(M)$, by Lemma 3.6. Thus there exists an integer $n \geq 1$ such that $\Phi(N_m) = \Phi(N_n)$ for each $m \geq n$ and so $N_m = N_n$ by Lemma 3.3. It follows that $\frac{N_n}{\text{Nil}(M)} = \frac{N_m}{\text{Nil}(M)}$ as well.

Conversely, suppose that $\frac{M}{\text{Nil}(M)}$ is a Mori module. Let $\{N_m\}$ be an ascending chain of non-nil divisorial submodules of $M$. Thus, by Lemma 3.6, $\{\frac{N_m}{\text{Nil}(M)}\}$ is an ascending chain of divisorial submodules of $\frac{M}{\text{Nil}(M)}$. Hence there exists an integer $n \geq 1$ such that $\frac{N_n}{\text{Nil}(M)} = \frac{N_m}{\text{Nil}(M)}$ for each $m \geq n$. As above, we have $N_n = N_m$ for each $m \geq n$. So $M$ is a $\Phi$-Mori module. □

Theorem 3.8. Let $R$ be a ring and $M$ be a finitely generated faithful multiplication $R$-module. The following statements are equivalent:

1. If $R \in \mathbb{H}$ is a $\phi$-Mori ring, then $M$ is a $\Phi$-Mori module.
2. If $M \in \mathbb{H}$ is a $\Phi$-Mori module, then $R$ is a $\phi$-Mori ring.

Proof. Since $\text{Nil}(R) \subseteq \text{Ann}(\frac{M}{\text{Nil}(M)}) = \text{Ann}(\frac{M}{\text{Nil}(M)})$, we have:

1. $\Rightarrow$ (2) Let $R \in \mathbb{H}$. Then, by [Yous, Proposition 3], $M \in \mathbb{H}$. If $R$ is a $\phi$-Mori ring, then by [BadaL06, Theorem 2.5], $\frac{R}{\text{Nil}(R)}$ is a Mori domain. So, by Theorem 2.2, $\frac{M}{\text{Nil}(M)}$ is a Mori module. Therefore, by Theorem 3.7, $M$ is a $\Phi$-Mori module.

2. $\Rightarrow$ (1) Let $M \in \mathbb{H}$. Then, by [Yous, Proposition 3], $R \in \mathbb{H}$. If $M$ is a $\Phi$-Mori module, then by Theorem 3.7, $\frac{M}{\text{Nil}(M)}$ is a Mori module. So, by Theorem 2.2, $\frac{R}{\text{Nil}(R)}$ is a Mori domain. Therefore, by [BadaL06, Theorem 2.5], $R$ is a $\phi$-Mori ring. □

Theorem 3.9 ([MY, Lemma 2.6]). Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module with $M \in \mathbb{H}$. Then $\frac{\Phi(M)}{\text{Nil}(\Phi(M))}$ is isomorphic to $\frac{\Phi(M)}{\text{Nil}(\Phi(M))}$ as $R$-module.

Corollary 3.10. Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module with $M \in \mathbb{H}$. Then $M$ is a $\Phi$-Mori module if and only if $\frac{\Phi(M)}{\text{Nil}(\Phi(M))}$ is a Mori module.
Lemma 3.11. Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module with $M \in \mathbb{H}$. Suppose that a nonnil submodule $N$ of $M$ is a divisorial submodule of $M$. Then $\Phi(N)$ is a divisorial submodule of $\Phi(M)$, i.e., $N$ is a $\Phi$-divisorial submodule of $M$.

Proof. We must show that $\Phi(N) = \Phi(N) \cdot \Phi(M)$. Since 
$$[\Phi(N) : R \Phi(M)] \subseteq [\Phi(N) : R \Phi(M)]_\nu,$$

and 
$$[\Phi(N) : R \Phi(M)] \Phi(M) \subseteq [\Phi(N) : R \Phi(M)]_\nu \Phi(M).$$

Hence 
$$\Phi(N) \subseteq \Phi(N)_\nu \Phi(M)$$

by [Ali09-a, Lemma 1]. Now, let $y \in \Phi(N)_\nu \Phi(M)$. Then $y = \sum a_im_i$ where $a_i \in \Phi(N)_\nu$ and $m_i = \Phi(m_i) \in \Phi(M)$. Since $\Phi(N)_\nu \subseteq R$, $a_i \in R$. If $x \in N^{-1}$ then $\Phi(x) \in \Phi(N)^{-1} = [\Phi(M) : R \Phi(N)]$. Therefore

$$y\Phi(x) = \left(\sum a_im_i\right)\Phi(x) = \left(\sum a_i\Phi(m_i)\right)\Phi(x) = \sum a_i\Phi(m_ix)$$

$$= \sum \Phi(a_im_ix) = \Phi\left(\sum a_im_ix\right).$$

Since $\Phi(N)_\nu \Phi(N)^{-1} \subseteq \Phi(M)$, $y\Phi(x) = \Phi(\sum a_im_ix) \in \Phi(M)$. Hence 
$$\left(\sum a_im_ix\right) \in M.$$ 

Since $N$ is a divisorial submodule and $x \in N^{-1}$ is arbitrary, 
$$\sum a_im_i \in N.$$ 

Thus $\Phi(\sum a_im_i) = \sum \Phi(a_im_i) = \sum a_i\Phi(m_i) \in \Phi(N)$. Therefore $y = \sum a_im_i \in \Phi(N)$ as well. □

Theorem 3.12. Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module with $M \in \mathbb{H}$. If $M$ is a $\Phi$-Mori module, then $M$ satisfies the A.C.C on nonnil divisorial submodules of $M$. In particular $M$ is a Mori module.

Proof. Let $N_m$ be an ascending chain of nonnil divisorial submodules of $M$. Hence, by Lemma 3.11, $\Phi(N_m)$ is an ascending chain of divisorial submodules of $\Phi(M)$. Since $\Phi(M)$ is a Mori module by Theorem 3.5, there exists an integer $n \geq 1$ such that $\Phi(N_n) = \Phi(N_m)$ for each $m \geq n$. Thus $N_n = N_m$ by Lemma 3.3. The "In particular" statement is now clear. □

Theorem 3.13. Let $M \in \mathbb{H}$ be a $\Phi$-Noetherian module. Then $M$ is a $\Phi$-Mori module.

Proof. It is clear by [Yous, Theorem 10]. □

Theorem 3.14. Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module with $M \in \mathbb{H}$. Let $M$ be a $\Phi$-Mori module and $N$ be a $\Phi$-divisorial submodule of $M$. Then $N$ contains a power of its radical.

Proof. Let $M$ be a $\Phi$-Mori module. Then, by Theorem 3.7, $\frac{M}{\text{Nil}(M)}$ is a Mori module and so $R$ is a Mori domain. Since $N$ is a $\Phi$-divisorial submodule of $M$, then $\frac{N}{\text{Nil}(M)}$ is a divisorial submodule of $\frac{M}{\text{Nil}(M)}$ by Lemma 3.6. Hence 
$$\left[\frac{N}{\text{Nil}(M)} : R \frac{M}{\text{Nil}(M)}\right]$$

is a divisorial ideal of $R$ and therefore contains a power of
its radical by [Raill75, Theorem 5]. In other words, there exists a positive integer \( n \) such that
\[
\left( \sqrt{\left[ \frac{N}{\text{Nil}(M)} : R \right] \frac{M}{\text{Nil}(M)}} \right)^n \subseteq \left[ \frac{N}{\text{Nil}(M)} : R \right] \frac{M}{\text{Nil}(M)}.
\]
Hence \( \sqrt{\frac{N}{\text{Nil}(M)}} \subseteq \frac{N}{\text{Nil}(M)} \). Since Nil\((M)\) is divided, \( N \) contains a power of its radical.

We will extend concepts of definition 2.6 to the module in \( H \).

**Definition 3.15.** Let \( M \in H \) and \( N \) be a nonnil submodule of \( M \). Then \( N \) is \( \Phi \)-strong if \( \Phi(N) \) is strong, i.e., \( \Phi(N)\Phi(N)^{-1} = \Phi(N) \). Also, \( N \) is strongly \( \Phi \)-divisorial if \( N \) is both \( \Phi \)-strong and \( \Phi \)-divisorial.

Obviously, \( N \) is \( \Phi \)-strong (or strongly \( \Phi \)-divisorial) if and only if \( \Phi(N) \) is strong (or strongly divisorial).

**Lemma 3.16.** Let \( M \in H \) be a \( \Phi \)-Mori module and \( N \) be a nonnil submodule of \( M \). Then the following hold:

1. \( N \) is a \( \Phi \)-strong submodule of \( M \) if and only if \( \frac{N}{\text{Nil}(M)} \) is a strong submodule of \( \frac{M}{\text{Nil}(M)} \).
2. \( N \) is strongly \( \Phi \)-divisorial if and only if \( \frac{N}{\text{Nil}(M)} \) is a strongly divisorial submodule of \( \frac{M}{\text{Nil}(M)} \).

**Proof.**

1. \( N \) is a \( \Phi \)-strong if and only if \( \Phi(N) \) is strong if and only if \( \frac{\Phi(N)}{\text{Nil}(M)} \frac{\Phi(N)^{-1}}{\text{Nil}(M)} = \Phi(N) \) if and only if \( \frac{\Phi(N)}{\text{Nil}(M)} \) is strong if and only if \( \frac{N}{\text{Nil}(M)} \) is strong.

2. \( N \) is strongly \( \Phi \)-divisorial if and only if \( N \) is both \( \Phi \)-strong and \( \Phi \)-divisorial if and only if \( \frac{\Phi(N)}{\text{Nil}(M)} \) is both strong and divisorial if and only if \( \frac{N}{\text{Nil}(M)} \) is a strongly divisorial.

Set \( P := (\text{Nil}(M) : R M) \). Then \( P \) is a prime ideal of \( R \) and we have
\[
\left( \frac{M}{\text{Nil}(M)} \right)_P = \frac{M_P}{\text{Nil}(M_P)},
\]
[MY].

**Theorem 3.17.** Let \( M \in H \) be a \( \Phi \)-Mori module. Then \( M_P \) is a \( \Phi \)-Mori module.

**Proof.** Let \( M \) be a \( \Phi \)-Mori module. Then, by Theorem 3.7, \( \frac{M}{\text{Nil}(M)} \) is a Mori module. Hence \( \left( \frac{M}{\text{Nil}(M)} \right)_P = \frac{M_P}{\text{Nil}(M_P)} \) is a Mori module by Theorem 2.5. Therefore, by Theorem 3.7, \( M_P \) is a \( \Phi \)-Mori module.
Theorem 3.18. Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module with $M \in \mathcal{H}$. Let $M$ be a $\Phi$-Mori module and $P$ be a nonnil prime submodule of $M$ minimal over a nonnil principal submodule $N$ of $M$. If $P$ is finitely generated, then $\text{ht}(P) = 1$.

Proof. Let $M$ be a $\Phi$-Mori module. Then, by Theorem 3.7, $\frac{M}{\text{nil}(M)}$ is a Mori module and so $R$ is a Mori domain. Also, by [MY, Theorem 2.8 and Corollary 2.9], we have $\frac{P}{\text{nil}(M)}$ is a minimal finitely generated prime submodule of $\frac{M}{\text{nil}(M)}$ over the principal submodule $\frac{N}{\text{nil}(M)}$ of $\frac{M}{\text{nil}(M)}$. Thus $[\frac{P}{\text{nil}(M)} : R \frac{M}{\text{nil}(M)}]$ is a minimal finitely generated prime ideal of $R$ over the principal ideal $[\frac{N}{\text{nil}(M)} : R \frac{M}{\text{nil}(M)}]$ of $R$. Then, by [BAD87, Theorem 3.4], $\text{ht}(\frac{P}{\text{nil}(M)}) = 1$. Therefore $\text{ht}(\frac{P}{\text{nil}(M)}) = 1$ and so $\text{ht}(P) = 1$. □

Proposition 3.19. Let $R$ be an integral domain and $M$ a faithful multiplication $R$-module with $M \in \mathcal{H}$. Let $M$ be a $\Phi$-Mori $R$-module and $P$ be a nonnil prime submodule of $M$ such that $\text{ht}(P) = 1$. Then $P$ is a $\Phi$-divisorial submodule of $M$. If $\text{ht}(P) \geq 2$, then either $P^{-1} = R$ or $P_\nu$ is a strong divisorial submodule of $M$.

Proof. Let $M$ be a $\Phi$-Mori $R$-module and $P$ be a nonnil prime submodule of $M$. Then, by Theorem 3.7, $\frac{M}{\text{nil}(M)}$ is a Mori module and $\frac{P}{\text{nil}(M)}$ is a prime submodule of $\frac{M}{\text{nil}(M)}$ with $\text{ht}(\frac{P}{\text{nil}(M)}) = 1$. Therefore, by Proposition 2.8, $\frac{P}{\text{nil}(M)}$ is a divisorial submodule of $\frac{M}{\text{nil}(M)}$ and so by Theorem 3.6, $P$ is a $\Phi$-divisorial submodule of $M$. Now, let $\text{ht}(P) \geq 2$. Then $\text{ht}(\frac{P}{\text{nil}(M)}) \geq 2$ and so by Proposition 2.8, $P^{-1} = R$ or $(\frac{P}{\text{nil}(M)})_\nu$ is a strong divisorial submodule of $M$. Therefore, $P^{-1} = R$ or $P_\nu$ is a strong divisorial submodule of $M$. □

Theorem 3.20. Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module with $M \in \mathcal{H}$. Then $M$ is a $\Phi$-Mori module if and only if for each nonnil submodule $N$ of $M$, there exists a nonnil finitely generated submodule $L \subset N$ such that $\Phi(N)^{-1} = \Phi(L)^{-1}$, equivalently $\Phi(N)_\nu = \Phi(L)_\nu$.

Proof. Suppose that $M$ is a $\Phi$-Mori module and $N$ is a nonnil submodule of $M$. Since by Theorem 3.7, $\frac{M}{\text{nil}(M)}$ is a Mori module and $F := \frac{N}{\text{nil}(M)}$ is a nonzero submodule of $\frac{M}{\text{nil}(M)}$, there exists a finitely generated submodule $L \subset F$ such that $F^{-1} = L^{-1}$. Since $L = \frac{K}{\text{nil}(M)}$ for some nonnil finitely generated submodule $K$ of $M$ by [MY, Theorem 2.8], and $\mathcal{T}(\frac{M}{\text{nil}(M)}) = \mathcal{T}(\frac{\Phi(M)}{\text{nil}(\Phi(M))})$, we conclude that $\Phi(N)^{-1} = \Phi(L)^{-1}$.

Conversely, suppose that for each nonnil submodule $N$ of $M$, there exists a nonnil finitely generated submodule $L \subset N$ such that $\Phi(N)^{-1} = \Phi(L)^{-1}$. Then for each nonzero submodule $F := \frac{N}{\text{nil}(M)}$ of $\frac{M}{\text{nil}(M)}$, there exists a finitely generated submodule $K \subset F$ such that $F^{-1} = K^{-1}$. Hence $\frac{M}{\text{nil}(M)}$ is a...
Mori module by Theorem 2.9. Therefore, by Theorem 3.7, \( M \) is a \( \Phi \)-Mori module.

**Corollary 3.21.** Let \( R \) be a ring and \( M \) a finitely generated faithful multiplication \( R \)-module with \( M \in \mathbb{H} \). If \( M \) is a \( \Phi \)-Mori module, then every \( \Phi \)-divisorial submodule of \( M \) is a \( \Phi \)-divisorial submodule of finite type.

**Proof.** Let \( M \) be a \( \Phi \)-Mori module and \( N \) be a \( \Phi \)-divisorial submodule of \( M \). Then, by Theorem 3.5, \( \Phi(M) \) is a Mori module and \( \Phi(N) \) is a divisorial submodule of \( \Phi(M) \). Thus, by Theorem 2.9, there is a finitely generated submodule \( \Phi(L) \subseteq \Phi(N) \) such that \( \Phi(N) = \Phi(L)_\nu \). Since \( \Phi(N) \) is divisorial, \( \Phi(N) = \Phi(L)_\nu \). Therefore \( N \) is a \( \Phi \)-divisorial submodule of finite type.

**Theorem 3.22.** Let \( R \) be a ring and \( M \) a finitely generated faithful multiplication \( R \)-module with \( M \in \mathbb{H} \). Then the following statements are equivalent:

1. \( M \) is a \( \Phi \)-Mori module.
2. \( R \) is a \( \phi \)-Mori ring.
3. \( \Phi(M) \) is a Mori module.
4. \( \frac{M}{\text{Nil}(M)} \) is a Mori module.
5. \( \frac{\Phi(M)}{\text{Nil}(\Phi(M))} \) is a Mori module.
6. For each nonnil submodule \( N \) of \( M \), there exists a nonnil finitely generated submodule \( L \subset N \) such that \( \Phi(N)^{-1} = \Phi(L)^{-1} \).
7. For each nonnil submodule \( N \) of \( M \), there exists a nonnil finitely generated submodule \( L \subset N \) such that \( \Phi(N)_\nu = \Phi(L)_\nu \).

**Acknowledgments.** We thank the referees for their careful reading of the whole manuscript and their helpful suggestions.

**References**


ON Φ-MORI MODULES


This paper is available via http://nyjm.albany.edu/j/2015/21-57.html.