A short proof that $\text{Diff}_c(M)$ is perfect

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Abstract. In this note, we follow the strategy of Haller, Rybicki and Teichmann to give a short, self contained, and elementary proof that $\text{Diff}_0(M)$ is a perfect group, given a theorem of Herman on diffeomorphisms of the circle.

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1. Introduction

Let $M$ be a smooth manifold of dimension $n > 1$ and let $\text{Diff}_c(M)$ denote the group of diffeomorphisms supported on compact sets and isotopic to the identity through a compactly supported isotopy. Note that if $M$ is compact, then $\text{Diff}_c(M) = \text{Diff}_0(M)$, the group of isotopically trivial diffeomorphisms. That $\text{Diff}_c(M)$ is a perfect group was first proved by Thurston, as announced in [Th74]. The proof relies on the relationship between the homology of $\text{Diff}_c(M)$ and certain classifying spaces of foliations. Recently, Haller, Rybicki and Teichmann gave a fundamentally different proof in [HT03] and [HRT13]. In fact, they prove a stronger form of “smooth perfection” and give bounds on commutator width of $\text{Diff}_c(M)$ for some manifolds.

Bounds on commutator width have also been given in [BIP08], [Ts09] and [Ts12]. In particular, given the result that $\text{Diff}_c(\mathbb{R}^n)$ is perfect, Burago, Ivanov, and Polterovich show in [BIP08, Lemma 2.2] that any element of $\text{Diff}_c(\mathbb{R}^n)$ can be written as a product of two commutators; an isotopically trivial diffeomorphism of a compact 3-manifold can be written as a product of 10 commutators, and an element of $\text{Diff}_0(S^n)$ as a product of 4 commutators. Related results were obtained by Tsuboi in [Ts08], who later gave general bounds on commutator width of $\text{Diff}_0(M)$, depending on $M$ ([Ts09], [Ts12]).
The purpose of this note is to show that if one only wants to show that $\text{Diff}_c(M)$ is perfect, then the techniques of Haller, Rybicki and Teichmann provide a remarkably simple proof. Our exposition closely follows the strategy of [HT03], but avoids discussion of the tame Fréchet manifold structure on $\text{Diff}_0(M)$. As the perfectness of these diffeomorphism groups is widely cited, we thought it worthwhile to make available this short and widely accessible proof. We show the following.

**Theorem 1.1.** Let $M$ be a smooth manifold, $M \neq \mathbb{R}$. Then $\text{Diff}_c(M)$ is perfect. In fact, any compactly supported diffeomorphism $g$ can be written as a product of commutators $g = [g_1, f_1][g_2, f_2] \cdots [g_r, f_r]$ where each $f_i$ is the time one map of a (time independent) vector field $X_i$ on $M$.

In particular, this result can then be fed into Lemma 2.2 of [BIP08] to obtain their bounds on commutator width. The assumption $M \neq \mathbb{R}$ seems essential to this proof, although $\text{Diff}_c(\mathbb{R})$ is also perfect. The proof uses only one deep theorem, a result of Herman on circle diffeomorphisms.

**Theorem 1.2** ([He79]). There is a neighborhood $U$ of the identity in $\text{Diff}_0(S^1)$ and a dense set of rotations $R_\theta$ by angles $\theta \in [0, 2\pi)$ such that any $g \in U$ can be written as $R_\lambda[g_0, R_\theta]$ for some rotation $R_\lambda$ and some $g_0 \in \text{Diff}_0(S^1)$. Moreover, $\lambda$ and $g_0$ can be chosen to vary smoothly in $g$, with $\lambda = 0$ and $g_0 = \text{id}$ at $g = \text{id}$.

"Vary smoothly in $g$" can be made precise with reference to the Fréchet structure on $\text{Diff}_0(M)$, but for our purposes the reader may take it to mean the following.

**Definition 1.3.** A smooth family in $\text{Diff}_c(M)$ is a family $\{g_t : t \in [0, 1]\}$ such that the map $(x, t) \mapsto (g_t(x), t)$ is a smooth diffeomorphism of $M \times [0, 1]$. A map $\phi : \text{Diff}_c(M) \to \text{Diff}_c(N)$ varies smoothly if it maps smooth families to smooth families.

A more general version of Herman’s theorem on diffeomorphisms of the $n$-dimensional torus is used in both Thurston’s original proof and the Haller-Rybicki-Teichmann proof, though Haller, Rybicki and Teichmann state that their methods work using only Herman’s theorem for $S^1$. This note provides the details.

2. Reduction to $M = \mathbb{R}^n$ and diffeomorphisms near identity

Recall that the support of a diffeomorphism $g$ is the closure of the set $\{x \in M \mid g(x) \neq x\}$. The first step in the proof of Theorem 1.1 is to reduce it to the case of compactly supported diffeomorphisms on $M = \mathbb{R}^n$. This reduction is a consequence of the well-known fragmentation property. For simplicity, we assume $M$ is compact.

**Lemma 2.1** (Fragmentation). Let $\{U_i\}$ be a finite open cover of $M$. Then any $g \in \text{Diff}_0(M)$ can be written as a composition $g_1 \circ g_2 \circ \cdots \circ g_n$ of
diffeomorphisms where each $g_i$ has support contained in some element of $\{U_i\}$.

**Proof.** The proof is straightforward, for completeness we outline it here, following [Ba97, Ch. 2]. Let $g_t$ be an isotopy from $g_0 = \text{id}$ to $g_1 = g$. By writing

$$g = g_{1/r} \circ (g_{1/r}^{-1} g_{2/r}) \circ \cdots \circ (g_{r-1/r}^{-1} g_1)$$

for $r$ large, and working separately with each factor $g_{k-1/r} g_{k/r}$, we may assume that $g$ and $g_t$ lie in an arbitrarily small neighborhood of the identity.

Take a partition of unity $\lambda_i$ subordinate to $\{U_i\}$ and define

$$\mu_k := \sum_{i \leq k} \lambda_i.$$

Now define $\psi_k(x) := g_{\mu_k(x)}(x)$. This is a $C^\infty$ map, and can be made as close to the identity as we like by taking $g_t$ close to the identity. Although $\psi_k$ is not a priori invertible, being invertible with smooth inverse is an open condition. Thus, $\psi_k$ being sufficiently close to the identity implies that it is a diffeomorphism. By definition, $\psi_k$ agrees with $\psi_{k-1}$ outside of $U_k$, and hence $g = (\psi_{0}^{-1} \psi_1)(\psi_1^{-1} \psi_2) \cdots (\psi_{n-1}^{-1} \psi_n)$ is the desired decomposition of $g$ with each diffeomorphism $\psi_{k-1}^{-1} \psi_k$ supported on $U_k$.

To prove Theorem 1.1, it is also sufficient to prove that some neighborhood of the identity in $\text{Diff}_c(\mathbb{R}^n)$ is perfect, because any neighborhood of the identity generates $\text{Diff}_c(\mathbb{R}^n)$. The strategy is to first prove perfectness of a neighborhood of the identity for $S^1$, move to $\mathbb{R}^2$, and then induct on dimension.

3. **Proof for $S^1$ and diffeomorphisms preserving vertical lines**

Perfectness of $\text{Diff}_0(S^1)$ is a consequence of Herman’s theorem together with the fact that $\text{PSL}(2,\mathbb{R})$ is perfect, so any rotation can be written as a commutator.

**Lemma 3.1 (Perfectness for $S^1$).** There is a neighborhood $\mathcal{U}$ of the identity in $\text{Diff}_0(S^1)$ and $f_1, \ldots, f_4 \in \text{Diff}_0(S^1)$ such that any $g \in \mathcal{U}$ can be written $g = [g_1, f_1] \cdots [g_4, f_4]$, with each $g_i$ depending smoothly on $g$.

Furthermore, we may take $f_i = \exp(X_i)$ to be the time one map of a vector field, and may take $g_i = \text{id}$ when $g = \text{id}$.

**Proof.** Let $\mathcal{U}$ be a neighborhood of the identity as in Herman’s theorem and let $g \in \mathcal{U}$. Then $g$ can be written as $R_\lambda [g_0, R_\lambda]$ with $\lambda$ and $g_0$ depending smoothly on $g$. Let $f_4 = R_\lambda$; this is indeed the time one map of a (constant) vector field. We need to write the rotation $R_\lambda$ as a product of commutators $[g_1, f_1][g_2, f_2][g_3, f_3]$ with $g_i$ depending smoothly on $\lambda$, and will do this working inside of $\text{PSL}(2,\mathbb{R}) \subset \text{Diff}_0(S^1)$. This can be done completely explicitly: take $f_1 = f_3 = \exp((0 1 \ 0 0))$ and $f_2 = \exp((0 1 \ 0 0))$, and define $g_0 = \left( \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right)$. 


Then
\[ [g_\alpha, f_1][g_\beta, f_2][g_\alpha, f_3] = \left( \frac{1}{0} \begin{pmatrix} \alpha^2 - 1 & 0 \\ \beta^2 - 1 & 1 \end{pmatrix} \frac{1}{0} \begin{pmatrix} \alpha^2 - 1 & 0 \\ \beta^2 - 1 & 1 \end{pmatrix} \right). \]

This is the matrix of rotation by \( \lambda := \sin^{-1}(\beta^2 - 1) \) provided that \(-\beta^2 - 1) = 2(\alpha^2 - 1) + (\alpha^2 - 1)^2(\beta^2 - 1)\). If \( \alpha \) is close to 1, then there exists \( \beta \) (close to 1) satisfying this equation, namely
\[
\beta = \left( \frac{-2(\alpha^2 - 1)}{1 + (\alpha^2 - 1)^2} + 1 \right)^{-1/2}.
\]

In fact, the inverse function theorem implies that \( \alpha \mapsto \left( \frac{-2(\alpha^2 - 1)}{1 + (\alpha^2 - 1)^2} + 1 \right)^{-1/2} \) is a local diffeomorphism of \( \mathbb{R} \) at \( \alpha = 1 \). Since \( \beta \mapsto \sin^{-1}(\beta^2 - 1) \) is also a local diffeomorphism at \( \beta = 1 \) onto a neighborhood of 0, this shows that \( \alpha \) and \( \beta \) can be chosen to smoothly depend on \( \lambda \), and approach \( 1 \) as \( \lambda \to 0 \).

Alternatively, one can see that such \( f_i \) exist from the fact that \( \text{PSL}(2, \mathbb{R}) \) is a three dimensional perfect Lie group. See [HT03, Sect. 4] for details and further generalizations.

**Remark 3.2.** Above, we showed that every diffeomorphism in a neighborhood of the identity could be written as a product of four commutators of a specific form. Relaxing this condition allows one to (easily) write every element \( g \) in a neighborhood of \( \text{id} \) in \( \text{Diff}(S^1) \) as a product of two commutators, \( g = [a_1, b_1][a_2, b_2] \) with \( a_i \) and \( b_i \) depending smoothly on \( g \). To do so, Herman’s theorem again implies that it suffices to write a rotation \( R_\lambda \) as \( [a_1, b_1] \), with \( a_1 \) and \( b_1 \) depending smoothly on \( \lambda \), and this can be done in \( \text{PSL}(2, \mathbb{R}) \), either explicitly or with an elementary argument using hyperbolic geometry as in [Gh01, Prop 5.11].

As a consequence, we now prove a perfectness result for compactly supported diffeomorphisms of \( \mathbb{R}^n \) that preserve vertical lines.

**Proposition 3.3.** Let \( U \subset \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \) be precompact, and \( V \) a neighborhood of the closure of \( U \). There exist vector fields \( Y_1, \ldots, Y_4 \) supported on \( V \) with the following property:
- If \( g \in \text{Diff}_c(\mathbb{R}^n) \) is supported on \( U \), sufficiently close to the identity, and preserves each vertical line \( \mathbb{R}^{n-1} \times \{x\} \), then \( g \) can be decomposed as
  \[ g = [g_1, \exp(Y_1)] \cdots [g_4, \exp(Y_4)] \]
  with \( g_i \) supported on \( V \) and depending smoothly on \( g \).

**Proof.** Let \( B^{n-1} \) be a ball in \( \mathbb{R}^{n-1} \). There exists an embedding \( \phi \) of \( S^1 \times B^{n-1} \) in \( \mathbb{R}^n \) with \( U \subset \phi(S^1 \times \{b\}) \subset V \), and such that for each \( b \in B^{n-1} \) the image \( \phi(S^1 \times \{b\}) \cap U \) is a vertical line segment as in Figure 1.
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If $g$ preserves vertical lines, then we can consider it as a diffeomorphism $\mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R} \times \mathbb{R}^{n-1}$ of the form $(x, y) \mapsto (x, \hat{g}(x, y))$. For each $x \in \mathbb{R}^{n-1}$ let $g_x(y)$ denote $\hat{g}(x, y)$. Then $g_x$ has support on a vertical line in $U$ so we can consider it as a diffeomorphism of $S^1$ by pulling it back to $S^1$ by $\phi$. Using Lemma 3.1, write the pullback $\phi^*(g_x) = [g_{x,1}, \exp(X_1)] \ldots [g_{x,4}, \exp(X_4)]$. Now push the vector fields $X_i$ on each $S^1 \times \{b\}$ forward to $\mathbb{R}^n$ to get vector fields on $\phi(S^1 \times B)$ tangent to $\phi(S^1 \times \{b\})$ and extend these smoothly to vector fields $Y_i$ with support in $V$. The smooth dependence of $g_{x,i}$ on $g_x$ and hence on $x$ means that the functions $\phi g_{x,i} \phi^{-1}$ on the vertical lines $\phi(S^1 \times \{b\})$ piece together to form smooth functions $g_i$ on the image of $\phi$. Since $g = \text{id}$ on the boundary of the image of $\phi$, Lemma 3.1 implies that $g_i = \text{id}$ as well, so it can be extended (trivially) to a diffeomorphism of $\mathbb{R}^n$. Now $g = [g_1, \exp(Y_1)] \ldots [g_4, \exp(Y_4)]$ on the image of $\phi$ by construction, and both are equal to the identity everywhere else. $\square$

4. Proof for $\mathbb{R}^n$

The proof of Theorem 1.1 for $\mathbb{R}^n$ will follow from a short inductive argument using Proposition 3.3 and the following lemma.

**Lemma 4.1.** There is a neighborhood $U$ of the identity in $\text{Diff}_c(\mathbb{R}^n)$ such that any $f \in U$ can be written as $g \circ h$, where $h$ preserves each vertical line and $g$ preserves each horizontal hyperplane. Moreover, $g$ and $h$ can be chosen to depend smoothly on $f$.

In other words, if $x = (x_1, \ldots, x_{n-1})$, this Lemma says that we may take $h$ to be of the form $h(x, y) = (x, \hat{h}(x, y))$ and $g$ of the form $g(x, y) = (\hat{g}(x, y), y)$.

**Proof.** Let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ denote projection to the $i^{th}$ coordinate. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is compactly supported and sufficiently $C^\infty$ close to the identity. Then for any point $(x, y) = (x_1, \ldots, x_{n-1}, y)$ the map $f_x : \mathbb{R} \to \mathbb{R}$ given by $f_x(y) = \pi_n f(x, y)$ is a diffeomorphism. (Injectivity follows from the fact that tangent vectors to vertical lines remain nearly vertical under a diffeomorphism close to the identity – if $\pi_n f(x, y_1) = \pi_n f(x, y_2)$ for some $y_1 \neq y_2$, then the image of $f_x$ has horizontal tangent at some point $y \in [y_1, y_2]$.)
Now given \( f \), define \( h \) and \( g \):

\[
R^{n-1} \times \mathbb{R} \to R^{n-1} \times \mathbb{R}
\]

by

\[
h(x, y) = (x, f_x(y)),
\]

\[
g(x, y) = (g_1(x, y), \ldots, g_{n-1}(x, y), y),
\]

where \( g_i(x, y) = \pi_i(x, f_x^{-1}(y)) \in \mathbb{R} \). Then \( f = g \circ h \) and \( g \) and \( h \) vary smoothly with \( f \). \( \square \)

**Proof of Theorem 1.1.** We induct on the dimension \( n \). The case \( n = 2 \) follows from Lemma 4.1 using \( n = 2 \), together with Proposition 3.3 applied to \( g \) and \( h \) in the decomposition (Proposition 3.3 works just as well for the diffeomorphism \( g \), which preserves horizontal rather than vertical lines).

Now suppose Theorem 1.1 holds for \( n = k \), and let \( f \in \text{Diff}_c(R^{k+1}) \) be close to the identity. By Lemma 4.1, \( f = g \circ h \), where \( h \) preserves each vertical line and \( g \) preserves each horizontal hyperplane in \( R^{k+1} \), and \( g \) and \( h \) are close to the identity. By our inductive assumption, there are smooth vector fields \( X_1, \ldots, X_r(k) \) tangent to each horizontal hyperplane such that \( g = [g_1, \exp(X_1)] \cdots [g_r, \exp(X_r(k))] \) where the \( g_i \) preserve horizontal hyperplanes as well. Technically speaking, our hypothesis gives vector fields \( X_i \) and diffeomorphisms \( g_i \) defined separately on each \( R^k \)-hyperplane, but the proof of Proposition 3.3 allows us to choose them so that they vary smoothly and form global vector fields or diffeomorphisms on \( R^{k+1} \). By Proposition 3.3, there are also vector fields \( Y_1, \ldots, Y_4 \) supported on a neighborhood of \( \text{supp}(h) \) so that \( h = [h_1, \exp(Y_1)] \cdots [h_4, \exp(Y_4)] \). Thus, \( f = g \circ h \) is a product of commutators as desired. \( \square \)

**References**


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