ON p-GROUPS OF SMALL ORDER

Theodoros Exarchakos

Abstract. We prove that if $G$ is a finite non-abelian $p$-group of order $p^n$ ($p$ a prime number), $\leq 6$, then the order of $G$ divides the order of the group of automorphisms of $G$.

Introduction and notation

The conjecture “if $G$ is a finite non-cyclic $p$-group of order $p^n$, $n \geq 2$, then the order of $G$ divides the order of the group of automorphisms of $G”$ has been an interesting subject of research for a long time. Although a great number of papers have appeared on this topic, the conjecture still remains open. However, it has been established for abelian $p$-groups [14], for $p$-groups of class two [8], for non-cyclic metacyclic $p$-groups, $p \neq 2$ [3] and for some other classes of finite $p$-groups ([4, 5, 6, 7, 11]). In this paper we show that this conjecture is also true for all finite non-abelian $p$-groups of order $p^n$, $n \leq 6$ for every prime number $p$.

Throughout this paper, $G$ stands for a finite non-abelian $p$-group, of order $p^n$ ($p$ a prime number), with commutator subgroup $G'$ and center $Z$. The order of a group $X$ is denoted by $|X|$. We take the lower and the upper central series of $G$ to be:

$$G = L_0 \subseteq L_1 = G' \subseteq L_2 \subseteq \cdots \subseteq L_c = 0 \quad \text{and} \quad 1 = Z_0 \subseteq Z_1 = Z \subseteq Z_2 \subseteq \cdots \subseteq Z_c = G,$$

where $c$ is the class of $G$, $P(G) = \{x^p \mid x \in G\}$ and $|X|_p$ is the greatest power of $p$ which divides $|X|$.

The invariants of $G/G'$ are taken to be:

$$m_1 \geq m_2 \geq \cdots \geq m_t \geq 1 \quad \text{and} \quad |G/G'| = p^{m_t}.$$

The number $t$ is the number of generators of $G$. We denote by $A(G), I(G)$, $A_c(G)$, the group of automorphisms, inner automorphisms, central automorphisms of $G$ respectively. Hom $(G, Z)$ is the group of homomorphisms of $G$ into $Z$. The group $G$ has maximal class $c$, if $|G| = p^n$ and $c = n - 1$. $G$ is called a $PN$-group.

AMS Subject Classification (1980): Primary 20D 15, 20D 45
if $G$ has no non-trivial abelian direct factor. $G$ is metacyclic if it has a normal subgroup $H$ such that both $H$ and $G/H$ are cyclic.

First we give some results which we shall use very often throughout the proof of the theorem.

**Lemma 1.** [6] (i) If $G = H \times K$, where $H$ is abelian and $K$ is a $PN$-group, then

$$\left| A_c(G) \right| = \left| A_c(K) \right| \cdot \left| A(H) \right| \cdot \left| \text{Hom}(K, H) \right| \cdot \left| \text{Hom}(H, Z(K)) \right| .$$

(ii) If $G$ is a $PN$-group of class $c$ and $s$ is the number of invariants of $Z$, then

$$\left| A(G) \right| \geq p^{2s+c-1} \quad \text{and} \quad \left| A(G) \right| \geq \left| A_c(G) \right| \cdot p^{c-1} .$$

(iii) If $G$ is a $PN$-group and $\exp(G/G') \leq |Z|$, then $\left| A_c(G) \right| \geq |G/G'|$.

(iv) If the Frattini subgroup $\Phi(G)$ of $G$ is cyclic, then $\left| A(G) \right| \geq |G|$.

**Lemma 2.** [5] If $m_1 \geq m_2 \geq \cdots \geq m_t \geq 1$ are the invariants of $G/G'$, then $\exp G \leq p^{m_1+m_2(c-1)}$. For $t = 2$, $\exp Z \leq p^{m_1+m_2(c-1)-2}$ and $Z_{c-1} \leq \Phi(G)$ where $\Phi(G)$ is the Frattini subgroup of $G$.

**Lemma 3.** [2] If $m_1 \geq m_2 \geq \cdots \geq m_t \geq 1$ are the invariants of $G/G'$, then

$$p^{m_2} \geq \exp L_1/L_2 \geq \exp L_2/L_3 \geq \cdots \geq \exp L_{t-1}/L_t .$$

For $t = 2$, $L_1/L_2$ is cyclic of order at most $p^{m_2}$.

Now we prove some useful lemmas.

**Lemma 4.** Let $G = H \times K$, where $H$ is abelian and $K$ is a $PN$-group. Let $A, B, C, D$ be as in Lemma 1 with $A = A_c(K)$, $B = \text{Hom}(K, H)$, $C = A(H)$ and $D = \text{Hom}(H, Z(K))$. Then (i) $\left| A(G) \right| \geq \left| A(K) \right| \cdot |B| \cdot |C| \cdot |D|$ and (ii) $\left| A(G) \right| \geq p \cdot |I(G)| \cdot |B| \cdot |C| \cdot |D|$.

**Proof.** (i) Let $\bar{A} = \{ \bar{h} \mid \bar{h}(h, k) = (h, \theta(k)) \}, h \in H, k \in K, \theta \in A(K)\}$. Then $\bar{\theta}$ is an automorphism of $G$ for every $\theta \in A(K)$. So $\bar{A} \leq A(G)$. Since $A_c(G) < A(G)$, we get that $\bar{A} \cdot A_c(G) = AABCD = ABCD \leq A(G)$. But $|\bar{A} \cap A_c(G)| = |A_c(K)| = |A|$ and so

$$\left| A(G) \right| \geq |\bar{A} \cdot A_c(G)| = |\bar{A}| \cdot |A_c(G)| / |A| = |\bar{A}| \cdot |B| \cdot |C| \cdot |D| = |A(K)| \cdot |B| \cdot |C| \cdot |D| .$$

(ii) $I(G) = G/Z \simeq K/Z(K) \simeq I(K)$ and by [9] $|A(K)/I(K)| \geq p$.

Hence the result follows from (i).

**Lemma 5.** If $G$ has order $2^n$ and class $c = n - 2$, then $\left| A(G) \right| \geq |G|$.

**Proof.** Since $G$ has class $c = n - 2$, $|G/G'| \leq 2^3$. We may assume that $|Z| > 2$; otherwise, Lemma 4 (ii) gives $\left| A(G) \right| \geq 2 \cdot |I(G)| = 2 \cdot |G/Z|$.
2 \cdot 2^{n-1} = 2^n. \text{ If } |G/G'| = 2^2, \text{ then } G \text{ has a maximal subgroup } M \text{ which is cyclic [2]. So } \Phi(G) \text{ is cyclic, as } \Phi(G) < M. \text{ Then by Lemma 1 (iv) the result follows. If } |G/G'| = 2^2, \text{ exp } G/G' \leq 2^2 \leq |Z|, \text{ and by Lemma 1 (iii), } |A_c(G)| \geq 2^3. \text{ Then } |A(G)|^2 \geq |A(G)| \cdot 2^{c-1} \geq 2^3 \cdot 2^{n-3} = 2^n.

Now we prove our theorem.

THEOREM. If } G \text{ is a finite non-abelian group of order } p^n, p \text{ a prime number and } n \leq 6, \text{ then } |A(G)| \geq |G|.

Proof. By Lemma 4(i) we may assume that } G \text{ is a } PN\text{-group. If } Z = p, \text{ Lemma 4 (ii) gives } |A(G)| \geq p \text{ if } I(G) = p \text{ if } G/Z = p^n. \text{ By Theorem 1 in [5], if } n = 5, \text{ then } |A(G)| \geq p^3. \text{ So } n = 6. \text{ If } G \text{ has class 5, then } |G/G'| = p^2, \text{ exp } G/G = p \text{ and by Lemma 1 (ii) we get } |A(G)| \geq |A_c(G)| \cdot p^{c-1} \geq p^2 \cdot p^4 = p^6. \text{ Therefore } c \leq 4. \text{ For } c = 2, |A(G)| \geq |G| \text{ by [8], and so, } 3 \leq c \leq 4. \text{ If } Z \text{ is non-cyclic and } s \text{ is the number of invariants of } Z, \text{ then } s > 1, \text{ and Lemma 1 (ii) gives } |A(G)| \geq p^{2s+1} \geq p^3, \text{ as } c \geq 3. \text{ Finally, if } Z \leq \Phi(G), \text{ then there exists a maximal subgroup } M \text{ of } G \text{ such that } Z \leq M. \text{ Then } G = MZ. \text{ But } |A(M)| \geq p^5, \text{ since } |M| = p^5, \text{ and so, } |A(G)| \geq p \cdot |A(<)| \geq p^6 \text{ from [11]. Therefore we may assume that:}

| G \text{ is } PN\text{-group of order } p^6, |

| Z \text{ is cyclic of order greater than } p, |

| Z \leq \Phi(G) \text{ and } |

| 3 \leq c \leq 4. |

Consider the following cases:

(a) Take } c = 4. \text{ Let } G = L_0 > L_1 > L_2 > L_3 > L_4 = 1 \text{ be the lower central series of } G. \text{ Since } |L_i/L_{i+1}| \geq p \text{ for all } i = 1, 2, 3, \text{ we have } p^2 \leq |G/L_1| \leq p^5. \text{ If } |G/L_1| = p^2, \text{ then it has type } (p, p) \text{ and by Lemma 2, exp } Z \leq p^2. \text{ Also by Lemma 3, } L_1/L_2 \text{ has order } p \text{ and exp } L_i/L_{i+1} = p \text{ for all } i = 1, 2, 3. \text{ For } p = 2, \text{ the result follows from Lemma 5. Therefore we may assume that } p \neq 2. \text{ If } |Z| > p^2, Z \text{ is not cyclic, as } Z \leq p^2; \text{ a contradiction. Hence } |Z| = p^2. \text{ Then } |G/Z_3| = p^2, |Z_3/Z_2| = p, \text{ where } G = Z_4 > Z_3 > Z_2 > Z_1 = Z > Z_0 = 1 \text{ is the upper central series of } G. \text{ Since } L_1 \leq Z_3 \text{ and } |L_3| = |Z_3| = p^4 \text{ we get } L_1 = Z_3. \text{ Also } |L_1/L_2| = p \text{ and } L_2 \leq Z_2 \text{ gives } L_2 = Z_2. \text{ Hence } Z < L_2. \text{ Let } H \text{ be a normal subgroup of } G \text{ of order } p^3 \text{ and exponent } p. \text{ Then } H < Z_0 = L_1 \text{ and } |L_1/H| = p. \text{ So } L_1/H \leq Z(G/H), \text{ which gives } L_2 = [G, L_1] \leq H. \text{ Since } |L_2| = p^3 = |H|, \text{ we get } L_2 = H, \text{ and so, } Z < L_2 = H. \text{ Therefore, exp } Z = p \text{ and } Z \text{ is not cyclic; a contradiction. So } G \text{ has no normal subgroup } H \text{ of order } p^3 \text{ and exponent } p. \text{ Then } G \text{ is metacyclic and the result follows by [4].}

If } |G/L_1| = p^3, \text{ then exp } G/L_1 \leq p^2 \leq |Z|, \text{ and Lemma 1 (iii) gives } |A_c(G)| \geq p^3. \text{ Then } |A(G)| \geq p^3 \cdot p^{c-1} = p^6.

(b) Take } c = 3. \text{ Let } G = L_0 > L_1 > L_2 > L_3 = 1 \text{ be the lower central series of } G. \text{ Then } p^2 \leq |G/L_1| \leq p^4. \text{ If } |G/L_1| = p^2, \text{ exp } Z \leq p^{c-2} = p \text{ and } Z \text{ is not cyclic; a contradiction. Hence } |G/L_1| \geq p^3 \text{ and so, } |A_c(G)| \geq p^3 \text{ in all cases as } G/L_1 \text{ is not cyclic.
Then $|A(G)|_p \geq |A_c(G)| \cdot |G/Z_2| \geq p^3 |G/Z_2|$. Therefore we may assume that $|G/Z_2| = p^2$; otherwise the theorem holds.

Let $|G/L_1| = p^2$. Then $G/L_1$ has either type $(p^2,p)$ or $(p,p,p)$. In the first case, Lemma 3 gives $|L_1/Z_2| = p$, $|L_2| = p^2$ and exp $L_2 = p$. Since $L_2 \leq Z$, $Z$ is not cyclic; a contradiction. If $G/L_1$ has type $(p,p,p)$ then exp $(L_1/L_2) = pL_2 = p$, so that $L_1 \leq p^2$. Also $L_1 = \Phi(G)$ and $Z \leq \Phi(G) = L_1$. Therefore, exp $Z \leq p^2$, and we may assume that $Z$ is cyclic of order $p^2$. Since $|G/L_1| = p^3$, $|L_1/L_2| \geq p$, we get that $|L_2| \leq p^2$. If $|L_2| = p$, then $Z$ is not cyclic, as $L_2 \leq Z$ and exp $L_2 = p$. Therefore we may assume that $L_2 = p$ and $L_2$ is the only subgroup of $Z$ of order $p$. Since $G/L_1$ has type $(p,p,p), G$ can be generated by 3 elements $a, b, c$ such that $\alpha^p, b^p, c^p$ are elements of $L_1$. But $|G/Z_2| = p^2$. So we can chose $a, b, c$ such that $G = \langle a, b, c \rangle$, $c^p \in Z$, $c \in Z_2$. Then $[\alpha, c], [b, c]$ are elements of $Z$ of order $p$, and so $[\alpha, c], [b, c]$ are elements of $L_2$. Since $x^p \in Z_2$, for every $x \in Z_1$, we have that $[\alpha, b]^p \in L_2$. If $[\alpha, b]^p = 1$, exp $L_1 = p$, and $Z$ is not cyclic, as $Z \leq L_1$. Let $[\alpha, b]^p \neq 1$. Then $L_2 = \langle [\alpha, b] \rangle$. But $L_1 = \langle [\alpha, b], [\alpha, c], [b, c], L_2 \rangle$ [1, Lemma 1.1] and so $L_1 = \langle [\alpha, b] \rangle$. Then $L_1$ is cyclic; a contradiction, as $|L_1| = p^2$ and exp $L_1 \leq p^2$.

Let $|G/L_1| = p^4$. If exp$(G/L_1) \leq p^2 \leq |Z|$, then $|A_c(G)| \geq p^4$ (Lemma 1 (iii)), and so, $|A(G)|_p \geq p^4 \cdot p^{-1} = p^3$. Therefore, we may assume that $G/L_1$ has type $(p^3,p)$ and $|Z| = p^2$. Then $|L_1/L_2| = p$ and $G$ can be generated by two elements $\alpha, b$ such that $\alpha^p \in L_1$, $b^p \in L_1$ and $\alpha^p \notin L_1$, $b \notin L_1$. Also $L_2 \leq Z$ and $L_2$ is the only subgroup of $G$ of order $p$. Since $G/Z_2$ is elementary abelian of order $p^2$, $\Phi(G) \leq Z_2$, and so, $\Phi(G) = Z_2$. But $L_1 Z \leq Z(Z_2)$ and $|Z_2/L_1| = p$ gives that $Z_2$ is abelian. As $G = \langle \alpha, b \rangle$ and $\alpha^p \in Z$, $b^p \in Z$, we get $Z_2 = \langle [\alpha, b], b^p, \alpha^p, Z \rangle$. If $\alpha^p \in Z$, then $\alpha^p \in L_1$; contradiction. Since $Z_2$ has order $p^4$ and $[\alpha, b] = b^p$, if and only if $b^p \in Z$, we have to assume that $b^p \in Z$. On the other hand, if $\alpha^p = 1$, then $\langle \alpha^p \rangle$ is the only subgroup of $Z$ of order $p$, and so, $L_2 = \langle \alpha^p \rangle$. Then $\alpha^p \in L_1$ a contradiction. So $\alpha^p \neq 1$ and since $Z$ is cyclic of order $p^2$ we get that $Z = \langle \alpha^p \rangle$, $L_2 = \langle \alpha^p \rangle$ and $\alpha$ has order $p^4$. Since $G$ has order $p^6$, $b^p \notin \langle \alpha^p \rangle$. This contradiction proves the theorem.

REFERENCES


Université of Athens
33, Ippocratus Street
Athens, Greece

(Received 01 10 1988)