NOTE ON GENERALIZING PREGROUPS

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Abstract. Let $P$ be a pree which satisfies the first four axioms of Stallings' pregroup. Then the following three axioms are equivalent:

[K] If $ab$, $bc$ and $cd$ are defined, and $(ab)(cd)$ is defined, then $(ab)c$ or $(bc)d$ is defined.

[L] Suppose $V = [x,y]$ is reduced and suppose $y = ab = cd$ where $xa$ and $xc$ are defined. Then $a^{-1}c$ is defined.

[M] Suppose $W = [x,y,z]$ is reduced. Then $W$ is not reducible to a word of length one.

1. Introduction. Let $P$ be a pree that is, let $P$ be a nonempty set with a partial operation $m : D \to P$ where $D \subset P \times P$. [We say $pq$ is defined if $(p,q) \in D$ and we will usual denote $m(p,q)$ by $pq$.] The universal group $G(P)$ of the pree $P$ is the group with the following presentation:

$$G(P) = gp[P; z = xy \text{ where } xy \text{ is defined and } z = m(x,y)].$$

In other words, the generators of $G(P)$ are the elements of $P$ and the defining relations of $G(P)$ come from the partial operation $m$ on $P$. A pree $P$ is said to be group-embeddable if $P$ can be embedded in its universal group $G(P)$. (See Rimlinger [2].)

Stallings in [4] defined a collection of press, called pregrouops, which guarantees such an embedding. Specifically, a pree $P$ is a pregroup if it satisfies the following five axioms:

[$P_1$] There exists an identity element $1 \in P$ such that, for all $p \in P$, $1p$ and $p1$ are defined and $1p = p = p1$.

[$P_2$] For each $p \in P$ there exists $p^{-1} \in P$ such that $pp^{-1}$ and $p^{-1}p$ are defined and $pp^{-1} = p^{-1}p = 1$.

[$P_3$] If $pq$ is defined, then $q^{-1}p^{-1}$ is defined and $(pq)^{-1} = q^{-1}p^{-1}$.

[$P_4$] Supposing $ab$ and $bc$ are defined, then $a(bc)$ is defined if and only if $(ab)c$ is defined, in which case the two are equal.

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If $ab, bc$ and $cd$ are defined, then either $(ab)c$ or $(bc)d$ are defined.

Kusner and Lipschutz in [1] generalized Stallings’ result by weakening his last axiom $[P_5]$. Specifically, they proved that a pre-group $P$ is group-embeddable if it satisfies axioms $[P_1]$ through $[P_4]$ and the following two axioms:

$[Q_5]$ If $a_1a_2$, $a_2a_3$, $a_3a_4$ and $a_4a_5$ are defined, then at least one of $(a_1a_2)a_3$, $(a_2a_3)a_4$, $(a_3a_4)a_5$ is defined.

$[K]$ If $ab$, $bc$ and $cd$ are defined and $(ab)(cd)$ is defined, then $(ab)c$ or $(bc)d$ is defined.

**Example 1.** Let $A$ and $B$ be groups which intersect in a subgroup $H$. The amalgam $P = A \cup_H B$ is the classical example of a pregroup. Here $G(P)$ is the free product of $A$ and $B$ with $H$ amalgamated.

**Example 2.** Let $A, B, C$ be groups where $A$ and $B$ intersect in a non-trivial subgroup $H$, and $B$ and $C$ intersect in a non-trivial subgroup $K$. Also, suppose $B = H \oplus K$. Consider the amalgam $P = A \cup_H B \cup_K C$. Then $P$ is not a pregroup. For example, let $a \in A - H$, $h \in H$, $k \in K$, $c \in B - K$ where $h \neq 1$ and $k \neq 1$. Then $ah, hk$ and $kc$ are each defined, but neither $(ah)k$ nor $(hk)c$ is defined. On the other hand, $P$ does satisfy axioms $[P_1]$ through $[P_4]$ and $[Q_5]$ and $[K]$. Moreover, $G(P)$ is the tree product of the $A, B$ and $C$ with $H$ and $K$ amalgamated.

**Example 3.** Let $T_n = (A_i; H_{rs})$ be a tree graph of groups with vertices $A_i$ with edges $H_{rs}$, and with diameter $n$. (Here $H_{rs}$ is a subgroup of $A_r$ and $A_s$). Let $P = \cup_i A_i$ be the amalgam of the groups in $T_n$. It is known (cf. Serre [3]) that $P$ is group-embeddable where $G(P)$ is tree product of the groups $A_i$ with the semigroups $H_{rs}$ amalgamated.

The group-embeddable pre-group $P$ in Example 3 is neither a pregroup nor satisfies axiom $[Q_5]$. However, it does satisfy axiom $[K]$ and the following axiom:

$[T_k]$ If $a_1a_2$, $a_2a_3$, $\ldots$, $a_{n+1}a_{n+2}$ are defined, then at least one of $(a_1a_2)a_3$, $(a_2a_3)a_4$, $\ldots$, $(a_na_{n+1})a_{n+2}$ is defined.

The question of whether axioms $[K]$ and $[T_n]$, together with axioms $[P_1]$ through $[P_4]$, are sufficient to guarantee that a pre-group $P$ is group-embeddable is still an open question. This paper concerns axiom $[K]$ and the following two axioms:

$[L]$ Suppose $V = [x, y]$ is reduced and suppose $y = ab = cd$ where $xa$ and $xc$ are defined. Then $a^{-1}c$ is defined.

$[M]$ Suppose $W = [x, y, z]$ is reduced. Then $W$ is not reducible to a word of length one.

Specifically, we prove the following theorem.

**Main Theorem:** Let $P$ be a pre-group which satisfies axioms $[P_1]$ through $[P_4]$. Then axioms $[K]$, $[L]$ and $[M]$ are equivalent.

In other words, if $P$ satisfies one of $[K]$, $[L]$, $[M]$, then it satisfies all three axioms. (We emphasize that the pre-group $P$ in the Main Theorem need not satisfy axiom $[P_5]$, $[Q_5]$ or $[T_n]$ for $[K]$, $[L]$ and $[M]$ to be equivalent).
2. Notation, Preliminaries. Throughout this section \( P \) denotes a preE which satisfies axioms \([P_1]\) through \([P_d]\).

Let \( X = [x_1, x_2, \ldots, x_n] \) be an \( n \)-tuple of elements of \( P \). Then \( X \) is called a word of length \( n \). The word \( X \) is said to be reduced if no pair \( x_i x_{i+1} \) is defined. On the other hand, if some \( x_i x_{i+1} \) is defined, then \( Y = [x_1, \ldots, x_i x_{i+1}, \ldots, x_n] \) is said to be obtained from \( X \) by an elementary reduction.

The triple \( abc \) is said to be defined if \( ab \) and \( bc \) are defined and either \((ab)c \) or \( a(bc) \) is defined. (By \([P_3]\), if \( abc \) is defined, then \( abc = (ab)c = a(bc) \).)

Suppose \( X = [x_1, x_2, \ldots, x_n] \) and \( A = [a_1, a_2, \ldots, a_{n-1}] \) are words such that each triple \( a_i^{-1} x_i a_i \) is defined (where \( a_0 = a_n = 1 \)). Then the interleaving of \( X \) by \( A \), denoted by \( X * A \), is said to be defined and

\[
X * A = [x_1 a_1, a_1^{-1} x_2 a_2, \ldots, a_{n-1}^{-1} x_n]
\]

We write \( X * A * B \) for \((X * A) * B \).

A word \( X \) is said to be reducible to a word \( Z \) if \( Z \) can be obtained from \( X \) by a sequence consisting of interleavings and elementary reductions. [Observe that if \( X \) is reducible to \( Z \) then \( X \) and \( Z \) represent the same element in the universal group \( G(P) \) of \( P \).]

Stallings proved the following in \([4]\):

**Lemma A.** Suppose \( P \) satisfies axioms \([P_1]\) through \([P_d]\). Then:

1. \( (x^{-1})^{-1} = x \) for every \( x \) in \( P \).
2. If \( ax \) is defined, then \( a^{-1}(ax) \) is defined and \( a^{-1}(ax) = x \). Dually, if \( xa \) is defined, then \((xa)a^{-1} \) is defined and \((xa)a^{-1} = x \).
3. If \( xa \) and \( a^{-1}y \) are defined, then \( xy \) is defined if and only if \( (xa)(a^{-1}y) \) is defined; in which case \( xy = (xa)(a^{-1}y) \).

A word \( X \) in \( P \) is said to be fully reduced if \( X \) is reduced and \( X * A_1 * A_2 * \ldots * A_m \) is reduced whenever defined. Every word \( X \) of length \( n = 1 \) is automatically reduced and fully reduced. Lemma A(3) immediately implies:

**Lemma B.** \( X = [x, y] \) is reduced if and only if \( X \) is fully reduced.

**Example 4.** Consider groups \( A = F \oplus G, B = G \oplus H \) and \( C = H \oplus F \), where \( F, G, H \) are nontrivial subgroups. Let \( P = A \cup B \cup C \). Then \( P \) is a preE which satisfies axioms \([P_1]\) through \([P_d]\). Let \( f_1, f_2 \in F, g \in G, h \in H \) be nontrivial elements. Then \( X = [f_1 g^{-1}, g h, h^{-1} f_2] \) is a reduced word of length three. Let \( A = [g, h^{-1}] \). Then \( X * A = [f_1, 1, f_2] \) is reducible to a word \( Z = f_1 f_2 \) of length one. Thus, by our Main Theorem, \( P \) does not satisfy any of the axioms \([K]\), \([L]\) or \([M]\).

3. Proof of Main Theorem. First we show that \([K]\) is equivalent to \([M]\). Suppose \([M]\) does not hold. Then there exists a reduced word \( X \) of length three which is reducible to a word \( Z \) of length one. Thus there exist words \( A_1, \ldots, A_m \) and \( B \) such that:
(1) $Y = X * A_1 \cdots * A_m$ is reduced (2) $Y * B$ is not reduced and, after an elementary reduction, $Y * B$ is reducible to $Z$.

Suppose $Y = [x, y, z]$ and $B = [a, b]$. Then $Y * B = [xa, a^{-1}yb, b^{-1}z]$ is not reduced. Say $(xa)(a^{-1}yb)$ is defined. By Lemma A(3), $(xa)(a^{-1}yb) = x(yb)$. By Lemma $B$,
\[(xa)(a^{-1}yb), b^{-1}z] = [x(yb), b^{-1}z]\]
is reducible to $Z$ if and only if $(x(yb))(b^{-1}z)$ is defined. The 4-tuple
\[[x, yb, b^{-1}, z]\]
satisfies the hypothesis of axiom $[K]$. If axiom $[K]$ holds then either
\[x(yb)b^{-1} = xy \text{ or } (yb)b^{-1}z = yz\]
is defined. This contradicts the fact that $Y$ is reduced. Thus $[K]$ cannot hold. Accordingly, $[K]$ implies $[M]$.

On the other hand, suppose $[K]$ does not hold. Then there exist $a, b, c, d$ such that $ab, bc, cd$ and $(ab)(cd)$ are defined but neither $(ab)c$ nor $(bc)d$ are defined. By [P3], $a(bc)$ is not defined. Thus $X = [a, bc, d]$ is reduced. Let $A = [b, c^{-1}]$. Then
\[X * A = [ab, b^{-1}(bc)c^{-1}, cd] = [ab, a, cd]\]
is reducible to a word of length one. Thus $[M]$ does not hold. Accordingly, $[K]$ and $[M]$ are equivalent.

Next we show that $[K]$ and $[L]$ are equivalent. Suppose $[K]$ holds. Furthermore, suppose $V = [x, y]$ is reduced and suppose $y = ab = cd$ where $xa$ and $xc$ are defined. By Lemma A(2), $c^{-1}(cd) = c^{-1}(ab)$ is defined. By [P3], $c^{-1}x^{-1}$ is defined, and by Lemma A(2), $(c^{-1}x^{-1})x = c^{-1}$ is defined. Thus the 4-tuple
\[[c^{-1}x^{-1}, x, a, b]\]
satisfies the hypothesis of axiom $[K]$. The second triple $xab = xy$ is not defined since $V$ is reduced. Thus the first triple $(c^{-1}x^{-1})xa = c^{-1}a$ is defined. Thus $[K]$ implies $[L]$.

On the other hand, suppose $[L]$ holds. Suppose $ab, bc, cd$ and $(ab)(cd)$ are defined, and $(ab)c$ is not defined. We need to show that $b(cd)$ is defined for $[K]$ to hold. Note that
\[X = [ab, c] = [ab, (cd)d^{-1}] = [ab, b^{-1}(bc)]\]
is reduced where $(ab)(cd)$ and $(ab)b^{-1}$ are defined. By axiom $[L]$, $(b^{-1})^{-1}(cd) = b(cd)$ is defined. Thus $[L]$ implies $[K]$. Therefore $[K]$ and $[L]$ are equivalent.

Accordingly, $[K]$, $[L]$ and $[M]$ are equivalent and our main theorem is proved.
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