FUNCTIONAL EQUATIONS OF GENERALIZED ASSOCIATIVITY,
BISYMMETRY, TRANSITIVITY AND DISTRIBUTIVITY

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In this paper we solve functional equations of generalized associativity (1),
bisymmetry (2), transitivity (3) and distributivity (4).

(1) \[ A(x, B(y, z)) = C(D(x, y), z) \]
(2) \[ A(B(x, y), C(u, v)) = D(E(x, u), F(y, v)) \]
(3) \[ A(B(x, y), C(y, z)) = D(x, z) \]
(4) \[ A(x, B(y, z)) = C(D(x, y), E(x, z)) \]

We say that functional equation is generalized if in this equation every
unknown function occurs exactly once. We assume that unknown functions are
grupoids, defined on some nonempty set \( S \).

Despite numerous literature (see for example [1]) about the subject, these
equations are not solved under such weak conditions.

Here we continue our work, started in [2], on various generalized functionel
equations on grupoids.

* * *

Following common practice, for a given function \( P : S \to S' \), we define an
equivalence relation \( \text{ker} P \) on \( S \) as

\[ x \text{ ker } P y \quad \text{iff} \quad Px = Py \]

Bijecton \( f_p : S/\text{ker} P \to P(S) \), defined by \( f_p(x^{\text{ker} P}) = Px \), is naturally
connected with \( P \).

If an equivalence \( \alpha \) on \( S^2 \) is given, then:

\[ (a, b, c) \alpha_1 (p, q, r) \quad \text{iff} \quad a = p \quad (b, c)\alpha (q, r) \]
\[ (a, b, c) \alpha_3 (p, q, r) \quad \text{iff} \quad (a, b)\alpha (p, q) \quad \text{and} \quad c = r \]
If \( \beta \) is also an equivalence on \( S^2 \), then:
\[
(a, b, c, d) \Delta_1 \beta (p, q, r, s) \iff (a, b) \alpha (p, q) \text{ and } (c, d) \beta (r, s)
\]
\[
(a, b, c, d) \Delta_2 \beta (p, q, r, s) \iff (a, c) \alpha (p, r) \text{ and } (b, d) \beta (q, s)
\]
\[
(a, b, c) \Delta_3 \beta (p, q, r) \iff (a, b) \alpha (p, q) \text{ and } (b, d) \beta (q, r)
\]
\[
(a, b, c) \Delta_4 \beta (p, q, r) \iff (a, b) \alpha (p, q) \text{ and } (a, c) \beta (p, r)
\]
If \( \alpha_1, \ldots, \alpha_n \) are equivalences on \( S^2 \), then:
\[
(a_1, \ldots, a_{n+1}) \Delta (\alpha_1, \ldots, \alpha_n)(b_1, \ldots, b_{n+1}) \iff 
\text{ for all } k = 1, \ldots, n \quad (a_k, a_{k+1}) \alpha_k (b_k, b_{k+1})
\]
Also let:
\[
(a, b, c) \delta (p, q, r) \iff a = p \text{ and } c = r
\]
\[
(a_1, \ldots, a_{n+1}) \gamma (b_1, \ldots, b_{n+1}) \iff a_1 = b_1 \text{ and } a_{n+1} = b_{n+1}
\]
Suppose \( \alpha, \beta, \alpha_1, \ldots, \alpha_n \) are equivalences on \( S^2 \), \( \tau \) and \( \pi \) are equivalences on \( S^3(S^4) \) and \( \sigma \) is an equivalence on \( S^{n+1} \) for some \( n \in \mathbb{N} \). The following functions are not always well defined but we can easily formulate conditions under which they are. Whenever we use these functions we will prove that they are well defined.
\[
\Gamma_A : (x, (y, z)^\alpha) \mapsto (x, y, z)^\tau
\]
\[
\Gamma_C : ((x, y)^\alpha, z) \mapsto (x, y, z)^\tau
\]
\[
\Gamma_1 : ((x, y)^\alpha, (u, v)^\beta) \mapsto (x, y, u, v)^\pi
\]
\[
\Gamma_2 : ((x, u)^\alpha, (y, v)^\beta) \mapsto (x, y, u, v)^\pi
\]
\[
\Gamma_3 : ((x, y)^\alpha, (y, z)^\beta) \mapsto (x, y, z)^\tau
\]
\[
\Gamma_4 : (x, y) \mapsto (x, a, y)^\tau \text{ for some } a \in S
\]
\[
\Gamma_5 : ((x, y)^\alpha, (x, z)^\beta) \mapsto (x, y, z)^\tau
\]
\[
\Gamma_6 : ((x_1, x_2)^\alpha_1, \ldots, (x_n, x_{n+1})^\alpha_n) \mapsto (x_1, \ldots, x_{n+1})^\sigma
\]
\[
\Gamma_7 : (x, y) \mapsto (x, a_2, \ldots, a_n, y)^\sigma \text{ for some } a_2, \ldots, a_n \in S
\]

The functional equation of generalized associativity (1) is solved in [2] (Th 5). The solution given here (Th 1) is different but similar to that in [2], so we do not give the proof here. The present solution has some advantages, it is simpler and similar in form to solutions of other equations.

**Theorem 1.** The general solution of the generalized associativity equation (1), is given by:

\[ B \text{ and } D \text{ are arbitrary grupoids on } S \]

\[
A(x, y) = \begin{cases} 
    fT\Gamma_A(x, f_B^{-1} y) & \text{for } (x, y) \in S \times B(S, S) \\
    \text{arbitrary} & \text{otherwise}
\end{cases}
\]

\[
C(x, y) = \begin{cases} 
    fT\Gamma_C(f_D^{-1} x, y) & \text{for } (x, y) \in D(S, S) \times S \\
    \text{arbitrary} & \text{otherwise}
\end{cases}
\]
where $T$ is an arbitrary 3-grupoid such that:

$$(\ker B)_1 \lor (\ker D)_3 \subset \ker T$$

**Theorem 2.** The general solution of the generalized bisymmetry equation (2) is given by:

$$B, C, E \text{ and } F \text{ are arbitrary grupoids on } S$$

$$A(x,y) = \begin{cases} f_T \Gamma_1(f_B^{-1}x, f_C^{-1}y) & \text{for } (x,y) \in B(S,S) \times C(S,S) \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

$$D(x,y) = \begin{cases} f_T \Gamma_2(f_E^{-1}x, f_F^{-1}y) & \text{for } (x,y) \in E(S,S) \times F(S,S) \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

where $T$ is an arbitrary 4-grupoid such that:

$$(\ker B \Delta_1 \ker C) \lor (\ker E \Delta_2 \ker F) \subset \ker T$$

**Proof:** (i) It can be easily checked that for given $B, C, E, F$ and $T$, sixtuple $(A, B, C, D, E, F)$, satisfying (6) is a solution of (2).

(ii) Let $A, B, C, D, E, F$ be grupoids satisfying (2) and let:

$$T(x,y,u,v) = A(B(x,y),C(u,v))$$

Then from $(a,b,c,d)(\ker B \Delta_1 \ker C)(p,q,r,s)$ it follows that $(a,b)\ker B(p,q)$ (i.e. $B(a,b) = B(p,q)$) and $(c,d) \ker C(r,s)$ (i.e. $C(c,d) = C(r,s)$).

It follows that $A(B(a,b),C(c,d)) = A(B(p,q),C(r,s))$ i.e. $(a,b,c,d) \ker T(p,q,r,s)$.

Similarly, from $(a,b,c,d)(\ker E \Delta_2 \ker F)(p,q,r,s)$ it follows that $(a,b,c,d) \ker T(p,q,r,s)$. This also means that $\Gamma_1$ and $\Gamma_2$ are well defined.

Let $x \in B(S,S)$ and $y \in C(S,S)$. Then there are $z,u,v,w$ such that $x = B(z,u)$ and $y = C(v,w)$ and $A(x,y) = \hat{A}(B(z,u),C(v,w)) = f_T((z,u,v,w)^{\ker T}) = f_T \Gamma_1((z,u)^{\ker B}(v,w)^{\ker C}) = f_T \Gamma_1(f_B^{-1}B(z,u), f_C^{-1}C(v,w)) = f_T \Gamma_1(f_B^{-1}x, f_C^{-1}y)$.

Analogously we can prove $D(x,y) = f_T \Gamma_2(f_E^{-1}x, f_F^{-1}y)$.

In [2] (example 1) it is proved that we can obtain the general solution of the generalized associativity on quasigroups, from the general solution in the grupoid case. We can prove that the same holds for the generalized bisymmetry (see also [3]).

**Theorem 3.** The general solution on quasigroups, of the generalized bisymmetry equation, is given by:

$$A(x,y) = A_1 x + A_2 y$$

$$B(x,y) = A_1^{-1}(A_1 B_1 x + A_1 B_2 y)$$

$$C(x,y) = A_2^{-1}(A_2 C_1 x + A_2 C_2 y)$$

$$D(x,y) = D_1 x + D_2 y$$

$$E(x,y) = D_1^{-1}(D_1 E_1 x + D_1 E_2 y)$$

$$F(x,y) = D_2^{-1}(D_2 F_1 x + D_2 F_2 y)$$
where $+$ is an arbitrary abelian group and $A_1, A_2, \ldots, F_1, F_2$ arbitrary permutations such that:
\[
A_1 B_1 = D_1 E_1 \\
A_1 B_2 = D_2 F_1 \\
A_2 C_1 = D_1 E_2 \\
A_2 C_2 = D_2 F_2
\]

**Theorem 4.** The general solution of the generalized transitivity equation (3) is given by:

\[
B \text{ and } C \text{ are arbitrary grupoids on } S \\
A(x,y) = \begin{cases} 
    f_T \Gamma_3(f_B^{-1} x, f_C^{-1} y) & \text{for } (x,y) \in \bigcup_{z \in S} (B(S,z) \times C(z,S)) \\
    \text{arbitrary} & \text{otherwise} 
\end{cases} \\
D(x,y) = f_T \Gamma_4(x,y)
\]

where $T$ is an arbitrary 3-grupoid such that:
\[
\ker B \Delta_3 \ker C \cup \delta \subset \ker T
\]

**Proof:** (i) For given $B, C$ and $T$ quadruple $(A,B,C,D)$ satisfying (7) is a solution of (3).

(ii) Let $A,B,C,D$ be grupoids satisfying (3) and let
\[
T(x,y,z) = A(B(x,y), C(y,z))
\]

Then from $(a,b,c)(\ker B \Delta_3 \ker C)(p,q,r)$ it follows that $(a,b)\ker B(p,q)$ (i.e. $B(a,b) = B(p,q)$) and $(b,c)\ker C(q,r)$ (i.e. $C(b,c) = C(q,r)$). It follows that $A(B(a,b), C(b,c)) = A(B(p,q)), C(q,r))$ i.e. $(a,b,c)\ker T(p,q,r)$.

Also, from $(a,b,c)\delta(p,q,r)$ it follows that $a = p$ and $c = r$. So $D(a,c) = D(p,r)$ and by (3), $A(B(a,b), C(b,c)) = A(B(p,q), C(q,r))$ i.e. $(a,b,c)\ker T(p,q,r)$.

Consequently $\Gamma_3$ and $\Gamma_4$ are well defined.

Let $x \in B(S,z)$ and $y \in C(z,S)$ for some $z \in S$. Then there are $u, v \in S$ such that $x = B(u,z)$ and $y = C(z,v)$. Also:
\[
A(x,y) = A(B(u,z), C(z,v)) = T(u,v,z) = f_T((u,z,v)^{\ker T}) = \\
= f_T \Gamma_3((u,z)^{\ker B}, (z,v)^{\ker C}) = f_T \Gamma_3(f_B^{-1} B(u,z), f_C^{-1} C(z,v)) = f_T \Gamma_3(f_B^{-1} x, f_C^{-1} y) \\
D(x,y) = A(B(x,z), C(z,y)) = T(x,z,y) = f_T((x,z,y)^{\ker T}) = f_T \Gamma_4(x,y)
\]

As in the case of associativity and bisymmetry equations, from the general solution of generalized transitivity equation in grupoid case, we can obtain the general solution in the quasigroup case (see [3]).
Theorem 5. The general solution on quasigroups, of the generalized transitivity equation is given by:

\[ A(x, y) = A_1 x \cdot A_2 y \]
\[ B(x, y) = A_1^{-1}(A_1 B_1 x - A_1 B_2 y) \]
\[ C(x, y) = A_2^{-1}(A_2 C_1 x - A_2 C_2 y) \]
\[ D(x, y) = D_1 x \cdot D_2 y \]

where \( \cdot \) is an arbitrary group (with an unit \( e \)) and \( A_1, A_2, \ldots, D_1, D_2 \) arbitrary permutations such that:

\[ A_1 B_1 = D_1 \]
\[ A_1 B_2 x \cdot A_2 C_1 x = e \]
\[ A_2 C_2 = D_2 \]

Contrary to generalized distributivity equation on quasigroups (see [4]), which is not solved, this equation in the grupoid case be solved easily.

Theorem 6. The general solution of the generalized distributivity equation is given by:

\[ B, D \text{ and } E \text{ are arbitrary on } S \]

\[ A(x, y) = \begin{cases} f_T \Gamma_A(x, f_B^{-1}) & \text{for } (x, y) \in S \times (S, S) \\ \text{arbitrary} & \text{otherwise} \end{cases} \]

\[ C(x, y) = \begin{cases} f_T \Gamma_0(f_D^{-1} x, f_E^{-1} y) & \text{for } (x, y) \in \bigcup_{z \in S} (D(z, S) \times E(z, S)) \\ \text{arbitrary} & \text{otherwise} \end{cases} \]

where \( T \) is an arbitrary 3-grupoid such that:

\[ (\ker B)_1 \lor (\ker D \Delta_1 \ker E) \subset \ker T \]

Proof: (i) For given \( B, D, E \) and \( T \), quintuple \( (A, B, C, D, E) \) satisfying (8) is a solution of (4).

(ii) Let \( A, B, C, D, E \) be grupoids satisfying (4) and let:

\[ T(x, y, z) = A(x, B(y, z)) \]

Then from \((a, b, c) (\ker B)_1 (p, q, r) \) it follows that \( a = p \) and \((b, c) \ker B(q, r) \) (i.e. \( B(b, c) = B(q, r) \)). Consequently \( A(a, B(b, c)) = A(p, B(q, r)) \) i.e. \((a, b, c) \ker T(p, q, r) \).

Also, from \((a, b, c) (\ker D \Delta_1 \ker E) (p, q, r) \) it follows that \((a, b) \ker D(p, q) \) (i.e. \( D(a, b) = D(p, q) \)) and \((a, c) \ker E(p, r) \) (i.e. \( E(a, c) = E(p, r) \)). It follows that \( C(D(a, b), E(a, c)) = C(D(p, q), E(p, r)) \) i.e. \( A(a, B(b, c)) = A(p, B(q, r)) \) which is the same as \((a, b, c) \ker T(p, q, r) \).

Consequently \( \Gamma_A, \Gamma_0 \) are well defined.
Let \( x \in S \) and \( y \in B(S,S) \). Then there are \( u,v \in S \) such that \( y = B(u,v) \) and:

\[
A(x,y) = A(x,B(u,v)) = T(x,u,v) = \]

\[
= f_T \Gamma_A(x,(u,v)^{ker B}) = f_T \Gamma_A(x,f_B^{-1}B(u,v)) = f_T \Gamma_A(x,f_B^{-1}y).
\]

If \( x \in D(z,S) \) and \( y \in E(z,S) \) for some \( z \in S \), then there are \( u,v \in S \) such that \( x = D(z,u) \) and \( y = E(z,v) \) and:

\[
C(x,y) = C(D(z,u),E(z,v)) = A(z,B(u,v)) = T(z,u,v) = \]

\[
= f_T((z,u,v)^{ker T}) = f_T \Gamma_5((z,u,v)^{ker D}(z,v)^{ker E}) = \]

\[
f_T \Gamma_5(f_D^{-1}D(z,u),f_E^{-1}E(z,v)) = f_T \Gamma_5(f_D^{-1}x,f_E^{-1}y).
\]

As an example that the above method of solving, can be extended to the case of \( n \)-ary groupoids, we give a general solution of so called generalized simple polytransitivity (see [5]):

\[
A(B^1(x_1,x_2),\ldots,B^n(x_n,x_{n+1})) = C(x_1,x_{n+1})
\]

**Theorem 7.** The general solution of the generalized simple polytransitivity (9) is given by:

\[
B^1, \ldots, B^n \text{ are arbitrary groupoids on } S
\]

\[
A(x_1,\ldots,x_n) = \begin{cases} 
 f_T \Gamma_6(f_B^{-1}x_1,\ldots,f_B^{-1}x_n) \text{ for } (x_1,\ldots,x_n) \in \partial_A \\
 \text{arbitrary otherwise}
\end{cases}
\]

\[
C(x,y) = f_T \Gamma_7(x,y)
\]

where \( \partial_A = \{(x_1,\ldots,x_n)|\exists z_1,\ldots,z_{n+1}(x_k = B^k(z_k,z_{k+1}) \text{ for all } k = 1,\ldots,n)\} \)

and \( T \) is an arbitrary \( n+1 \)-groupoid such that:

\[
\Delta(\ker B^1,\ldots,\ker B^n) \in \gamma \subset \ker T
\]

**Proof:** Similar to the proof of Th 4.

REFERENCES


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