FIXED-POINT MAPPINGS ON COMPACT METRIC SPACES

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Let \((M, d)\) be a metric space and \(T\) a selfmapping of \(M\) into itself. If

\[
d(Tx, Ty) < d(x, y)
\]

holds for every \(x, y\) in \(M\) with \(x \neq y\), then \(T\) is called a contractive mapping. On complete metric spaces contractive mappings may be without fixed-point. However, if \(M\) is compact, then every contractive selfmapping on \(M\) has a unique fixed point.

D. Bailey in [1] has proved that if \(M\) is compact and \(T\) is continuous and such that for every \(x, y \in M\) with \(x \neq y\) there exists a positive integer \(n(x, y)\) such that

\[
d(T^{n(x,y)}x, T^{n(x,y)}y) < d(x, y),
\]

then \(T\) has a unique fixed point in \(M\).

In the following we will extend the result of Bailey to mappings which satisfy a contractive condition which is weaker than (2).

We now prove the following theorem.

THEOREM 1. Let \(T\) be a continuous mapping on the compact metric space \(M\) into itself satisfying the inequality

\[
d(T^n x, T^n y) < \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} \left[ d(x, Ty) + d(y, Tx) \right] \right\}
\]

for all \(x, y\) in \(M\) with \(x \neq y\) where \(n = n(x, y)\) is a positive integer. Then \(T\) has a unique fixed point in \(M\).
Proof. Define on $M$ a real-valued function $F$ by $F(x) = d(x, Tx)$. Since $T$ is continuous, it follows that $F$ is continuous too. Therefore, $F$ on $M$ attains its maximum and minimum. Let $u$ in $M$ be such that

$$F(u) = \min \{F(x) : x \in M\}.$$  

We will show that $u$ is a fixed-point of $T$. If we assume that $F(u) = d(u, Tu) > 0$, then by (3), for $n = n(u, Tu)$, we have

$$F(T^n u) = d(T^n u, T^n Tu) = d(T^n u, T^n Tu)$$

$$< \max \left\{ d(u, Tu), d(u, Tu), d(Tu, TTu), \frac{1}{2}\left[d(u, T^2u) + 0\right] \right\}$$

$$\leq \max \left\{ F(u), F(Tu), \frac{1}{2}\left[F(u) + F(Tu)\right] \right\}.$$  

Since by (4) $\max\{F(u), F(Tu), \frac{1}{2}\left[F(u) + F(Tu)\right]\} = F(u)$, we have $F(T^n u) < F(u)$, which is a contradiction with (4). Therefore, $u$ is a fixed-point of $T$. The uniqueness of $u$ follows easily from (3). This completes the proof of the theorem.

Theorem 1 holds if some of the conditions are relaxed. So we have

**Theorem 2.** Let $T$ be an orbitally continuous mapping on the compact metric space $M$ into itself satisfying (3). Then $T$ has a unique fixed-point in $M$.

The following example shows that the continuity conditions of $T$ in the theorems 1 and 2 cannot be removed.

**Example.** If $M$ is the closed interval $[0,1]$ and $T : M \rightarrow M$ is defined by $T(x) = \frac{2}{3}x$ if $x \neq 0$ and $T(0) = 1$, then $T$ satisfies (3) with $n(x, y) = 2$, as $d(T^2 0, T^2 x) = \frac{1}{3} - \frac{2}{3} \cdot 2 < \frac{1}{3} \cdot 1 = \frac{1}{2}d (0, T0)$ and $d(T^2 y, T^2 x)\frac{1}{2}d(y, x)$ for $y \neq 0$. However, $T$ has not fixed points.

Now we will show that the condition (3) may be much more weakened.

**Theorem 3.** Let $T$ be a continuous mapping on the compact metric space $M$ into itself satisfying the inequality

$$d(T^n x, T^n y) < \max_{0 \leq p, q, r, s, t \leq n} \left\{ d(T^p x, T^p y), d(T^q x, T^{q+1} x), d(T^r y, T^{r+1} y), \frac{1}{2}[d(T^s x, T^{s+1} y) + d(T^t y, T^{t+1} x)] \right\}$$

for some positive integer $n = n(x, y)$ and $x, y$ in $M$ for which the right-hand side of inequality is positive. Then $T$ has a unique fixed-point in $M$.

**Proof.** We may assume that the right-hand side of inequality (5) is positive for each $x, y$ in $M$. For if it is not positive, then $x = y = Tx$, which means that $T$ has a fixed-point. Let the mapping $F$ and the point $u$ be defined as in the proof of the theorem 1. and let $n = n(u, Tu)$. Then by (5) it follows

$$F(T^n u) = d(T^n u, T^n Tu) < \max_{0 \leq p, q, r, s, t \leq n} \left\{ d(T^p u, TT^p u), d(T^q u, TT^q u), d(T^r u, TT^r u), \frac{1}{2}[d(T^s u, T^2 T^s u) + 0] \right\}.$$
Using the triangle inequality and (4) we obtain $F(T^m) < F(u)$, which is a contradiction with (4). Therefore, the right-hand side of (5) is zero for $x = u$ and $y = Tu$. Hence $Tu = u$. The uniqueness of the fixed-point follows easily. This completes the proof of the theorem.

REFERENCES

