A STRUCTURAL THEOREM FOR DISTRIBUTIONS HAVING S-ASYMPTOTIC

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Abstract. We prove that a distribution $T$ with an $S$-asymptotic related to $c(h)$ and to the cone $\Gamma$ has on the set $B + \Gamma$ a restriction which is a finite sum of derivatives of the functions $F_i$, continuous in $B + \Gamma$ and having some properties which imply that as $h \to \infty$ the $F_i(x + h)/c(h)$ converge uniformly for $x \in B_i$ when $h \in \Gamma$ and $\|h\| \to \infty$. If we know more about the distribution $T$ or about the cone $\Gamma$, then we can say more about the properties of $F_i$, $B$ is the ball $B(0, r)$.

1. Introduction. In the last few years many papers were published concerning the asymptotical behaviour of distributions at infinity. Of the reasons lies in the usefulness of these results in quantum field theory. $S$-asymptotic is one of the notions related to the asymptotical behaviour of distributions, introduced and elaborated in [2]. As $S$-asymptotic can be profitable in many studies and applications (see, for example, [4] and [5]), it is very useful to know the analytical expression of a distribution having $S$-asymptotic, especially if it is given by continuous functions and their derivatives S. Plišović proved two theorems of this kind [1], but in the one dimensional case. Our theorem gives a different result and a different method of proof.

2. Notations and definitions. We denote by $\Gamma$ a cone in $\mathbb{R}^n$ with the vertex at zero and $\sum(\Gamma)$ the set of real valued functions $c(h)$, $h \in \Gamma$, continuous and different from zero when $h \in \Gamma$, $B = B(0, r)$ will be the ball $\{x \in \mathbb{R}^n, \|x\| < r\}$. The notations of the spaces of distributions are the same as in [3]; we use the $n$-dimensional case of distributions.

Definition. A distribution $T \in (D')$ has an $S$-asymptotic in the cone $\Gamma$ related to some $c(h) \in \sum(\Gamma)$ and with the limit $U \in (D')$ if the following limit exists

$$ \lim_{h \in \Gamma, \|h\| \to \infty} \langle T(x + h)/c(h), \varphi(x) \rangle = \langle U, \varphi \rangle, \quad \varphi \in (D). $$

Then we write $T(x + h) \sim^S c(h) \cdot U(x)$, $h \in \Gamma$.

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We know that if a filter $A$ has a countable base, then from the weak convergence in $(D')$, related to $A$, the strong convergence in $(D')$ follows as well. In this way relation (1) can be written in the form

$$\lim_{h \in \Gamma, ||h|| \to \infty} T(x + h)/c(h) = U \text{ in } (D').$$

If $T \in (D')$ has an $S$-asymptotic related to $c(h)$, the set of distributions $Q \equiv \{T(x + h)/c(h), \ h \in \Gamma\}$ is weakly bounded in $(D')$. If this were not true, we would have a sequence $h_n \in \Gamma$ and $\varphi_0 \in (D)$ such that

$$| \langle T(x + h_n)/c(h_n), \varphi_0(x) \rangle - \langle U, \varphi_0(x) \rangle | \geq n, \quad n \in N$$

which is not possible because of (1).

Now, from the weak boundedness of the set $Q$ it follows that $Q$ is bounded in $(D')$, as well.

3. **Structural theorem. Theorem.** If $T \in (D')$ has an $S$-asymptotic related to $c(h) \in \sum(\Gamma)$, then for the ball $B(0, r)$ there exist numerical functions $F_i, i \leq m$, continuous on $B(0, r) + \Gamma$, such that, for every $i \leq m$, $F_i(x + h)/c(h)$ converges uniformly for $x \in B(0, r)$ when $h \in \Gamma$, $||h|| \to \infty$, and the restriction of the distribution $T$ on $B(0, r) + \Gamma$ can be given in the form $T = \sum_{|i| \leq m} D^i F_i$.

For the proof of the theorem we need the following:

**Lemma.** If $T \in (D')$ and $T(x + h) \sim c(h) \cdot U(x)$, $h \in \Gamma$, then for a $B(0, r)$ and a $\Omega$ which is a relatively compact open neighbourhood of zero in $R^n$, there exists an $m \geq 0$ such that for every $\varphi, \psi \in (D^n_{\Omega})$ the function $(T * \varphi * \psi)(x)$ is continuous for $x \in B(0, r) + \Gamma$ and the set of functions $\{ (T_h * \varphi * \psi)(x), \ h \in \Gamma \}$ converges uniformly for $x \in B(0, r)$ to $(U * \varphi * \psi)(x)$, when $h \in \Gamma$, $||h|| \to \infty$: $T_h = T(x + h)/c(h)$.

**Proof.** Suppose that $T$ has an $S$-asymptotic related to $c(h)$; then the set $Q = \{T(x + h)/c(h) \equiv T_h, \ h \in \Gamma\}$ is weakly bounded in $(D')$ and consequently bounded in $(D')$, as well. A necessary and sufficient condition that a set $B' \subset (D')$ is bounded in $(D')$ is: for every $\alpha \in (D)$ the set of functions $\{T_{\alpha}, T \in B'\}$ is bounded on every compact set $C$ belonging to $R^n$. Hence $\{T_{\alpha}, T \in B'\}$ defines a bounded set of regular distributions.

We denote by $\Omega$ an open neighbourhood of zero in $R^n$ which is relatively compact, $C(\Omega) = K$, $K$ is a compact set. For a fixed $\alpha$, sup $\alpha \subset K$, the linear mappings $\beta \to (T_{h} * \alpha) * \beta$ are continuous mappings of $(D_K)$ into $(E)$ because of the separate continuity of the convolution. As the set $\{T_{h} * \alpha, T_{h} \in Q\}$ is a bounded set in $(D')$, then for every ball $B(0, r)$ the set of mappings $\beta \to \{T_{h} * \alpha) * \beta, T_{h} \in Q\}$ is the set of equicontinuous mappings of $(D_K)$ into $(L^n_{B})$, $B = B(0, r)$. Now there exists a $m \geq 0$ such that the linear mappings $(\alpha, \beta) \to T_{h} * \alpha * \beta$ which map $(D_K) \times (D_K)$ into $(L^n_{B})$ can be extended to $(D^n_{B}) \times (D^n_{B})$ in such a way that
$(\alpha, \beta) \to T_h * \alpha * \beta, T_h \in Q$, are equicontinuous mappings of $(D_m^0) \times (D_m^0)$ into $(L_B^m)$ (see for example the proof of Theorem XXII, p. 51 in [3]).

We saw that for every $\varphi, \psi \in (D_m^0)$ and every $x \in B(0, r), h \in \Gamma$ the functions $(T_h * \varphi * \psi)(x)$ are continuous functions in $x$. From the relation $(T_h * \varphi * \psi)(x) = (T * \varphi * \psi)(x + h) / c(h)$ and from the properties of $c(h)$ it follows that $(T * \varphi * \psi)(y)$ is a continuous function for $y \in B(0, r) + \Gamma$ and $\varphi, \psi \in (D_m^0)$.

It remains to prove that $T_h * \varphi * \psi$ converges to $U * \varphi * \psi$ in $(L_B^m)$ for $\varphi, \psi \in (D_m^0)$. We know that $(D)$ is a dense subset of $(D_m^0)$, $m \geq 0$. We can construct a subset $A$ of $(D)$ to be dense $(D_m^0)$, $C_l(\Omega) = K$. The set of functions $T_h * \alpha * \beta$ converges in $(L_B^m)$ for $\alpha, \beta \in A$, when $h \in \Gamma$, $||h|| \to \infty$. Taking care of the equicontinuity of the mappings $(D_m^0) \times (D_m^0)$ into $(D_m^0)$, defined by $T_h * \varphi * \psi$, we can use the Banach-Stenhaus theorem to prove that $T_h * \varphi * \psi$ converges in $(L_B^m)$ when $h \in \Gamma$, $||h|| \to \infty$.

**Proof of the Theorem.** We shall use, now relation (VI, 6; 23) from [3]

$$
\Delta^{2k} (\gamma E * \gamma E * T) - 2\Delta^k \gamma E * \xi * T + (\xi * \xi * T) = T
$$

where $E$ is a solution of the iterated Laplace equation $\Delta^k E = \delta; \gamma, \xi \in (D_m^0)$. We have only to choose the number $k$ large enough so that $\gamma E$ belongs to $(D_m^0)$. Now, it is possible to take $F_1 = \gamma E * \gamma E * T, F_2 = \gamma E * \xi * T$ and $F_3 = \xi * \xi * T$. All of these functions are of the form $F_i = T * \varphi_i * \psi_i, \varphi_i, \psi_i \in (D_m^0)$, $i = 1, 2, 3$.

By the property of the convolution:

$$
F_i(x + h) / c(h) - (F_i(x) / c(h)) * \tau_{-h} = (T * \varphi_i * \psi_i) * \tau_{-h} / c(h)
$$

$$
= (T * \tau_{-h} / c(h)) * (\varphi_i * \psi_i) = T_h * \varphi_i * \psi_i.
$$

Hence, by the Lemma it follows that $F_i(x + h) / c(h)$ converges uniformly for $x \in B(0, r)$ when $h \in \Gamma$, $||h|| \to \infty$.

**4. Consequences of the theorem 1.** It the functions $F_i, |i| \leq m$, have the property given in the theorem and if $\Gamma = R^m$, the regular distributions defined by the functions $F_i / c$ have an S-asymptotic related to $c_1(h) = 1$. In the case $\Gamma = R^n$ the functions $F_i(x) / c(x)$ are continuous and $F_i(x)$ converge when $||x|| \to \infty$ to the numbers $C_i$. Now,

$$
\lim_{||h|| \to \infty} \langle F_i(x + h) / c(x + h), \varphi(x) \rangle = \lim_{||h|| \to \infty} \int_{R^n} (F_i(x + h) / c(x + h)) \varphi(x) dx, \quad \varphi \in (D)
$$

$$
= \langle C_i, \varphi \rangle.
$$

**2.** If $\Gamma$ is a convex cone with nonempty interior, int $\Gamma \neq \emptyset$, and if $T(x + h) \sim c(h) \cdot U(x)$, $h \in \Gamma$, then for $x \in B(0, r)$

$$
\lim_{h \in \Gamma, ||h|| \to \infty} F_i(x + h) / c(h) = a_i U(x),
$$
where $\alpha_i$ are constants, $\alpha_i \neq 0$.

**Proof.** If $\Gamma$ is a convex cone and $\int \Gamma \neq 0 \rho$, then $U$ is of the form: $U(x) = b \exp(\ll x_0, x \gg)$, where $b \in R$, $x_0$ is a fixed element from $R^n$ and $\ll x, y \gg = \sum_{i=1}^{n} x_i y_i$ (see [2]).

All the functions $F_i$, $|i| \leq m$ are of the form $F_i = T \ast \varphi_i * \psi_i$, $\psi_i \in (D^m_{\Omega})$. By the Lemma, for $x \in B(0, r)$, we have

$$\lim_{h \in \Gamma, ||h|| \to \infty} F_i(x + h)/c(h) = \lim_{h \in \Gamma, ||h|| \to \infty} (T(x + h)/c(x)) * \varphi_i * \psi_i = U \ast \varphi_i * \psi_i.$$

If $U(x) = b \exp(\ll x_0, x \gg)$, then $U \ast \varphi_i * \psi_i = \alpha_i b \exp(\ll x_0, x \gg)$ which proves our assertion.

3. If $T$ belongs to a subspace $A'$ of $(D')$, then sometimes we can say more about $F_i$ from relation (3). As an illustration we shall examine the case $T \in (S')$.

In we suppose in our Theorem that $T \in (S') \subset (D')$, then we know not only that all the functions $F_i$, $|i| \leq m$, are continuous for $x \in B(0, r) + \Gamma$, but continuous functions of the slow growth. That means that $F_i(x) = (1 + r^2)^z f_i(x)$, where $r = ||x||$, $q \in R$ and $f_i$ are continuous and bounded functions.

**Proof.** If $T \in (S')$, then there exists a real number $q$ such that the set of distributions $\{T(x + h)/(1 + ||h||^2)^q, h \in R^n\} = W$ is bounded in $(D')$ (Theorem VI, p. 95 in [3]). We can repeat the first part of the Lemma’s proof but with $c(h) = (1 + ||h||^2)^q$, $\Gamma = R^n$ and we shall obtain that there exists a $p \geq 0$ such that for $\varphi, \psi \in (D^m_{\Omega})$ and $x \in R^n$ the function $(T \ast \varphi \ast \psi)(x)$ is continuous and $(T \ast \varphi \ast \psi)(x)/(1 + ||x||^2)^q$ is a bounded and continuous function. It remains only to choose the number $k$ in (4) large enough so that $\gamma E \in (D^m_{\Omega})$.

REFERENCES


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