On smoothly superslice knots

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Abstract. We find smoothly slice (in fact doubly slice) knots in the 3-sphere with trivial Alexander polynomial that are not superslice, answering a question posed by Livingston and Meier.

1. Introduction

A recent paper of Livingston and Meier raises an interesting question about superslice knots. Recall [3] that a knot $K$ in $S^3$ is said to be superslice if there is a slice disk $D$ for $K$ such that the double of $D$ along $K$ is the unknotted 2-sphere $S$ in $S^4$. We will refer to such a disk as a superslicing disk. In particular, a superslice knot is slice and also doubly slice, that is, a slice of an unknotted 2-sphere in $S^4$. Livingston and Meier ask about the converse in the smooth category.

Problem 4.6 (Livingston–Meier [10]). Find a smoothly slice knot $K$ with $\Delta_K(t) = 1$ that is not smoothly superslice.

The corresponding question in the topological (locally flat) category is completely understood [10, 12], for a knot $K$ with $\Delta_K(t) = 1$ is topologically superslice.

In this note we give a simple solution to Problem 4.6, making use of Taubes’ proof [16] that Donaldson’s diagonalization theorem [5] holds for certain noncompact manifolds. For $K$ a knot in $S^3$, we write $\Sigma_k(K)$ for a $k$-fold cyclic branched cover of $S^3$ branched along $K$. The same notation will be used for the corresponding branched cover along an embedded disk in $B^4$ or sphere in $S^4$.

Theorem 1.1. Suppose that $J$ is a knot with Alexander polynomial 1, so that $\Sigma_k(J) = \partial W$, where $W$ is simply connected and the intersection form on $W$ is definite and not diagonalizable. Then the knot $K = J\# - J$ is smoothly doubly slice, but is not smoothly superslice.

An unpublished argument of Akbulut says that the positive Whitehead double of the trefoil is a knot $J$ satisfying the hypotheses of the theorem, with $k = 2$. The construction is given as [1, Exercise 11.4] and is also

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documented, along with some generalizations, in the paper [4]. Hence $J$ gives an answer to Problem 4.6. We remark that for the purposes of the argument, it doesn’t matter if $W$ is positive or negative definite, as one could replace $J$ by $-J$ and change all the signs.

We need a simple and presumably well-known algebraic lemma.

**Lemma 1.2.** Suppose that

\[ \begin{array}{ccc}
  A & \xrightarrow{i_1} & B \\
  \downarrow{j_1} & & \downarrow{j_2} \\
  C & \xleftarrow{i_2} & D
\end{array} \]

is a pushout of groups, and that $i_1 = i_2$. Then $C$ surjects onto $B$.

**Proof.** This follows from the universal property of pushouts; the identity map $id_B$ satisfies $id_B \circ i_1 = id_B \circ i_2$, and hence defines a homomorphism $C \to B$ with the same image as $id_B$. □

Applying Lemma 1.2 to the decomposition of the complement of the unknot in $S^4$ into two disk complements, we obtain the following useful facts. (The first of these was presumably known to Kirby and Melvin; compare [8, Addendum, p. 58], and the second is due to Gordon and Sumners [6].)

**Corollary 1.3.** If $K$ is superslice and $D$ is a superslicing disk, then

\[ \pi_1(B^4 - D) \cong \mathbb{Z} \quad \text{and} \quad \Delta_K(t) = 1. \]

**Proof.** The lemma says that there is a surjection

\[ \mathbb{Z} \cong \pi_1(S^4 - S) \to \pi_1(B^4 - D). \]

Hence $\pi_1(B^4 - D)$ is abelian and so must be isomorphic to $\mathbb{Z}$. This condition implies, using Milnor duality [13] in the infinite cyclic covering, that the homology of the infinite cyclic covering of $S^3 - K$ vanishes, which is equivalent to saying that $\Delta_K(t) = 1$. □

**Proof of Theorem 1.1.** It is standard [15] that any knot of the form $J \# -J$ is doubly slice. In fact, it is a slice of the 1-twist spin of $J$, which was shown by Zeeman [17] to be unknotted.

Suppose that $K$ is superslice and let $D$ be a superslicing disk, so $D \cup_K D = S$, an unknotted sphere. Then $S^4 = \Sigma_k(S) = V \cup_Y V$, where we have written $Y = \Sigma_k(K)$ and $V = \Sigma_k(D)$. By Corollary 1.3, the $k$-fold cover of $B^4 - D$ has $\pi_1 \cong \mathbb{Z}$, so the branched cover $V$ is simply connected.

Note that $\Sigma_k(K) = \Sigma_k(J) \# -\Sigma_k(J)$. Since $\Delta_J(t) = 1$, the same is true for $\Delta_K(t)$; moreover this implies that both $\Sigma_k(J)$ and $\Sigma_k(K)$ are homology spheres. An easy Mayer-Vietoris argument says that $V = \Sigma_k(D)$ is a homology ball; in fact Claim 1.3 implies that it is contractible. Adding a
3-handle to $V$, we obtain a simply-connected homology cobordism $V'$ from $\Sigma_k(J)$ to itself. By hypothesis, there is a manifold $W$ with boundary $\Sigma_k(J)$ and nondiagonalizable intersection form. Stack up infinitely many copies of $V'$, and glue them to $W$ to make a definite periodic-end manifold $M$, in the sense of Taubes [16]. Since $\pi_1(V)$ is trivial, $M$ is admissible (see [16, Definition 1.3]), and Taubes shows that its intersection form (which is the same as that of $W$) is diagonalizable. This contradiction proves the theorem. □

The fact that $\pi_1(B^4 - D) \cong \mathbb{Z}$ for a superslicing disk leads to a second obstruction to supersliceness, based on the Ózsváth–Szabó $d$-invariant [14]. Recall from [11] (for degree 2 covers) and [7] in general that for a knot $K$ and prime $p$, that one denotes by $\delta_p^n(K)$ the $d$-invariant of a particular spin structure $s$ on $\Sigma_{p^n}(K)$ pulled back from the 3-sphere. The fact that a $p^n$ fold branched cover of a slicing disk is a rational homology ball implies that if $K$ slice then $\delta_p^n(K) = 0$. For a non-prime-power degree $k$, the invariant $\delta_k(K)$ might not be defined, because $\Sigma_k(K)$ is not a rational homology sphere. (One might define such an invariant using Floer homology with twisted coefficients as in [2, 9], but there’s no good reason that it would be a concordance invariant.)

**Theorem 1.4.** If $K$ is superslice, then for any $k$, the $d$-invariant

$$d(\Sigma_k(K), s_0)$$

is defined and vanishes.

**Proof.** Since by Corollary 1.3 the Alexander polynomial is trivial, so $\Sigma_k(K)$ is a homology sphere, and hence $d(\Sigma_k(K), s_0)$ is defined. (There is only the one spin structure.) As in the proof of Theorem 1.1, the branched cover $\Sigma_k(D)$ is contractible, and hence [14, Theorem 1.12], $d(\Sigma_k(K), s_0) = 0$. □

Sadly, we do not know any examples of a slice knot where Theorem 1.4 provides an obstruction to it being superslice. For such a knot would not be ribbon, so we would also have a counterexample to the slice-ribbon conjecture!

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**References**


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