On the uniqueness of algebraic curves passing through $n$-independent nodes

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Abstract. A set of nodes in the plane is called $n$-independent if for arbitrary data at those nodes, there is a (not necessarily unique) polynomial of degree at most $n$ that matches the given information. We proved in a previous paper (Hakopian–Toroyan, 2015) that the minimal number of $n$-independent nodes determining uniquely the curve of degree $k \leq n$ passing through them equals to $D := (1/2)(k - 1)(2n + 4 - k) + 2$. In this paper we bring a characterization of the case when at least two curves of degree $k$ pass through the nodes of an $n$-independent node set of cardinality $D - 1$. Namely, we prove that the latter set has a very special construction: All its nodes but one belong to a (maximal) curve of degree $k - 1$. We show that this result readily yields the above cited one. At the end, an important application to the Gasca–Maeztu conjecture is presented.

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1. Introduction

Denote the space of all bivariate polynomials of total degree $\leq n$ by $\Pi_n$:

$$\Pi_n = \left\{ \sum_{i+j \leq n} a_{ij} x^i y^j \right\}.$$
We have that
\[ N := N_n := \dim \Pi_n = (1/2)(n + 1)(n + 2). \]

Consider a set of \( s \) distinct nodes
\[ X_s = \{(x_1, y_1), (x_2, y_2), \ldots, (x_s, y_s)\}. \]
The problem of finding a polynomial \( p \in \Pi_n \) which satisfies the conditions
\[ p(x_i, y_i) = c_i, \quad i = 1, \ldots, s, \]
is called interpolation problem.

A polynomial \( p \in \Pi_n \) is called an \( n \)-fundamental polynomial for a node \( A = (x_k, y_k) \in X_s \) if
\[ p(x_i, y_i) = \delta_{ik}, \quad i = 1, \ldots, s, \]
where \( \delta \) is the Kronecker symbol. We denote this fundamental polynomial by \( p^*_{A, X_s} \). Sometimes we call fundamental also a polynomial that vanishes at all nodes of \( X_s \) but one, since it is a nonzero constant times a fundamental polynomial.

Next, let us consider an important concept of \( n \)-independence (see [7], [11]).

**Definition 1.1.** A set of nodes \( X \) is called \( n \)-independent if all its nodes have \( n \)-fundamental polynomials. Otherwise, if a node has no \( n \)-fundamental polynomial, \( X \) is called \( n \)-dependent.

Fundamental polynomials are linearly independent. Therefore a necessary condition of \( n \)-independence of \( X_s \) is \( s \leq N \).

Suppose a node set \( X_s \) is \( n \)-independent. Then by the Lagrange formula we obtain a polynomial \( p \in \Pi_n \) satisfying the interpolation conditions (1.1):
\[ p = \sum_{i=1}^{s} c_i p^*_{i, X_s}. \]
In view of this, we get readily that the node set \( X_s \) is \( n \)-independent if and only if the interpolating problem (1.1) is solvable, meaning that for any data \((c_1, \ldots, c_s)\) there is a polynomial \( p \in \Pi_n \) (not necessarily unique) satisfying the interpolation conditions (1.1).

**Definition 1.2.** The interpolation problem with a set of nodes \( X_s \) and \( \Pi_n \) is called \( n \)-poised if for any data \((c_1, \ldots, c_s)\) there is a unique polynomial \( p \in \Pi_n \) satisfying the interpolation conditions (1.1).

The conditions (1.1) give a system of \( s \) linear equations with \( N \) unknowns (the coefficients of the polynomial \( p \)). The poisedness means that this system has a unique solution for arbitrary right side values. Therefore a necessary condition of poisedness is \( s = N \). If this condition holds then we obtain from the linear system:
Proposition 1.3. A set of nodes $\mathcal{X}_N$ is $n$-poised if and only if

$$p \in \Pi_n \text{ and } p|_{\mathcal{X}_N} = 0 \implies p = 0.$$ 

Thus, geometrically, the node set $\mathcal{X}_N$ is $n$-poised if and only if there is no curve of degree $n$ passing through all its nodes.

It is worth mentioning:

Proposition 1.4. For any set $\mathcal{X}_{N-1}$, i.e., set of cardinality $N - 1$, there is a curve of degree $n$ passing through all its nodes.

Indeed, the existence of the curve reduces to a system of $N - 1$ linear homogeneous equations with $N$ unknowns – the coefficients of the polynomial of degree $n$.

It follows from Proposition 1.3 also that a node set of cardinality $N$ is $n$-poised if and only if it is $n$-independent.

Next, let us describe the main result of this paper. Suppose we have an $n$-poised set $\mathcal{X}_N$. From what was said above we can conclude readily that through any $N - 1$ nodes of $\mathcal{X}$ there pass a unique curve of degree $n$. Indeed, this curve is given by the fundamental polynomial of the missing node. Next, through any $N - 2$ nodes of $\mathcal{X}$ there pass more than one curve of degree $n$, for example the curves given by the fundamental polynomials of two missing nodes. Thus we have that the minimal number of $n$-independent nodes determining uniquely the curve of degree $n$ equals to $N - 1$.

In [14] we considered this problem in the case of arbitrary degree $k$, $k \leq n - 1$. We proved that the minimal number of $n$-independent nodes determining uniquely the curve of degree $k \leq n - 1$ equals

$$D := (1/2)(k - 1)(2n + 4 - k) + 2.$$ 

Or, more precisely, for any $n$-independent set of cardinality $D$ there is at most one curve of degree $k \leq n - 1$ passing through its nodes, while there are $n$-independent node sets of cardinality $D - 1$ through which pass at least two such curves. Let us mention that the case $k = n - 1$ of the above described problem is considered in [2].

In this paper we bring a characterization of the sets of cardinality $D - 1$ through which pass at least two curves of degree $k$. Namely, we prove that in this case all the nodes of $\mathcal{X}$ but one belong to a curve of degree $k - 1$. Moreover, this latter curve is a maximal curve meaning that it passes through maximal possible number of $n$-independent nodes (see Section 3).

As we will see in Section 5, this result readily yields the above mentioned result of [14].

At the end let us bring a well-known Berzolari–Radon construction of $n$-poised sets (see [3], [15]).

Definition 1.5. A set of $N = 1 + \cdots + (n + 1)$ nodes is called Berzolari–Radon set for degree $n$, or briefly $BR_n$ set, if there exist lines $\ell_1, \ell_2, \ldots, \ell_{n+1}$, such that the sets $\ell_1, \ell_2 \setminus \ell_1, \ell_3 \setminus (\ell_1 \cup \ell_2), \ldots, \ell_{n+1} \setminus (\ell_1 \cup \cdots \cup \ell_n)$ contain exactly $(n + 1), n, n - 1, \ldots, 1$ nodes, respectively.
2. Some properties of \(n\)-independent nodes

Let us start with the following simple (see, e.g., [12], Lemma 2.3; [11] Lemma 2.2) lemma.

**Lemma 2.1.** Suppose that a node set \(X\) is \(n\)-independent and a node \(A \notin X\) has \(n\)-fundamental polynomial with respect to the set \(X \cup \{A\}\). Then the latter node set is \(n\)-independent, too.

Indeed, one can get readily the fundamental polynomial of any node \(B \in X\) with respect to the set \(Y := X \cup \{A\}\) by using a linear combination of the given fundamental polynomial \(p_A^*\) and the fundamental polynomial of \(B\) with respect to the set \(X\).

Evidently, any subset of \(n\)-poised set is \(n\)-independent. According to the next lemma any \(n\)-independent set is a subset of some \(n\)-poised set:

**Lemma 2.2** (e.g., [9], Lemma 2.1). Any \(n\)-independent set \(X\) with \(#X < N\) can be enlarged to an \(n\)-poised set.

**Proof.** It suffices to show that there is a node \(A\) such that the set \(X \cup \{A\}\) is \(n\)-independent. By Proposition 1.4 there is a nonzero polynomial \(q \in \Pi_n\) such that \(q|_X = 0\). Now, in view of Lemma 2.1, we may choose a desirable node \(A\) by requiring only that \(q(A) \neq 0\). Indeed, then \(q\) is a fundamental polynomial of \(A\) with respect to the set \(X \cup \{A\}\).

Denote the linear space of polynomials of total degree at most \(n\) vanishing on \(X\) by
\[
P_{n,X} = \{ p \in \Pi_n : p|_X = 0 \}.
\]
The following is well-known.

**Proposition 2.3** (e.g., [9], [11]). For any node set \(X\) we have that
\[
\dim P_{n,X} \geq N - \#X.
\]
Moreover, equality takes place here if and only if the set \(X\) is \(n\)-independent.

From Lemma 2.1 one gets readily:

**Corollary 2.4** (e.g., [12], Corollary 2.4). Let \(Y\) be a maximal \(n\)-independent subset of \(X\), i.e., \(Y \subset X\) is \(n\)-independent and \(Y \cup \{A\}\) is \(n\)-dependent for any \(A \in X \setminus Y\). Then we have that
\[
P_{n,Y} = P_{n,X}.
\]

**Proof.** We have that \(P_{n,X} \subset P_{n,Y}\), since \(Y \subset X\). Now, suppose that \(p \in \Pi_n, p|_Y = 0\) and \(A\) is any node of \(X\). Then \(Y \cup \{A\}\) is dependent and therefore, in view of Lemma 2.1, \(p(A) = 0\).

From (2.1) and Proposition 2.3 (part “moreover”) we have that
\[
\dim P_{n,X} = N - \#Y,
\]
where \( Y \) is any maximal \( n \)-independent subset of \( X \). Thus, all the maximal \( n \)-independent subsets of \( X \) have the same cardinality, which is denoted by \( \mathcal{H}_n(X) \) — the Hilbert \( n \)-function of \( X \). Hence, according to (2.2), we have that

\[
\dim \mathcal{P}_{n,X} = N - \mathcal{H}_n(X).
\]

3. Maximal curves

An algebraic curve in the plane is the zero set of some bivariate polynomial of degree at least 1. We use the same letter, say \( p \), to denote the polynomial \( p \in \Pi_k \setminus \Pi_{k-1} \) and the corresponding curve \( p \) of degree \( k \) defined by equation \( p(x, y) = 0 \).

According to the following well-known statement there are no more than \( n + 1 \) \( n \)-independent points in any line:

**Proposition 3.1.** Assume that \( \ell \) is a line and \( \chi_{n+1} \) is any subset of \( \ell \) containing \( n + 1 \) points. Then we have that

\[
p \in \Pi_n \quad \text{and} \quad p|_{\chi_{n+1}} = 0 \implies p = \ell r,
\]

where \( r \in \Pi_{n-1} \).

Denote

\[
d := d(n, k) := N_n - N_{n-k} = (1/2)k(2n + 3 - k).
\]

The following is a generalization of Proposition 3.1.

**Proposition 3.2** ([16], Prop. 3.1). Let \( q \) be an algebraic curve of degree \( k \leq n \) without multiple components. Then the following hold:

(i) Any subset of \( q \) containing more than \( d(n, k) \) nodes is \( n \)-dependent.

(ii) Any subset \( \chi_d \) of \( q \) containing exactly \( d = d(n,k) \) nodes is \( n \)-independent if and only if the following condition holds:

\[
p \in \Pi_n \quad \text{and} \quad p|_{\chi_d} = 0 \implies p = qr,
\]

where \( r \in \Pi_{n-k} \).

Suppose that \( \chi \) is an \( n \)-poised set of nodes and \( q \) is an algebraic curve of degree \( k \leq n \). Then of course any subset of \( \chi \) is \( n \)-independent too. Therefore, according to Proposition 3.2(i), at most \( d(n,k) \) nodes of \( \chi \) can lie in the curve \( q \). Let us mention that a special case of this when \( q \) is a set of \( k \) lines is proved in [6].

This motivates the following definition.

**Definition 3.3** ([16], Def. 3.1). Given an \( n \)-independent set of nodes \( \chi_s \), with \( s \geq d(n,k) \). A curve of degree \( k \leq n \) passing through \( d(n,k) \) points of \( \chi_s \), is called maximal.
Note that the maximal line, as a line passing through \( n + 1 \) nodes, is defined in [4]. Let us mention that \( q = \ell_1 \cdots \ell_k \) is a maximal curve of degree \( k \) of the node set \( BR_n \) (see Def. 1.5), where \( k = 1, \ldots, n \).

We say that a node \( A \in X \) uses a polynomial \( q \in \Pi_k \) if the latter divides the fundamental polynomial \( p = p_A^\star \), i.e., \( p = qr \), for some \( r \in \Pi_{n-k} \).

Next, we bring a characterization of maximal curves:

**Proposition 3.4** ([16], Prop. 3.3). Let a node set \( X \) be \( n \)-poised. Then a polynomial \( \mu \) of degree \( k, k \leq n \), is a maximal curve if and only if it is used by any node in \( X \setminus \mu \).

Note that one side of this statement follows from Proposition 3.2(ii). In the case of degree one this was proved in [4]. For other properties of maximal curves we refer reader to [16], [13].

**Proposition 3.5.** Assume that \( \sigma \) is an algebraic curve of degree \( k \), without multiple components, and \( X_s \subset \sigma \) is any \( n \)-independent node set of cardinality \( s, s < d(n,k) \). Then the set \( X_s \) can be extended to a maximal \( n \)-independent set \( X_d \subset \sigma \) of cardinality \( d = d(n,k) \).

**Proof.** It suffices to show that there is a point \( A \in \sigma \setminus X_s \) such that the set \( X_{s+1} := X_s \cup \{A\} \) is \( n \)-independent. Assume to the contrary that there is no such point, i.e., the set \( X_{s+1} := X_s \cup \{A\} \) is \( n \)-dependent for any \( A \in \sigma \). Then, in view of Lemma 2.1, \( A \) has no fundamental polynomial with respect to the set \( X_s \). In other words we have

\[
p \in \Pi_n \text{ and } p\big|_{X_s} = 0 \implies p(A) = 0 \text{ for any } A \in \sigma.
\]

From here we obtain that

\[
\mathcal{P}_{n,X_s} \subset \mathcal{P}_{n,\sigma} := \{q\sigma : q \in \Pi_{n-k}\}.
\]

Now, in view of Proposition 2.3, we get from here

\[
N - s = \dim \mathcal{P}_{n,X_s} \leq \dim \mathcal{P}_{n,\sigma} = N_{n-k}.
\]

Therefore \( s \geq d(n,k) \), which contradicts the hypothesis.

Let us mention that, as it follows from the above proof, the condition (3.1) does not hold if \( d < d(n,k) \).

The following lemma follows readily from the fact that the Vandermonde determinant, i.e., the main determinant of the linear system described just after Definition 1.2, is a continuous function of the nodes of \( X_N \).

**Lemma 3.6** (e.g., [8], Remark 1.14). Suppose that \( X_N = \{(x_i,y_i)\}_{i=1}^N \) is an \( n \)-poised set. Then there is a positive number \( \epsilon \) such that any set

\[
X'_N = \{(x'_i,y'_i)\}_{i=1}^N,
\]

with the property that the distance between \( (x'_i,y'_i) \) and \( (x_i,y_i) \) is less than \( \epsilon \), for each \( i \), is \( n \)-poised too.

From here, in view of Lemma 2.2 we get readily:
Corollary 3.7. Suppose that $X_s = \{(x_i, y_i)\}_{i=1}^s$ is an $n$-independent set. Then there is a positive number $\epsilon$ such that any set $X'_s = \{(x'_i, y'_i)\}_{i=1}^s$, with the property that the distance between $(x'_i, y'_i)$ and $(x_i, y_i)$ is less than $\epsilon$, for each $i$, is $n$-independent too.

Finally, let us bring a well-known lemma:

Lemma 3.8. Suppose that two different curves of degree at most $k$ pass through all the nodes of $X$. Then for any node $A \notin X$ there is a curve of degree at most $k$ passing through $A$ and all the nodes of $X$.

Indeed, if the given curves are $\sigma_1$ and $\sigma_2$ then the desired curve can be found easily in the form of linear combination $c_1\sigma_1 + c_2\sigma_2$.

4. Main result

In a previous paper we determined the minimal number of $n$-independent nodes that uniquely determine the curve of degree $k$, $k \leq n$, passing through them:

Theorem 4.1 ([14], Thm. 1). Assume that $X$ is an $n$-independent set of $d(n, k-1) + 2$ nodes lying in a curve of degree $k$ with $k \leq n$. Then the curve is determined uniquely by these nodes. Moreover, there is an $n$-independent set of $d(n, k-1) + 1$ nodes such that more than one curves of degree $k$ pass through all its nodes.

Let us mention that this result is obvious in the case $k = n$, while in the case $k = n-1$ it was established in [2].

In this section we characterize the case when more than one curve of degree $k$, $k \leq n-1$, passes through the nodes of an $n$-independent set $X$ of cardinality $d(n, k-1) + 1$.

As we will see later in Section 5 this result yields readily Theorem 4.1.

Theorem 4.2. Assume that $X$ is an $n$-independent set of $d(n, k-1) + 1$ nodes with $k \leq n-1$. Then two different curves of degree $k$ pass through all the nodes of $X$ if and only if all the nodes of $X$ but one lie in a maximal curve of degree $k-1$.

Proof. Let us start with the inverse implication. Assume that $d(n, k-1)$ nodes of $X$ are located in a curve $\mu$ of degree $k-1$. Therefore the curve $\mu$ is maximal and the remaining node of $X$, which we denote by $A$, is outside of it: $A \notin \mu$.

Now assume that $\ell_1$ and $\ell_2$ are two different lines passing through $A$. Then it is easily seen that $\ell_1\mu$ and $\ell_2\mu$ are two different curves of degree $k$ passing through all the nodes of $X$.

Now let us prove the direct implication. Assume that there are two curves of degree $k$: $\sigma_1$ and $\sigma_2$ that pass through all the nodes of the $n$-independent set $X$ with $\#X = d(n, k-1) + 1$. Let us start by choosing a node $B \notin X$ such that the following three conditions are satisfied:
(i) The set \( \mathcal{X} \cup \{ B \} \) is \( n \)-independent.

(ii) \( B \) does not lie in any line passing through two nodes of \( \mathcal{X} \).

(iii) \( B \) does not lie in the curves \( \sigma_1 \) and \( \sigma_2 \).

Let us verify that one can find such a node. Indeed, in view of Lemma 2.2, we can start by choosing a node \( B' \) satisfying the condition (i).

Then, according to Corollary 3.7, for some positive \( \epsilon \) all the nodes in \( \epsilon \)-neighborhood of \( B' \) satisfy the condition (i). Finally, from this neighborhood we can choose a node \( B \) satisfying the conditions (ii) and (iii), too.

Next, in view of Lemma 3.8, there is a curve \( \sigma \) of degree at most \( k \) passing through all the nodes of \( \mathcal{X}' := \mathcal{X} \cup \{ B \} \). According to the condition (iii) \( \sigma \) is different from \( \sigma_1 \) and \( \sigma_2 \).

Then notice that the curve \( \sigma \) passes through more than \( d(n,k − 1) \) nodes and therefore its degree equals to \( k \) and it has no multiple component.

Now, by using Proposition 3.5, let us extend the set \( \mathcal{X}' \) till a maximal \( n \)-independent set \( \mathcal{X}'' \subset \sigma \). Notice that, since \( \# \mathcal{X}'' = d(n,k) \), we need to add

\[
\sum_{j=1}^{n-k} \mathcal{C}_j \quad \mathcal{X}'' := \mathcal{X} \cup \{ B \} \cup \{ \mathcal{C}_j \}_{i=1}^{n-k}.
\]

Thus the curve \( \sigma \) becomes maximal with respect to this set.

Then let us consider \( n-k-1 \) lines \( \ell_1, \ell_2, \ldots, \ell_{n-k-1} \) passing through the nodes \( C_1,C_2,\ldots,C_{n-k-1} \), respectively. We require that each line passes through only one of the mentioned nodes and therefore the lines are distinct. We require also that none of these lines is a component (factor) of \( \sigma \). Finally let us denote by \( \tilde{\ell} \) the line passing through the nodes \( B \) and \( C_{n-k} \).

Now notice that the following polynomial

\[
\sigma_1 \tilde{\ell} \ell_1 \ell_2 \ldots \ell_{n-k-1}
\]

of degree \( n \) vanishes at all the \( d(n,k) \) nodes of \( \mathcal{X}'' \subset \sigma \). Consequently, according to Proposition 3.2, \( \sigma \) divides this polynomial:

\[
\sigma_1 \tilde{\ell} \ell_1 \ell_2 \ldots \ell_{n-k-1} = \sigma q, \quad q \in \Pi_{n-k}.
\]

The distinct lines \( \ell_1, \ell_2, \ldots, \ell_{n-k-1} \) do not divide the polynomial \( \sigma \in \Pi_k \), therefore all they have to divide \( q \in \Pi_{n-k} \). Thus \( q = \ell_1 \ldots \ell_{n-k-1} \ell' \), where \( \ell' \in \Pi_1 \). Therefore, we get from (4.1):

\[
\sigma_1 \tilde{\ell} = \sigma \ell'.
\]

If the lines \( \tilde{\ell}, \ell' \) coincide then the curves \( \sigma_1, \sigma \) coincide, which is impossible. Therefore the line \( \tilde{\ell} \) has to divide \( \sigma \in \Pi_k \):

\[
\sigma = \tilde{\ell} r, \quad r \in \Pi_{k-1}.
\]

Now, we are going to derive from this relation that the curve \( r \) passes through all the nodes of the set \( \mathcal{X} \) but one. Indeed, \( \sigma \) passes through all the nodes of \( \mathcal{X} \). Therefore these nodes are either in the curve \( r \) or in the line \( \tilde{\ell} \). But the latter line passes through \( B \), and according to the condition (ii), it passes through at most one node of \( \mathcal{X} \). Thus \( r \) passes through at least \( d(n,k-1) \)
nodes of $\mathcal{X}$. Since $r$ is a curve of degree $k - 1$ we conclude that $r$ is a maximal curve and passes through exactly $d(n, k - 1)$ nodes of $\mathcal{X}$. \hfill \Box$

It is worth mentioning that for any $n$-independent node set $\mathcal{X}$ of cardinality $d(n, k - 1) + 1$, where $k \leq n - 1$, we have that
\[
\dim \mathcal{P}_{k, \mathcal{X}} \leq 2,
\]
where an equality takes place if only if all the nodes of $\mathcal{X}$ but one lie in a maximal curve of degree $k - 1$.

Indeed, if
\[
\dim \mathcal{P}_{k, \mathcal{X}} \geq 2
\]
then according to Theorem 4.2 we have that all the nodes of $\mathcal{X}$ but one lie in a maximal curve $\mu$ of degree $k - 1$. Now, according to Proposition 3.2, we have that
\[
\mathcal{P}_{k, \mathcal{X}} = \{\alpha \mu | \alpha \in \Pi_1, \alpha(A) = 0\},
\]
where $A \in \mathcal{X}$ is the node outside of $\mu$. Therefore we get readily
\[
\dim \mathcal{P}_{k, \mathcal{X}} = \dim \{\alpha | \alpha \in \Pi_1, \alpha(A) = 0\} = 2.
\]

5. A corollary

Here we verify that our main result yields Theorem 4.1, which in view of Theorem 4.2, states that for any $n$-independent set $\mathcal{X}$ of cardinality
\[
d(n, k - 1) + 2
\]
there is at most one curve of degree $k$, $k \leq n$, passing through all its nodes.

**Proof of Theorem 4.1.** Note that the case $k = n$ is evident, since
\[
d(n, n - 1) + 2 = N - 1.
\]
Now assume that $k \leq n - 1$. Choose a node $A \in \mathcal{X}$ and consider the set $\mathcal{Y} := \mathcal{X} \setminus \{A\}$. If there is at most one curve of degree which passes through all the nodes of $\mathcal{Y}$ then the same is true also for the set $\mathcal{X}$ and we are done. Thus assume that there are at least two curves of degree $k$ which pass through all the nodes of the set $\mathcal{Y}$. Then, according to Theorem 4.2, all the nodes of $\mathcal{Y}$ but one, denoted by $B$, lie in a maximal curve $\mu$ of degree $k - 1$. Therefore, all the nodes of $\mathcal{X}$ but $A$ and $B$ lie in the curve $\mu$. Now, in view of Proposition 3.2, any curve of degree $k$ passing through all the nodes of $\mathcal{X}$ has the following form
\[
p = \ell \mu,
\]
where $\ell \in \Pi_1$. Finally notice that the line $\ell$ passes through the nodes $A$ and $B$ and therefore is determined in a unique way. Hence $p$ is determined uniquely, too. \hfill \Box
6. An application to the Gasca–Maeztu conjecture

Let us recall that a node \( A \in X \) uses a line \( \ell \) means that \( \ell \) is a factor of the fundamental polynomial \( p = p_A^* \), i.e., \( p = \ell r \), for some \( r \in \Pi_{n-1} \).

A \( GC_n \)-set in plane is an \( n \)-poised set of nodes where the fundamental polynomial of each node is a product of \( n \) linear factors. Note that this always takes place in the univariate case.

The Gasca–Maeztu conjecture states that any \( GC_n \)-set possesses a subset of \( n + 1 \) collinear nodes.

It was proved in [5] that any line passing through exactly 2 nodes of a \( GC_n \)-set \( X \) can be used at most by one node from \( X \), provided that the Gasca–Maeztu conjecture is true for all degrees not exceeding \( n \).

Recently, it was announced in [1], that this result holds for any poised set \( X \), without other restrictions. By the way it follows readily also from Theorem 4.2.

Below we consider the case of lines passing through exactly 3 nodes.

**Corollary 6.1.** Let \( X \) be an \( n \)-poised set of nodes and \( \ell \) be a used line which passes through exactly 3 nodes. Then \( \ell \) is used either by exactly one or by exactly three nodes from \( X \). Moreover, if it is used by three nodes, then they are noncollinear.

**Proof.** Assume that \( \ell \cap X = \{A_1, A_2, A_3\} \). Assume also that there are two nodes \( B, C \in X \) using the line \( \ell \):

\[ p_B^* = \ell q_1, \quad p_C^* = \ell q_2, \]

where \( q_1, q_2 \in \Pi_{n-1} \).

Both the polynomials \( q_1, q_2 \) vanish at \( N - 5 \) nodes of the set

\[ X' := X \setminus \{A_1, A_2, A_3, B, C\}. \]

Hence these \( N - 5 = d(n, n - 2) + 1 \) nodes do not uniquely determine the curve of degree \( n - 1 \) passing through them. By Theorem 4.2 there exists a maximal curve \( \mu \) of degree \( n - 2 \) passing through \( N - 6 \) nodes of \( X' \) and the remaining node denoted by \( D \) is outside of it. Now, according to Proposition 3.4, the node \( D \) uses \( \mu \):

\[ p_D^* = \mu q, \quad q \in \Pi_2. \]

This quadratic polynomial \( q \) has to vanish at the three nodes \( A_1, A_2, A_3 \in \ell \). Therefore, in view of Proposition 3.1, we have that \( q = \ell \ell' \) with \( \ell' \in \Pi_1 \).

Hence the node \( D \) uses the line \( \ell' \):

\[ p_D^* = \mu \ell \ell', \quad \ell' \in \Pi_1. \]

Thus if two nodes \( B, C \in X \) use the line \( \ell \) then there exists a third node \( D \in X' \) using it and all the nodes of \( Y := X \setminus \{A_1, A_2, A_3, B, C, D\} \) lie in a maximal curve \( \mu \) of degree \( n - 2 \):

\[ Y \subset \mu. \]
Next, let us show that there is no fourth node using $\ell$. Assume by way of contradiction that except of the nodes $B, C, D$, there is a fourth node $E$ using $\ell$. Of course we have that $E \in \mathcal{Y}$.

Then $B$ and $E$ are using $\ell$ therefore, as was proved above, there exists a third node $F \in \mathcal{X}$ (which may coincide or not with $C$ or $D$) using it and all the nodes of $\tilde{\mathcal{Y}} := \mathcal{X} \setminus \{A_1, A_2, A_3, B, E, F\}$ are located in a maximal curve $\mu$ of degree $n - 2$. We have also that

(6.2) \[ p_E^* = \tilde{\mu} \tilde{q}, \quad \tilde{q} \in \Pi_2. \]

Now, notice that both $\mu$ and $\tilde{\mu}$ pass through all the nodes of the set $Z := \mathcal{X} \setminus \{A_1, A_2, A_3, B, C, D, E, F\}$ with $\#Z \geq N - 8$.

Then, we get from Theorem 4.1, with $k = n - 2$, that $N - 8 = d(n, n - 3) + 2$ nodes determine the curve of degree $n - 2$ passing through them uniquely. Thus $\mu$ and $\tilde{\mu}$ coincide.

Therefore, in view of (6.1) and (6.2), $p_E^*$ vanishes at all the nodes of $\mathcal{Y}$, which is a contradiction since $E \in \mathcal{Y}$.

Now, let us verify the last “moreover” statement. Suppose three nodes $B, C, D \in \mathcal{X}$ use the line $\ell$. Then, as we obtained earlier, the nodes $\mathcal{Y} := \mathcal{X} \setminus \{A_1, A_2, A_3, B, C, D\}$ are located in a maximal curve $\mu$ of degree $n - 2$. Suppose conversely that the nodes $B, C$ and $D$ are lying in a line $\ell_1$. Then we have that all the nodes of the set $\mathcal{X}$ are lying in the curve $\mu \ell \ell_1$ of degree $n$. This, in view of Proposition 1.3, is a contradiction. □

References


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