ON CONVERGENCE OF DERIVATIVES OF LINEAR COMBINATIONS
OF MODIFIED LUPAS OPERATORS

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Abstract. We study some direct theorems in the simultaneous approximation by certain linear combinations of modified Lupas operators. We also consider a class of unbounded functions with growth of order of $t^q$.

1. Introduction

Motivated by Derriennic [1], Sahai and Prasad [4] proposed modified Lupas operators defined, for functions integrable on $[0, \infty)$ by

$$(L_n f)(x) = (n - 1) \sum_{\nu=0}^{\infty} p_{n, \nu}(x) \int_{0}^{\infty} p_{n, \nu}(t) f(t) dt,$$

where

$$p_{n, \nu}(x) = \binom{n + \nu - 1}{\nu} x^{\nu} (1 + x)^{-1 - \nu}.$$  

It turns out that the order of approximation by these operators is at best $O(1/n)$, howsoever smooth the function may be. With the aim of bettering the said rate of approximation, May [2] and Rathore [3] have described a method for forming linear combination linear of positive operators. The approximation process follows:

$$L_n f, k, x = \sum_{j=0}^{k} C(j, k) L_{d_j n} f, x,$$

where $d_0, d_1, d_2, \ldots d_k$ are arbitrary but fixed distinct positive integers. We define

$$C(j, k) = \prod_{i=0}^{k} \frac{d_j}{d_j - d_i}, \quad k \neq 0 \text{ and } C(0, 0) = 1.$$  

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The object of the present paper is to study the problem of simultaneous approximation by the above linear combination of modified Lupas operators.

Throughout this paper \((a, b) \subset [0, \infty)\) denotes an open interval containing the closed interval \([a, b]\). The superscript \((r)\), \([\lambda]\) and \(\|\cdot\|\) stand for the \(r\)-th derivative of the function, maximum integer not exceeding \(\lambda\) and the sup-norm on \([a, b]\) respectively.

2. Auxiliary results

We shall need the following results:

**Lemma 2.1.** [4]. Let

\[
T_{n,m} = (n - r - 1) \sum_{\nu=0}^{\infty} p_{n+r,\nu}(x) \int_{0}^{\infty} p_{n-r,\nu+r}(t)(t - x)^m dt.
\]

Then

\[
T_{n,0} = 1, \quad T_{n,1} = \frac{(r + 1)(1 + 2x)}{(n - r - 2)}, \quad n > (r + 2)
\]

\[
(n - m - r - 2)T_{n,m+1} = x(1 + x)T_{n,m}^1 + 2mT_{n,m-1} + (m + r + 1)(1 + 2x)T_{n,m}; \quad n > m + r + 2.
\]

And hence \(T_{n,m} = O(n^{-(m+1)/2})\).

**Lemma 2.2.** [4]. For \(r = 0, 1, 2, \ldots\) we have

\[
(L_n^{(r)}f)(x) = \frac{(n - r - 1)(n + r - 1)}{(n - 1)(n - 2)} \sum_{\nu=0}^{\infty} p_{n+r,\nu}(x) \int_{0}^{\infty} p_{n-r,\nu+r}(t)f^{(r)}(t)dt.
\]

**Lemma 2.3.** [2]. If \(C(j,k), j = 0, 1, 2 \ldots k\) are defined as in (1.3), then

\[
\sum_{j=0}^{k} C(j,k)d_j^m = \begin{cases} 1 & m = 0 \\ 0 & m = 1, 2, \ldots k. \end{cases}
\]

3. Main results

**Theorem 3.1.** Let \(f\) be integrable on \([0, \infty)\) admitting \((2k + r + 2)\)-th derivative at a point \(x \in [0, \infty)\) with \(f^{(r)}(x) = O(x^\alpha)\), where \(\alpha\) is a positive integer not less that \(2k + 2\), as \(x \to \infty\). Then

\[
\lim_{n \to \infty} n^{k+1} [L_n^{(r)}(f, k, x) - f^{(r)}(x)] = \sum_{i=1}^{2k+2} Q(i, k, r, x) f^{(i+r)}(x), \quad (3.1)
\]

\[
\lim_{n \to \infty} n^{k+1} [L_n^{(r)}(f, k + 1, x) - f^{(r)}(x)] = 0, \quad (3.2)
\]
where \( Q(i,k,r,x) \) are certain polynomials in \( x \) of degree at most \( i \). Furthermore if \( f^{(2k+r+2)} \) exists and is continuous on \( (1,b) \) then (3.1) and (3.2) hold uniformly on \([a,b]\).

**Proof.** By Lemma 2.2. and Taylor’s expansion of \( f \), we are led to

\[
\sum_{j=0}^{k} C(j,k) \frac{(d_jn - 1)! (d_jn - 2)!}{(d_jn + r - 1)! (d_jn - r - 2)!} L_{d_jn}(f;x) - f^{(r)}(x)
\]

\[
= \sum_{j=0}^{k} C(j,k) \left[ (d_jn - r - 1) \sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x) \int_{0}^{\infty} p_{d_jn-r,\nu+r}(t) \left[ f^{(r)}(t) - f^{(r)}(x) \right] dt \right]
\]

\[
= \sum_{j=0}^{k} C(j,k)(d_jn - r - 1) \sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x) \int_{0}^{\infty} p_{d_jn-r,\nu+r}(t) \left\{ \sum_{i=1}^{2k+2} \frac{f^{(i+r)}(x)}{i!} (t-x)^i + \varepsilon(t-x)(t-x)^{2k+2} \right\} dt
\]

\[
= \sum_{i=1}^{2k+2} \frac{f^{(i+r)}(x)}{i!} \sum_{j=0}^{k} C(j,k)(d_jn - r - 1) \sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x) \int_{0}^{\infty} p_{d_jn-r,\nu+r}(t) (t-x)^i dt
\]

\[+ \sum_{j=0}^{k} C(j,k)(d_jn - r - 1) \sum_{\nu=0}^{\infty} p_{d_jn+r,\nu}(x) \int_{0}^{\infty} p_{d_jn-r,\nu+r}(t) \varepsilon(t-x)(t-x)^{2k+2} dt
\]

\[= \sum_{i=1}^{2k+2} \frac{f^{(i+r)}(x)}{i!} \sum_{j=0}^{k} C(j,k) T_{d_jn,i}(x) + E_{n,r,k}(x),
\]

where

\[
\varepsilon(t-x) = (t-x)^{2k+2} \left( f^{(r)}(t) - \sum_{i=0}^{2k+2} \frac{f^{(i+r)}(x)}{i!} (t-x)^i \right) \text{ for } t \neq x
\]

\[= 0, \text{ otherwise.}
\]

Using Lemma 2.1 and 2.3,

\[
\sum_{i=1}^{2k+2} \frac{f^{(i+r)}(x)}{i!} \sum_{j=0}^{k} C(j,k) T_{d_jn,i}(x)
\]

\[= \sum_{i=0}^{2k+2} \frac{f^{(i+r)}(x)}{i!} \sum_{j=0}^{k} C(j,k) O \left( \frac{1}{(d_jn)^{(i+1)}/2} \right)
\]

\[= n^{-k-1} \sum_{i=1}^{2k+2} Q(i,k,r,x) f^{(i+r)}(x),
\]
where $Q(i, k, r, x)$ are certain polynomials in $x$ of degree at most $i$.

To prove (3.1) it suffices to show that $n^{k+1} E_{n, r, k}(x) \to 0$ for sufficiently large $n$. For arbitrary $\varepsilon > 0, A > 0$, there exists a $\delta > 0$ such that $|\varepsilon(T - X)| < \varepsilon$ for $x \leq A$ and $|t - x| < \delta$. Now

$$
E_{n, r, k}(x) = \sum_{j=0}^{k} C(j, k) (d_{jn} - r - 1) \sum_{\nu=0}^{\infty} p_{d_{jn} + r, \nu}(x) \\
\left( \int_{|t-x|<\delta} p_{d_{jn} - r, \nu + r}(t) \varepsilon(t-x)(t-x)^{2k+2} dt \\
+ \int_{|t-x|>\delta} p_{d_{jn} - r, \nu + r}(t) \varepsilon(t-x)(t-x)^{2k+2} dt \right) \\
= I_1 + I_2 \quad \text{(say)}.
$$

To estimate $I_1$, using Lemma 2.1 we get

$$
|I_1| \leq \sum_{j=0}^{k} \left| C(j, k) \right| (d_{jn} - r - 1) \sum_{\nu=0}^{\infty} p_{d_{jn} + r, \nu}(x) \int_{|t-x|<\delta} p_{d_{jn} - r, \nu + r}(t) \\
|\varepsilon(t-x)| (t-x)^{2k+2} dt \\
< \sum_{j=0}^{k} \left| C(j, k) \right| (d_{jn} - r - 1) \sum_{\nu=0}^{\infty} p_{d_{jn} + r, \nu}(x) \int_{0}^{\infty} p_{d_{jn} - r, \nu + r}(t) (t-x)^{2k+2} dt \\
= \varepsilon \sum_{j=0}^{k} \left| C(j, k) \right| T_{d_{jn}, 2k+2}(x) \\
= \varepsilon \sum_{j=0}^{k} \left| C(j, k) \right| o((d_{jn})^{-k-1}) \\
= \varepsilon O(n^{-k-1}).
$$

Finally,

$$
I_2 = \sum_{j=0}^{k} C(j, k) (d_{jn} - r - 1) \sum_{\nu=0}^{\infty} p_{d_{jn} + r, \nu}(x) \int_{|t-x|\geq\delta} p_{d_{jn} - r, \nu + r}(t) \varepsilon(t-x)(t-x)^{2k+2} dt \\
= \sum_{j=0}^{k} C(j, k) \left( (d_{jn} - r - 1) \sum_{\nu=0}^{\infty} p_{d_{jn} + r, \nu}(x) \int_{|t-x|\geq\delta} p_{d_{jn} - r, \nu + r}(t) t^\alpha dt \right) \\
= \sum_{j=0}^{k} C(j, k) \left( (d_{jn} - r - 1) \sum_{\nu=0}^{\infty} p_{d_{jn} + r, \nu}(x) \int_{|t-x|\geq\delta} p_{d_{jn} - r, \nu + r}(t) \cdot \left( \sum_{i=0}^{\alpha} \frac{(\alpha)!}{i!(\alpha-i)!} dt \right) \right) 
$$

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\[
= \sum_{j=0}^{k} C(j,k) \left( (d_j n - r - 1) \sum_{\nu=0}^{\infty} p d_j n + r, p (x) \int_{|t-x|\geq \delta} p d_j n - r, p + r(t) \frac{(t-x)^{2k+3}}{\delta^{2k+3}} \left( \sum_{i=0}^{\infty} \left( \frac{q}{i} \right) (t-x)^i z^{-i} \right) dt \right) \\
= \sum_{j=0}^{k} C(j,k) \sum_{i=0}^{n} \left( \frac{q}{i} \right) x^{-i} \cdot O(T_d_j n, 2k+3+3(x)) = O \left( \frac{1}{n^{k+2}} \right) 
\]
and (3.1) follows.

The assertion (3.2) can be proved along similar lines using \( L_n((t-x)^i, k + 1, x) = O(n^{-(k+2)}) \), \( i = 1, 2, \ldots \) which follows from Lemma 2.3.

The last assertion follows due to the uniform continuity of \( f^{(2k+r+2)} \) on \([a,b]\) (enabling \( \delta \) to become independent of \( x \in [a,b] \)).

This completes the proof.

Remark. We may note here that \( \frac{(d_j n-1)!(d_j n-2)!}{(d_j n+r-1)!(d_j n+r-2)!} \to 1 \) as \( n \to \infty \).

Theorem 3.2. Let \( 1 \leq p \leq 2k+2 \) and \( f \) be integrable on \([0,\infty)\). If \( f^{(p+r)} \) exists and is continuous on \([a,b] \) having the modulus of continuity \( \omega_f^{(p+r)}(\delta) \) on \((a,b)\) and \( f^{(r)}(x) = O(x^\alpha)(\alpha \text{ is a positive integer} \geq p) \) then for \( n \) sufficiently large

\[
||L_n^{(p)}(f,k,x) - f^{(r)}|| \leq \text{Max}\{C_1 n^{-p/2} \omega_f^{(p+r)}(n^{-1/2}), C_2 n^{-k+1}\}
\]

where \( C_1 = C_1(k, p, r) \) and \( C_2 = C_2(k, p, r, f) \).

Proof. For every \( t \in [0,\infty) \) and \( x \in [a,b] \) we have

\[
f^{(r)}(t) = \sum_{i=0}^{p} \frac{f^{(i+r)}(x)}{i!} (t-x)^i + \frac{f^{(p+r)}(\xi) - f^{(p+r)}(x)}{p!} (t-x)^p + h(t,x)\chi(t),
\]

where \( \xi \) lies between \( t \) and \( x \) and \( \chi(t) \) is the characteristic function of the set \([0,\infty)\setminus(a,b)\). The function \( h(t,x) \) for \( x \in [a,b] \) is bounded by \( M t^\alpha \| t - x \|^p \) for some constant \( M \). Using (3.3) we get

\[
\sum_{j=0}^{k} C(j,k) \frac{(d_j n-1)!(d_j n-2)!}{(d_j n+r-1)!(d_j n+r-2)!} T_d_j n^{(r)}(f;x) \\
= \sum_{j=0}^{k} C(j,k)(d_j n - r - 1) \sum_{\nu=0}^{\infty} p d_j n + r, p (x) \int_{0}^{\infty} p d_j n - r, p + r(t) f^{(r)}(t) dt \\
= \sum_{j=0}^{k} C(j,k)(d_j n - r - 1) \sum_{\nu=0}^{\infty} p d_j n + r, p (x) \int_{0}^{\infty} p d_j n - r, p + r(t) \left\{ \sum_{i=0}^{p} \frac{f^{(i+r)}(x)}{i!} (t-x)^i + \frac{f^{(p+r)}(\xi) - f^{(p+r)}(x)}{p!} (t-x)^p + h(t,x)\chi(t) \right\} dt 
\]
\[\begin{align*}
= & \sum_{j=0}^{k} C(j, k)(d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j n + r, \nu}(x) \int_{0}^{\infty} p_{d_j n - r, \nu + r}(t) \sum_{i=0}^{r} \frac{f^{(i+r)}(x)}{i!} (t - x)^i dt + \\
& \sum_{j=0}^{k} C(j, k)(d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j n + r, \nu}(x) \int_{0}^{\infty} p_{d_j n - r, \nu + r}(t) \left( \frac{f^{(p+r)}(\xi) - f^{(p+r)}(x)}{p!} \right) (t - x)^p dt \\
+ & \sum_{j=0}^{k} C(j, k)(d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j n + r, \nu}(x) \int_{0}^{\infty} p_{d_j n - r, \nu + r}(t) h(t, x) \chi(t) dt \\
= & I_1 + I_2 + I_3 \quad (say)
\end{align*}\]

An application of Lemma 2.1 gives us \( I_1 = f^{(r)}(x) + O(n^{-k+1}) \) uniformly in \( x \in [a, b] \).

To estimate \( I_2 \), for every \( \delta > 0 \), we have

\[\begin{align*}
| f^{(p+r)}(\xi) - f^{(p+r)}(x) | & \leq \omega_f^{(p+r)}(| x - \xi |) \\
& \leq (1 + | t - x |) \omega_f^{(p+r)}(\delta).
\end{align*}\]

Hence

\[\begin{align*}
| I_2 | & \leq \frac{1}{p!} \sum_{j=0}^{k} | C(j, k) | (d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j n + r, \nu}(x) \int_{0}^{\infty} p_{d_j n - r, \nu + r}(t) \\
& \quad \times (1 + | t - x |) | t - x |^p \omega_f^{(p+r)}(\delta) dt \\
= & \frac{\omega_f^{(p+r)}(\delta)}{p!} \sum_{j=0}^{k} | C(j, k) | (d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j n + r, \nu}(x) \int_{0}^{\infty} p_{d_j n - r, \nu + r}(t) \cdot | t - x |^p + | t + x |^{p+1} \omega_f^{(p+r)}(\delta) dt.
\end{align*}\]

Using Schwarz inequality for summation and then for integration we find that

\[\begin{align*}
\sum_{\nu=0}^{\infty} p_{d_j n + r, \nu}(x) \int_{0}^{\infty} p_{d_j n - r, \nu + r}(t) | t - x |^p dt & \leq \left\{ \sum_{\nu=0}^{\infty} p_{d_j n + r, \nu}(x) \right\} \left( \int_{0}^{\infty} p_{d_j n - r, \nu + r}(t) | t - x |^p dt \right)^{1/2} \\
& \leq \left\{ \sum_{\nu=0}^{\infty} p_{d_j n + r, \nu}(x) \right\}^{1/(d_j n - r - 1)} \left( \int_{0}^{\infty} p_{d_j n - r, \nu + r}(t) | t - x |^{2p} dt \right)^{1/2}.
\end{align*}\]
It may be remarked that (3.4) is true when $p$ is replaced by $p+1$ and consequently

$$
|I_2| \leq \frac{\omega_{f^{(p+1)}}(\delta)}{p!} \sum_{j=0}^{k} C(j, k) \left\{ (d_j n - r - 1) \sum_{\nu=0}^{\infty} p_dj,n+r,\nu(x) \int_{0}^{\infty} p_{d_j,n-r,\nu+r}(t)((t-x)^{2p} + (t-x)^{2(p+1)} / \delta)dt \right\}^{1/2}
$$

$$
= \frac{\omega_{f^{(p+1)}}(\delta)}{p!} \sum_{j=0}^{k} C(j, k) \left\{ O(T_d j n, 2p + \delta^{-1} T_d j n, 2(p+1)) \right\}^{1/2}
$$

$$
= \frac{\omega_{f^{(p+1)}}(\delta)}{p!} \sum_{j=0}^{k} C(j, k) \left\{ O((d_j n)^{-p}) + \delta^{-1} O((d_j n)^{-p-1}) \right\}^{1/2},
$$

Choosing $\delta = n^{-1/2}$ we get $|I_2| \leq C_1 n^{-p/2} \omega_{f^{(p+1)}}(n^{-1/2})$.

Finally, choosing a positive number $\eta$ such that $|t-x| \geq \eta$ we get

$$
|I_3| \leq \sum_{j=0}^{k} C(j, k) \left\{ O((d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j,n+r,\nu}(x) \int_{|t-x| \geq \eta} p_{d_j,n-r,\nu+r}(t) |t-x|^p dt \right\}
$$

$$
= \sum_{j=0}^{k} C(j, k) \left\{ O((d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j,n+r,\nu}(x) \int_{|t-x| \geq \eta} p_{d_j,n-r,\nu+r}(t) \left( \sum_{\xi=0}^{\alpha} (\eta^2)^{\xi} |t-x|^\alpha \right) |t-x|^p dt \right\}
$$

$$
= \sum_{j=0}^{k} C(j, k) \left\{ O((d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j,n+r,\nu}(x) \int_{|t-x| \geq \eta} p_{d_j,n-r,\nu+r}(t) \left( \sum_{\xi=0}^{\alpha} (\eta^2)^{\xi} |t-x|^\alpha \right) |t-x|^m dt \right\}, m > k+1
$$

$$
= \sum_{j=0}^{k} \frac{C(j, k)}{\eta^{2n-p}} O((d_j n - r - 1) \sum_{\nu=0}^{\infty} p_{d_j,n+r,\nu}(x) \int_{0}^{\infty} p_{d_j,n-r,\nu+r}(t) \left( \sum_{\xi=0}^{\alpha} (\eta^2)^{\xi} |t-x|^\alpha \right) dt)
$$

$$
= \sum_{j=0}^{k} \frac{C(j, k)}{\eta^{2n-p}} \cdot O((d_j n)^{-m}) = C_3 n^{-m}
$$

uniformly in $x \in [a, b]$. Combining the estimates of $I_1 - I_3$, we obtain the required result.
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