On Banach spaces of universal disposition

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Abstract. We present: i) an example of a Banach space of universal disposition that is not separably injective; ii) an example of a Banach space of universal disposition with respect to finite dimensional polyhedral spaces with the Separable Complementation Property; iii) a new type of space of universal disposition nonisomorphic to the previous existing ones.

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1. Introduction

The monograph [1] contains a study of separably injective spaces, among which one encounters two somewhat unexpected classes: ultrapowers of spaces of type \( L_{\infty,\lambda} \) and spaces of universal disposition with respect to the class of separable spaces. Recall from [9] that given a class \( \mathcal{M} \) of Banach spaces the space \( U \) is said to be of almost universal disposition for \( \mathcal{M} \) if given and \( \varepsilon > 0 \), \( A, B \in \mathcal{M} \) and isometries \( u : A \to U \) and \( i : A \to B \) there is an \( \varepsilon \)-isometry \( u' : B \to U \) such that \( u = u' i \). The space \( U \) is said to be of universal disposition for \( \mathcal{M} \) (sometimes called \( \mathcal{M} \)-universal disposition) if the condition above also holds for \( \varepsilon = 0 \).

We are particularly interested in the classes \( \mathcal{M} = \mathcal{S} \) of separable Banach spaces and \( \mathcal{M} = \mathcal{F} \) of finite dimensional Banach spaces. Spaces of (almost) universal disposition for \( \mathcal{S} \) will simply be called spaces of (almost) universal disposition.

A Banach space \( E \) is said to be separably injective if for every separable Banach space \( X \) and each subspace \( Y \subset X \), every operator \( t : Y \to E \) extends to an operator \( T : X \to E \). In [1, Thm. 3.5] it is established that spaces of \( \mathcal{S} \)-universal disposition are separably injective, as well as the ultrapowers of \( L_{\infty,\lambda} \)-spaces [1, Thm. 4.4]. Which raises the question, not considered in [1], of whether spaces of universal...
disposition must also be separably injective. Our first result is to show that such is not the case.

2. A space of universal disposition that is not separably injective

To proceed with the example, we need to recall the construction of $F_{\omega 1}(\mathbb{R})$, the only known example of space that is of universal disposition with respect to finite dimensional spaces but not with respect to separable spaces. Recall first the push-out construction; which, given an isometry $u : A \to B$ and an operator $t : A \to E$ will provides us with an extension of $t$ through $u$ at the cost of embedding $E$ in a larger space as it is showed in the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{u} & B \\
\downarrow t & & \downarrow t' \\
E & \xrightarrow{u'} & PO
\end{array}
$$

where $t'u = u't$. It is important to realize that $u'$ is again an isometry and that $t'$ is a contraction or an isometry if $t$ is. Once a starting Banach space $X$ has been fixed, the input data we need for our construction are:

- a class $\mathcal{M}$ of Banach spaces;
- the family $\mathcal{J}$ of all isometries acting between the elements of $\mathcal{M}$;
- a family $\mathcal{L}$ of norm one $X$-valued operators defined on elements of $\mathcal{M}$.

For any operator $s : A \to B$, we establish $\text{dom}(s) = A$ and $\text{cod}(s) = B$. Notice that the codomain of an operator is usually larger than its range, and that the unique codomain of the elements of $\mathcal{L}$ is $X$. Set $\Gamma = \{(u,t) \in \mathcal{J} \times \mathcal{L} : \text{dom}u = \text{dom}t\}$ and consider the Banach spaces of summable families $\ell_1(\Gamma, \text{dom}u)$ and $\ell_1(\Gamma, \text{cod}u)$. We have an obvious isometry

$$
\oplus \mathcal{J} : \ell_1(\Gamma, \text{dom}u) \longrightarrow \ell_1(\Gamma, \text{cod}u)
$$

defined by $(x_{(u,t)})_{(u,t)\in \Gamma} \longmapsto (u(x_{(u,t)}))_{(u,t)\in \Gamma}$; and a contractive operator

$$
\Sigma \mathcal{L} : \ell_1(\Gamma, \text{dom}u) \longrightarrow X,
$$

given by $(x_{(u,t)})_{(u,t)\in \Gamma} \longmapsto \sum_{(u,t)\in \Gamma} t(x_{(u,t)})$. Observe that the notation is slightly imprecise since both $\oplus \mathcal{J}$ and $\Sigma \mathcal{L}$ depend on $\Gamma$. We can form their push-out diagram

$$
\begin{array}{ccc}
\ell_1(\Gamma, \text{dom}u) & \xrightarrow{\oplus \mathcal{J}} & \ell_1(\Gamma, \text{cod}u) \\
\downarrow \Sigma \mathcal{L} & & \downarrow \\
E & \xrightarrow{i} & PO
\end{array}
$$

We obtain in this way an isometric enlargement of $X$ such that for every $t : A \to X$ in $\mathcal{L}$, the operator $u$ can be extended to an operator $t' : B \to PO$ through any embedding $u : A \to B$ in $\mathcal{J}$ provided $\text{dom}u = \text{dom}t = A$. In the next step we keep the family $\mathcal{J}$ of isometries, replace the starting space $X$ by $PO$ and $\mathcal{L}$ by a family of norm one operators $\text{dom}u \to PO$, $u \in \mathcal{J}$, and proceed again.
We start with $\mathscr{S}^0(X) = X$. The inductive step is as follows. Suppose we have constructed the directed system $(\mathscr{S}^\alpha(X))_{\alpha < \beta}$, including the corresponding linking maps $t(\alpha, \gamma) : \mathscr{S}^\alpha(X) \to \mathscr{S}^\gamma(X)$ for $\alpha < \gamma < \beta$. To define $\mathscr{S}^\beta(X)$ and the maps $t(\alpha, \beta) : \mathscr{S}^\alpha(X) \to \mathscr{S}^\beta(X)$ we consider separately two cases, as usual: if $\beta$ is a limit ordinal, then we take $\mathscr{S}^\beta(X)$ as the direct limit of the system $(\mathscr{S}^\alpha(X))_{\alpha < \beta}$ and $t(\alpha, \beta) : \mathscr{S}^\alpha(X) \to \mathscr{S}^\beta(X)$ the natural inclusion map. Otherwise $\beta = \alpha + 1$ is a successor ordinal and we construct $\mathscr{S}^\beta(X)$ applying the push-out construction as above with the following data: $\mathscr{S}^\alpha(X)$ is the starting space, $\mathfrak{S}$ keeps being the set of all isometries acting between the elements of $\mathfrak{M}$ and $\mathcal{L}_\alpha$ is the family of all isometries $t : S \to \mathscr{S}^\alpha(X)$, where $S \in \mathfrak{M}$.

We then set $\Gamma_\alpha = \{(u, t) \in \mathfrak{S} \times \mathcal{L}_\alpha : \text{dom } u = \text{dom } t\}$ and make the push-out

$$
\begin{array}{ccc}
\ell_1(\Gamma_\alpha, \text{dom } u) & \xrightarrow{\oplus \mathfrak{S}_\alpha} & \ell_1(\Gamma_\alpha, \text{cod } u) \\
\Sigma \mathcal{L}_\alpha \downarrow & & \downarrow \\
\mathscr{S}^\alpha(X) & \longrightarrow & \text{PO}
\end{array}
$$

(1)

thus obtaining $\mathscr{S}^{\alpha+1}(X) = \text{PO}$. The embedding $t(\alpha, \beta)$ is the lower arrow in the above diagram; by composition with $t(\alpha, \beta)$ we get the embeddings

$$t(\gamma, \beta) = t(\alpha, \beta) t(\gamma, \alpha),$$

for all $\gamma < \alpha$.

Set now as input data: $\mathfrak{M} = \mathfrak{S}$ the family of all finite dimensional spaces, $\mathfrak{S}$ the set of all isometries between elements of $\mathfrak{S}$ and $\mathcal{L}_\alpha$ all $X$-valued isometries defined on elements of $\mathfrak{S}$. Proceeding inductively until $\omega_1$ we get the space $\mathcal{F}^{\omega_1}(X)$. This space is of universal disposition (cf. [1, Chapter 3]). We prove first a structure theorem.

**Theorem 2.1.** The space $\mathcal{F}^{\omega_1}(\mathbb{R})$ is not separably injective.

**Proof.** In [1, Thm. 3.23 (2)] it is proved that for all separable $X$, all the spaces $\mathcal{F}^{\omega_1}(X)$ are isometric; thus isometric to $\mathcal{F}^{\omega_1}(\mathbb{R})$.

**Claim.** The space $C[0, 1]$ is $(1 + \varepsilon)$-complemented in $\mathscr{S}^\alpha(C[0, 1])$ for all $\alpha < \omega_1$ and all $\varepsilon > 0$.

**Proof.** Recall that a convex body is said to be a polyhedron if it is the convex hull of a finite set of points. A Banach space is said to be polyhedral if the unit ball of each of its finite dimensional subspaces is a polyhedron. It is a well-known fact that $C[0, 1]$-valued operators defined on finite-dimensional polyhedral spaces can be extended with the same norm. A simple proof for this result can be derived from Kalman’s theorem [10], which in turn can be easily proved by triangularisation. See also [11, 12] for the state-of-the-art about the problem of extension of $C(K)$-valued Lipschitz maps. Since every norm on a finite dimensional space can, for every $\varepsilon > 0$, be $\varepsilon$-approximated by a polyhedral norm, it follows that every norm one $C[0, 1]$-valued operator defined on a finite dimensional Banach space can, for every $\varepsilon > 0$, be extended with norm at most $1 + \varepsilon$. 
Thus, if $1_C$ denotes the identity of $C[0,1]$ and $\varepsilon > 0$ has been fixed, as well as a limit ordinal $\beta < \omega_1$, pick a sequence $(\varepsilon_j)_{j<\beta}$ with all $\varepsilon_j > 0$ so that $\sum_j \varepsilon_j < \varepsilon$. If $\beta$ is nonlimit, pick the sequence $(\varepsilon_j)_{j\leq\beta}$ so that each $\varepsilon_j > 0$ and $\sum_j \varepsilon_j < \varepsilon$. Now, all the elements in the composition $1_C \sum \Sigma \varepsilon$ are norm one finite rank operators that extend to $C[0,1]$-valued operators with norm at most $(1 + \varepsilon_1)$; thus, $1_C \sum \Sigma \varepsilon$ extends to an operator $\ell_1(\Gamma, \text{cod} u) \to C[0,1]$ with norm at most $(1 + \varepsilon_1)$, hence to an operator $\mathcal{F}^1(C[0,1]) \to C[0,1]$ with norm at most $(1 + \varepsilon_1)$. I.e., $C[0,1]$ is $(1 + \varepsilon_1)$-complemented in $\mathcal{F}^1(C[0,1])$. Iterating the argument, one gets that $C[0,1]$ is actually $(1 + \varepsilon)$-complemented in $\mathcal{F}^\beta(C[0,1])$. The Claim follows. □

Assume now that $\mathcal{F}^{\omega_1}(C[0,1])$ is separable injective. Let $S \to S'$ be an injective isometry between two separable spaces and let $\tau : S \to C[0,1]$ be an operator. As an operator $S \to C[0,1] \to \mathcal{F}^{\omega_1}(C[0,1])$, it can be extended to an operator $T : S' \to \mathcal{F}^{\omega_1}(C[0,1])$.

However, the uncountable cofinality of $\omega_1$ means that $T$ actually has its range contained in some $\mathcal{F}^\beta(C[0,1])$. A composition with the projection $\mathcal{F}^\beta(C[0,1]) \to C[0,1]$ provides an extension $S' \to C[0,1]$ of $\tau$. In other words, $C[0,1]$ would be separably injective, which it is not. The proof of Theorem 2.1 is complete. □

The result above can be improved for separable Lindenstrauss spaces. Recall that a Banach space is called a Lindenstrauss space if it is an isometric predual of some $L_1(\mu)$.

**Corollary 2.1.** Every separable Lindenstrauss space is $(1 + \varepsilon)$-complemented in $\mathcal{F}^\alpha(C[0,1])$ for all $\alpha < \omega_1$ and all $\varepsilon > 0$.

**Proof.** Indeed, one can skip using Kalman’s theorem and use instead the fact that norm one finite rank operators with values on a Lindenstrauss space can be extended, for every $\varepsilon > 0$, with norm at most $1 + \varepsilon$. □

**Proposition 2.1.** The space $\mathcal{F}^{\omega_1}(\mathbb{R})$ contains $1$-complemented copies of all isometric preduals of $\ell_1$.

**Proof.** Let $X$ be an isometric $\ell_1$-predual; it can therefore be renormed as follows to be a polyhedral Lindenstrauss space [8]: Let $(e_n)$ be the canonical basis of $\ell_1 = X^*$ and let $(e_n) \in c_0$ be a sequence of positive scalars. Set $\|x\| = \sup_n (1 + e_n)|e_n(x)|$. See [8] for details. A result of Lazar [13] shows that every compact operator with values on a polyhedral Lindenstrauss space can be extended to any separable superspace with the same norm. The paper [2] claims that the additional condition

$$(+) \quad \forall x \in X; \|x\| = 1 \quad \dim\{f \in X^* : f(x) = 1\} < +\infty$$

has to be added to Lazar’s result. But the renorming above mentioned satisfies condition $(+)$. 

Consider the space $\mathcal{F}^0(X) = \mathcal{F}^0(\mathbb{R})$ and let $1_X$ be the identity of $X$. All the elements in the composition $1_C \sum L$ are norm one finite rank operators that extend to norm one $X$-valued operators; thus, $1_C \sum L$ extends to a norm one operator $\ell_1(\Gamma, \text{cod} u) \to X$; hence, to a norm one operator $\mathcal{F}^1(X) \to X$. Iterating the argument, one gets that $X$ is actually 1-complemented in $\mathcal{F}^0(X)$. □

3. Universal disposition and the Separable Complementation Property

As we have seen, the key point seems to be that $c_0$ is the only possible separable complemented subspace of a separably injective space, which means that there are few complemented separable subspaces in separably injective spaces. Let us consider the case of spaces of universal disposition. A property that somehow means the existence of many separable complemented subspaces is the so-called Separable Complementation Property (in short, SCP). Recall from [14], see also [6], that a Banach space $X$ is said to have SCP if every separable subspace of $X$ is contained in a separable subspace complemented in $X$. Recall from [1, Def. 2.25] that a Banach space $X$ is said to be upper-$c_0$-saturated if every separable subspace is contained in a copy of $c_0$ contained in $X$. A few straightforward facts are:

Lemma 3.1. A separably injective space with SCP is upper-$c_0$-saturated. In particular, it is $c_0$-saturated. A space of universal disposition with respect to separable spaces cannot have SCP.

Proof.

(1) Indeed, every separable subspace must be contained in a separable separably injective subspace; i.e., in $c_0$.

(2) Observe that every copy of $c_0$ in a space with SCP must be complemented. But a space of universal disposition with respect to separable spaces must contain isometric copies of all spaces with density character $\aleph_1$ (see [1, Prop. 3.13 (2)]), which prevents them to have SCP since there are spaces with density character $\aleph_1$ containing uncomplemented copies of $c_0$: The simplest example being the space $C(\Delta)$ of continuous functions on the compact dyadic tree — the compact space having three types of points: the nodes of the dyadic tree, which are isolated points; the branches, each branch is the limit of its nodes, and the infinity point that appears by one-point compactification (see [4] for details). This space can be represented as a nontrivial twisted sum $0 \to c_0 \to C(\Delta) \to c_0(\aleph_1) \to 0$ and thus it is separably injective (see [1, Prop. 2.11]; also [3]). □

A different question is whether spaces of universal disposition can have SCP. Since, as we remarked in (2) above, every copy of $c_0$ in a space with SCP is complemented, picking a space $E$ with density character $\aleph_1$ containing an uncomplemented copy of $c_0$ immediately yields a space $\mathcal{F}^0(E)$ that is of universal disposition, which has density character $\aleph_1$ and without SCP. We have not been able to settle the question of whether $\mathcal{F}^0(c_0)$ has SCP. We however show:
Lemma 3.2. Under CH, every subspace of $\mathcal{F}^0(\mathbb{R})$ is complemented.

Proof. Under CH, $c = \aleph_1$, which means that a set $C$ of size $c$ can be written as a union $\bigcup_{\alpha < \omega_1} C_\alpha$ of countable sets $C_\alpha$. Do this and write the set of all finite-dimensional spaces as $F = \bigcup_{\alpha < \omega_1} F_\alpha$ and the set of all isometries between them as $I = \bigcup_{\alpha < \omega_1} I_\alpha$. Do the same with the set of isometric embeddings between elements of $F$ and $R$, say $L_0 = \bigcup_{\alpha < \omega_1} L_0^\alpha$ and keep in mind that when $S_\mu$ has been obtained one has to work with the set of isometric embeddings between elements of $F$ and $S_\mu$, say $L_\mu = \bigcup_{\alpha < \omega_1} L_\mu^\alpha$. Now proceed with the construction of $F(\mathbb{R})$ with the restriction that at step $\alpha < \omega_1$ one will only consider to make push-out the elements of $F_\alpha$ and $I_\alpha$ and $\bigcup_{\nu, \mu \leq \alpha} L_\nu^\mu$. The immediate consequence of acting this way is that each space $S_\alpha$ is separable.

Now, pick a $\lambda$-isomorphic copy of $Y$ of $c_0$ inside $F(\mathbb{R})$ and let $\phi : Y \to c_0$ be a $\lambda$-isomorphism. Since $\omega_1$ has uncountable cofinality, there must be some $\alpha < \omega_1$ so that $Y \subset S_\alpha$. By Sobczyk’s theorem, $\phi$ can be extended to an operator $\Phi : \mathcal{F}^\alpha \to c_0$ with norm at most $2\lambda$. Since $c_0$-valued finite-rank operators can be extended with the same norm everywhere (a well-known fact, see [1, Lemma 3.14] for details; it follows that the composition $\Phi \Sigma L_\alpha$ (see diagram (1)) can be extended to $\ell_1(\Gamma_\alpha, \text{cod} u)$ with the same norm, hence to $\mathcal{F}^{\alpha+1}$. Proceeding inductively, one gets an extension $\hat{\Phi} : \mathcal{F}^0(\mathbb{R}) \to c_0$ of $\Phi$. The operator $\phi^{-1} \hat{\Phi}$ is a projection onto $Y$. □

For smaller classes $\mathcal{M}$ it is however possible to enjoy SCP. Let $\mathcal{P}$ denote the class of finite dimensional polyhedral spaces.

Proposition 3.1. There is a space of universal disposition for the class $\mathcal{P}$ enjoying the SCP; precisely, such that every separable subspace is contained in a 1-complemented copy of $C[0,1]$.

Proof. Set the controls of the device at: $\mathcal{M} = \mathcal{P}$; the starting space will be $C[0,1]$ and we will call $X_\alpha = C(B_{\mathcal{F}_\alpha^*})$. Then proceed inductively. The first step is

\[
\ell_1(\Gamma_\alpha, \text{dom} u) \xrightarrow{\oplus 3_\alpha} \ell_1(\Gamma_\alpha, \text{cod} u)
\]

$\Sigma \varepsilon_\alpha \downarrow \quad \downarrow$

\[
X \quad \Longrightarrow \quad \text{PO} \quad \delta \quad C(B_{\mathcal{P}_\alpha^*})
\]

where $\delta$ is the canonical isometric embedding. I.e., instead of replacing $X$ by $\mathcal{F}^1$ at the first step, set $C(B_{\mathcal{F}_1^*})$, and so on. Thus, step $\alpha$ will then be

\[
\ell_1(\Gamma_\alpha, \text{dom} u) \xrightarrow{\oplus 3_\alpha} \ell_1(\Gamma_\alpha, \text{cod} u)
\]

$\Sigma \varepsilon_\alpha \downarrow \quad \downarrow$

\[
X_\alpha \quad \Longrightarrow \quad \mathcal{F}^{\alpha+1} \quad \delta \quad C(B_{\mathcal{F}_\alpha^{\alpha+1}}).
\]
Let us call \( \mathcal{F}^{\text{oh}}(C[0,1]) \) the outcome of the device at \( \omega_1 \).

To show that \( \mathcal{F}^{\text{oh}}(C[0,1]) \) has the desired property, observe that every operator from a finite-dimensional polyhedral space into a \( C(K) \)-space can be extended with the same norm to any separable superspace. □

Observe that the space \( \mathcal{F}^{\text{oh}}(C[0,1]) \) is of almost universal disposition.

### 4. A new space of universal disposition

In [1, Section 6.6. Problem 16] is formulated the conjecture that there is a continuum of mutually nonisomorphic spaces of universal disposition having density character \( c \). So far only two types are known: one is the space \( \mathcal{F}^{\text{oh}}(\mathbb{R}) \); as for the other, one has to pick as family \( \mathcal{M} \) that of separable spaces, and as \( \mathcal{I} \) that of into isometries between separable spaces. With this choice and the same construction as for \( \mathcal{F}^{\text{oh}}(\mathbb{R}) \) one gets the space \( S^{\text{oh}}(\mathbb{R}) \), which is of universal disposition for separable spaces and therefore separably injective. Actually, under \( \text{CH} \) it is isometric to the Fraïssé limit in the category of separable spaces and into isometries [1, Prop. 3.3 and Thm. 3.23]. Here we construct a third type. Recall that \( C(\Delta) \) represents here the space of continuous functions on the compact dyadic tree space as described above.

**Proposition 4.1.** Under \( \text{CH} \), for every separable \( C(K) \) the space

\[
\mathcal{F}^{\text{oh}}(C(\Delta) \oplus C(K))
\]

is a space of universal disposition that is not isomorphic to either \( \mathcal{F}^{\text{oh}}(\mathbb{R}) \) or \( S^{\text{oh}}(\mathbb{R}) \).

**Proof.** We settle first the case \( C(K) \simeq c_0 \), in which case \( C(\Delta) \oplus C(K) \simeq C(\Delta) \). The space \( \mathcal{F}^{\text{oh}}(C(\Delta)) \) is a space of universal disposition for exactly the same reason as \( \mathcal{F}^{\text{oh}}(\mathbb{R}) \). It cannot be isomorphic to \( \mathcal{F}^{\text{oh}}(\mathbb{R}) \) because it contains an uncomplemented copy of \( c_0 \). Let us show that it cannot be isomorphic to \( S^{\text{oh}}(\mathbb{R}) \) either:

From [7, Thm. 11] it follows that \( C(\Delta) \) admits a polyhedral Lindenstrauss renorming with property \((*)\). Reasoning now as in Proposition 2.1, it follows that \( C(\Delta) \) is 1-complemented in \( \mathcal{F}^{\text{oh}}(C(\Delta)) \). Assume that \( \mathcal{F}^{\text{oh}}(C(\Delta)) \) is isomorphic to \( S^{\text{oh}}(\mathbb{R}) \). This is a Grothendieck space [1, Thm 3.5 and Prop. 2.31] — operators into \( c_0 \) are weakly compact — hence \( \mathcal{F}^{\text{oh}}(C(\Delta)) \) should also be; which is impossible since it contains a complemented copy of \( c_0 \) (the subspace of continuous functions on \( \Delta \) with support contained in a given branch).

If \( C(K) \) is not isomorphic to \( c_0 \) then it is not separably injective, by Zippin’s theorem. The space \( \mathcal{F}^{\text{oh}}(C(\Delta) \oplus C(K)) \) is a space of universal disposition not isomorphic to \( \mathcal{F}^{\text{oh}}(\mathbb{R}) \) exactly as before. Let us show that it cannot be isomorphic to \( S^{\text{oh}}(\mathbb{R}) \) either:

Reasoning now as in the Claim of Theorem 2.1, \( C(\Delta) \oplus C(K) \) is complemented in every \( \mathcal{F}^{\alpha}(C(\Delta) \oplus C(K)) \). Since \( S^{\text{oh}}(\mathbb{R}) \) is separably injective, an isomorphism between \( \mathcal{F}^{\text{oh}}(C(\Delta) \oplus C(K)) \) and \( S^{\text{oh}}(\mathbb{R}) \) would imply that \( C(\Delta) \oplus C(K) \) is separably injective; in particular \( C(K) \) should be separably injective, which is not. □
And, taking Corollary 2.1 into account one gets:

**Corollary 4.1.** Under CH, for every separable Lindenstrauss space \( L \),

\[
\mathcal{F}^{0h}(\mathbb{C}(\Delta) \oplus L)
\]

is a space of universal disposition that is not isomorphic to either \( \mathcal{F}^{0h}(\mathbb{R}) \) or \( S^{0h}(\mathbb{R}) \).

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