The rational homology of the outer automorphism group of $F_7$

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Abstract. We compute the homology groups $H_\ast(\text{Out}(F_7); \mathbb{Q})$ of the outer automorphism group of the free group of rank 7.

We produce in this manner the first rational homology classes of $\text{Out}(F_n)$ that are neither constant ($\ast = 0$) nor Morita classes ($\ast = 2n - 4$).

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1. Introduction

The homology groups $H_k(\text{Out}(F_n); \mathbb{Q})$ are intriguing objects. On the one hand, they are known to “stably vanish”, i.e., for all $n \in \mathbb{N}$ we have $H_k(\text{Out}(F_n); \mathbb{Q}) = 0$ as soon as $k$ is large enough [3]. Hatcher and Vogtmann prove that the natural maps $H_k(\text{Out}(F_n); \mathbb{Q}) \to H_k(\text{Aut}(F_n); \mathbb{Q})$ and $H_k(\text{Aut}(F_n); \mathbb{Q}) \to H_k(\text{Aut}(F_{n+1}); \mathbb{Q})$ are isomorphisms for $n \geq 2k + 2$ respectively $n \geq 2k + 4$, see [4, 5]. On the other hand, $H_k(\text{Out}(F_n); \mathbb{Q}) = 0$ for $k > 2n - 3$, since $\text{Out}(F_n)$ acts geometrically on a contractible space (the “spine of outer space”, see [2]) of dimension $2n - 3$. Combining these results, the only $k \geq 1$ for which $H_k(\text{Out}(F_n); \mathbb{Q})$ could possibly be nonzero are in the range $\frac{n}{2} - 2 < k \leq 2n - 3$. Morita conjectures in [9, page 390] that $H_{2n-3}(\text{Out}(F_n); \mathbb{Q})$ always vanishes; this
would improve the upper bound to $k = 2n - 4$, and $H_{2n-4}(\text{Out}(F_n); \mathbb{Q})$ is also conjectured to be nontrivial.

We shall see that the first conjecture does not hold. Indeed, the first few values of $H_k(\text{Out}(F_n); \mathbb{Q})$ may be computed by a combination of human and computer work, and yield

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<tr>
<th>$n \setminus k$</th>
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The values for $n \leq 6$ were computed by Ohashi in [12]. They reveal that, for $n \leq 6$, only the constant class ($k = 0$) and the Morita classes $k = 2n - 4$ yield nontrivial homology. The values for $n = 7$ are the object of this Note, and reveal that the picture changes radically:

**Theorem.** The nontrivial homology groups $H_k(\text{Out}(F_7); \mathbb{Q})$ occur for $k \in \{0, 8, 11\}$ and are all 1-dimensional.

Previously, only the rational Euler characteristic

$$
\chi_{\mathbb{Q}}(\text{Out}(F_7)) = \sum (-1)^k \dim H_k(\text{Out}(F_7); \mathbb{Q})
$$

was known, and shown to be 1 by Morita, Sakasai and Suzuki [10]. These authors computed in fact the rational Euler characteristics up to $n = 11$ in that paper and the sequel [11].

2. Methods

We make fundamental use of a construction of Kontsevich [6], explained in [1]. We follow the simplified description from [12].

Let $F_n$ denote the free group of rank $n$. This parameter $n$ is fixed once and for all, and will in fact be omitted from the notation as often as possible. An admissible graph of rank $n$ is a graph $G$ that is 2-connected ($G$ remains connected even after an arbitrary edge is removed), without loops, with fundamental group isomorphic to $F_n$, and without vertices of valency $\leq 2$. Its degree is

$$
deg(G) := \sum_{v \in V(G)} (\deg(v) - 3).
$$

In particular, $G$ has $2n - 2 - \deg(G)$ vertices and $3n - 3 - \deg(G)$ edges, and is trivalent if and only if $\deg(G) = 0$. If $\Phi$ is a collection of edges in a graph $G$, we denote by $G/\Phi$ the graph quotient, obtained by contracting all edges in $\Phi$ to points.

A forested graph is a pair $(G, \Phi)$ with $\Phi$ an oriented forest in $G$, namely an ordered collection of edges that do not form any cycle. We note that
the symmetric group $\text{Sym}(k)$ acts on the set of forested graphs whose forest contains $k$ edges, by permuting the forest’s edges.

For $k \in \mathbb{N}$, let $C_k$ denote the $\mathbb{Q}$-vector space spanned by isomorphism classes of forested graphs of rank $n$ with a forest of size $k$, subject to the relation

$$(G, \pi \Phi) = (-1)^\pi (G, \Phi) \text{ for all } \pi \in \text{Sym}(k).$$

Note, in particular, that if $(G, \Phi) \sim (G, \pi \Phi)$ for an odd permutation $\pi$ then $(G, \Phi) = 0$ in $C_k$. These spaces $(C_*)$ form a chain complex for the differential $\partial = \partial_C - \partial_R$, defined respectively on $(G, \Phi) = (G, \{e_1, \ldots, e_p\})$ by

$$\partial_C(G, \Phi) = \sum_{i=1}^{p} (-1)^i (G/e_i, \Phi \{e_i\}),$$

$$\partial_R(G, \Phi) = \sum_{i=1}^{p} (-1)^i (G, \Phi \{e_i\}),$$

and the homology of $(C_*, \partial)$ is $H_*(\text{Out}(F_n); \mathbb{Q})$.

The spaces $C_k$ may be filtered by degree: let $F_p C_k$ denote the subspace spanned by forested graphs $(G, \Phi)$ with $\text{deg}(G/\Phi) \leq p$. The differentials satisfy respectively

$$\partial_C(F_p C_k) \subseteq F_p C_{k-1}, \quad \partial_R(F_p C_k) \subseteq F_{p-1} C_{k-1}.$$  

A spectral sequence argument gives

$$H_p(\text{Out}(F_n); \mathbb{Q}) = E^2_{p,0} = \frac{\ker(\partial_C|_{F_p C_p} \cap \ker(\partial_R|_{F_p C_p})}{\partial_R(\ker(\partial_C|_{F_{p+1} C_{p+1}}))}.$$

Note that if $(G, \Phi) \in F_p C_p$ then $G$ is trivalent. We compute explicitly bases for the vector spaces $F_p C_p$, and matrices for the differentials $\partial_C, \partial_R$, to prove the theorem.

3. Implementation

We follow for $n = 7$ the procedure sketched in [12]. Using the software program nauty [8], we enumerate all trivalent graphs of rank $n$ and vertex valencies $\geq 3$. The libraries in nauty produce a canonical ordering of a graph, and compute generators for its automorphism group. We then weed out the non-2-connected ones.

For given $p \in \mathbb{N}$, we then enumerate all $p$-element oriented forests in these graphs, and weed out those that admit an odd symmetry. The remaining ones are stored as a basis for $F_p C_p$. Let $a_p$ denote the dimension of $F_p C_p$.

For $(G, \Phi)$ a basis vector in $F_p C_p$, the forested graphs that appear as summands in $\partial_C(G, \Phi)$ and $\partial_R(G, \Phi)$ are numbered and stored in a hash table as they occur, and the matrices $\partial_C$ and $\partial_R$ are computed as sparse matrices with $a_p$ columns.

The nullspace $\ker(\partial_C|_{F_p C_p})$ is then computed: let $b_p$ denote its dimension; then the nullspace is stored as a sparse $(a_p \times b_p)$-matrix $N_p$. The computation
is greatly aided by the fact that $\partial C$ is a block matrix, whose row and column blocks are spanned by $\{(G, \Phi) : G/\Phi = G_0\}$ for all choices of the fully contracted graph $G_0$. The matrices $N_p$ are computed using the linear algebra library linbox [7], which provides exact linear algebra over $\mathbb{Q}$ and finite fields.

Finally, the rank $c_p$ of $\partial R \circ N_p$ is computed, again using linbox. By (1), we have

$$\dim H_p(\text{Out}(F_n); \mathbb{Q}) = b_p - c_p - c_{p+1}.$$  

For memory reasons (the computational requirements reached 200GB of RAM at its peak), some of these ranks were computed modulo a large prime ($65521$ and $65519$ were used in two independent runs).

Computing modulo a prime can only reduce the rank; so that the values $c_p$ we obtained are underestimates of the actual ranks of $\partial R \circ N_p$. However, we also know a priori that $b_p - c_p - c_{p+1} \geq 0$ since it is the dimension of a vector space; and none of the $c_p$ we computed can be increased without at the same time causing a homology dimension to become negative, so our reduction modulo a prime is legal.

For information, the parameters $a_p, b_p, c_p$ for $n = 7$ are as follows:

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<tbody>
<tr>
<td>$a_p$</td>
<td>365</td>
<td>3712</td>
<td>23227</td>
<td>$\approx 105k$</td>
<td>$\approx 348k$</td>
<td>$\approx 854k$</td>
<td>$\approx 1.6m$</td>
<td>$\approx 2.3m$</td>
<td>$\approx 2.6m$</td>
<td>$\approx 2.1m$</td>
<td>$\approx 1.2m$</td>
<td>$\approx 376k$</td>
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<tr>
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<td>$c_p$</td>
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<td>17275</td>
<td>11313</td>
<td>5427</td>
<td>1504</td>
<td>178</td>
</tr>
</tbody>
</table>

The largest single matrix operations that had to be performed were computing the nullspace of a $2038511 \times 53647$ matrix (16 CPU hours) and the rank modulo $65519$ of a (less sparse) $1355531 \times 16741$ matrix (10 CPU hours).

The source files used for the computations are available as supplemental material. Compilation requires g++ version 4.7 or later, a functional linbox library, available from the site http://www.linalg.org, as well as the nauty program suite, available from the site http://pallini.di.uniroma1.it.

It may also be directly downloaded and installed by typing

'make nauty25r9' 

in the directory in which the supplemental material was downloaded. Beware that the calculations required for $n = 7$ are prohibitive for most desktop computers.

Conclusion

Computing the dimensions of the homology groups is only the first step in understanding them; much more interesting would be to know visually, or graph-theoretically, where these nontrivial classes come from.

It seems almost hopeless to describe, via computer experiments, the nontrivial class in degree 8, unless it is somehow related to the nontrivial class
in $H_8(\text{Out}(F_6); \mathbb{Q})$. It may be possible, however, to arrive at a reasonable understanding of the nontrivial class in degree 11.

This class may be interpreted as a linear combination $w$ of trivalent graphs on 12 vertices, each marked with an oriented spanning forest. There are 376365 such forested graphs that do not admit an odd symmetry. The class $w \in \mathbb{Q}^{376365}$ is a $\mathbb{Z}$-linear combination of 70398 different forested graphs, with coefficients in $\{\pm 1, \ldots, \pm 16\}$. For illustration, eleven graphs occur with coefficient $\pm 13$; four of them have indices 25273, 53069, 53239, 53610 respectively, and are, with the spanning tree in bold,

![Graphs](image)

The coefficients of $w$, and corresponding graphs, are distributed as ancillary material in the file $w$-cycle, in format

`coefficient [edge1 edge2 ...]`,

where each edge is ‘x-y’ or ‘x+y’ to indicate whether the edge is absent or present in the forest. Edges always satisfy $x < y$, and the forest is oriented so that its edges are lexicographically ordered. Edges are numbered from 0. There are no loops nor multiple edges.

**Acknowledgments**

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that took place during this workshop; and to Jim Conant for having checked
the cycle \( w \) (after finding a mistake in its original signs) with an independent
program.

Links to ancillary material

The source and data files are stored at http://arxiv.org/src/1512.03075v2/anc; here are some direct links to them, embedded in the PDF
document.

The cycle in degree 11: w_cycle.

The source files of the program that computed the \( b_p \) and \( c_p \):

- Makefile
- homology.h
- homology_boundary.C
- homology_graphs.C
- homology_print.C
- murmur3/README.md
- murmur3/example.c
- murmur3/makefile
- murmur3/murmur3.c
- murmur3/murmur3.h
- murmur3/test.c.

References

Geom. Topol. 3 (2003), 1167–1224. MR2026331 (2004m:18006), Zbl 1063.18007,
[2] Culler, Marc; Vogtmann, Karen. Moduli of graphs and automorphisms of
mology stability for outer automorphism groups of free groups [Algebr. Geom.
187. MR1247289 (94i:58212), Zbl 0821.58018.
[7] LinBox – Exact Linear Algebra over the Integers and Finite Rings, Version 1.1.6, The
REFERENCES


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