ABSOLUTE AND ORDINARY KÖTHE-TOEPLITZ DUALS
OF SOME SETS OF SEQUENCES
AND MATRIX TRANSFORMATIONS

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Abstract. We determine the ordinary Köthe-Toeplitz dual of the set $\Delta l_\infty(p)$ and the absolute Köthe-Toeplitz duals of the sets $\Delta l_\infty(p)$, $\Delta c_0(p)$ and $\Delta c(p)$ defined by Ahmad and Mursaleen. Further we investigate matrix transformations in these spaces and give a characterization of the class $(\Delta l_\infty(p), l_\infty)$.

1. Introduction

By $\omega$ we denote the set of all complex sequences $x = (x_k)_{k=1}^\infty$. Throughout the paper $p = (p_k)_{k=1}^\infty$ shall always be an arbitrary sequence of positive reals. The following sets were introduced and investigated by various authors:

$$l_\infty(p) := \left\{ x \in \omega : \sup_k |x_k|^{p_k} < \infty \right\},$$

$$c(p) := \left\{ x \in \omega : |x_k - l|^{p_k} \to 0 \text{ for some complex } l \right\},$$

$$c_0(p) := \left\{ x \in \omega : |x_k|^{p_k} \to 0 \right\},$$

$$l(p) := \left\{ x \in \omega : \sum_{k=1}^\infty |x_k|^{p_k} < \infty \right\} \quad (\text{cf. } [2], [3], [5], \text{ and } [7]).$$

Given any sequence $x \in \omega$ we shall write $\Delta x := (x_k - x_{k+1})$. In a recent paper (cf. [1]), Ahmad and Mursaleen defined the following sets:

$$\Delta l_\infty(p) := \left\{ x \in \omega : \Delta x \in l_\infty(p) \right\},$$

$$\Delta c(p) := \left\{ x \in \omega : \Delta x \in c(p) \right\},$$

$$\Delta c_0(p) := \left\{ x \in \omega : \Delta x \in c_0(p) \right\}.$$

In the determination of the absolute Köthe-Toeplitz duals of $\Delta l_\infty(p)$ and $\Delta c_0(p)$, they applied some arguments which do not seem to hold:

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(i) $x \in \Delta l_{\infty}(p)$ does not imply in general the existence of a finite number $N > \sup_{k} k^{-1} |x_k|$, as the following counterexample will show: If we put $p_k := k^{-1}$ and $x_k := k^2 (k = 1, 2, \ldots)$ then $|\Delta x_k|^{p_k} \to 1 (k \to \infty)$, hence $x \in \Delta l_{\infty}(p)$, and $\sup_{k} k^{-1} |x_k| = \infty$.

(ii) If $a$ is a sequence such that
\[
\sum_{k=1}^{\infty} k |a_k| N^{1/p_k} = \infty \quad \text{for some } N > 1,
\]
then the sequence $x$ defined by $x_k := k N^{1/p_k} \text{sgn } a_k$ is not in $\Delta l_{\infty}(p)$, in general. In order to see this, we put $p_k := k$ and $a_k := (-1)^k (k = 1, 2, \ldots)$. Then $a$ satisfies (1.1) for all $N > 1$ and $|\Delta x_k|^{p_k} \to \infty$, hence $x \not\in \Delta l_{\infty}(p)$.

In this paper, we shall determine the absolute Köthe-Toeplitz duals of the sets $\Delta l_{\infty}(p)$ and $\Delta \alpha_0(p)$, and give new proofs for the characterizations of the matrix transformations considered in [1]. Further we shall state some new results.

2. Köthe-Toeplitz duals

For arbitrary set $X$ of sequences, we define the ordinary and absolute Köthe-Toeplitz duals by
\[
X^{\dagger} := \left\{ a \in \omega : \sum_{k=1}^{\infty} a_k x_k \text{ converges for all } x \in X \right\} \quad \text{and}
\]
\[
X^{\|\dagger\|} := \left\{ a \in \omega : \sum_{k=1}^{\infty} |a_k x_k| < \infty \text{ for all } x \in X \right\}
\]
respectively; we shall write $X^{\|\dagger\|} := (X^{\dagger})^{\|\dagger\|}$ and $X^{\|\dagger\|\|\dagger\|} := (X^{\|\dagger\|})^{\|\dagger\|\|\dagger\|}$.

**Theorem 2.1.** For every strictly positive sequence $p = (p_k)$, we have

(a) $\left( \Delta l_{\infty}(p) \right)^{\|\dagger\|} = D^{(1)}(p) := \bigcup_{N=2}^{\infty} \left\{ a \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{1/p_j} < \infty \right\},$

(b) $\left( \Delta l_{\infty}(p) \right)^{\|\dagger\|\|\dagger\|} = D^{(2)}(p) := \bigcup_{N=2}^{\infty} \left\{ a \in \omega : \sup_{k \geq 2} \left| a_k \right| \left( \sum_{j=1}^{k-1} N^{1/p_j} \right)^{-1} < \infty \right\},$

(c) $\left( \Delta \alpha_0(p) \right)^{\|\dagger\|} = D^{(1)}_0(p) := \bigcup_{N=2}^{\infty} \left\{ a \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{-1/p_j} < \infty \right\},$

(d) $\left( \Delta \alpha_0(p) \right)^{\|\dagger\|\|\dagger\|} = D^{(2)}_0(p) := \bigcup_{N=2}^{\infty} \left\{ a \in \omega : \sup_{k \geq 2} \left| a_k \right| \left( \sum_{j=1}^{k-1} N^{-1/p_j} \right)^{-1} < \infty \right\}.$

(We adopt the usual convention that $\sum_{j=1}^{m} y_j = 0$ (m < 1) for arbitrary $y_i$.)
Proof: (a) Let \( a \in D^{(1)}_{\infty}(p) \) and \( x \in \Delta l_\infty(p) \). We choose \( N > \max\{1, \sup |\Delta x_k|^{p_k}\} \). Then

\[
\sum_{k=1}^{\infty} |a_k x_k| \leq \sum_{k=1}^{\infty} |a_k| \left( \sum_{j=1}^{k-1} |\Delta x_j| \right) + |x_1| \sum_{k=1}^{\infty} |a_k| \tag{2.1}
\]

\[
\leq \sum_{k=1}^{\infty} |a_k| \left( \sum_{j=1}^{k-1} N^{1/p_j} \right) + |x_1| \sum_{k=1}^{\infty} |a_k| < \infty.
\]

(Note: Since \( \sum_{j=1}^{k-1} N^{1/p_j} \geq 1 \) for arbitrary \( N > 1 \) \( (k = 2, 3, \ldots) \), \( a \in D^{(1)}_{\infty}(p) \) implies \( \sum_{k=1}^{\infty} |a_k| < \infty \).

Conversely let \( a \not\in D^{(1)}_{\infty}(p) \). Then we have \( \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{1/p_j} = \infty \) for some integer \( N > 1 \).

We define the sequence \( x \) by \( x_k := \sum_{j=1}^{k-1} N^{1/p_j} \) \( (k = 1, 2, \ldots) \). Then it is easy to see that \( x \in \Delta l_\infty(p) \) and \( \sum_{k=1}^{\infty} |a_k x_k| = \infty \), hence \( a \not\in (\Delta l_\infty(p))^{\|\|} \).

(b) Let \( a \in D^{(2)}_{\infty}(p) \) and \( x \in (\Delta l_\infty(p))^{\|\|} = D^{(1)}_{\infty}(p) \), by part (a). Then for some \( N > 1 \), we have

\[
\sum_{k=2}^{\infty} |a_k x_k| = \sum_{k=2}^{\infty} |a_k| \left( \sum_{j=1}^{k-1} N^{1/p_j} \right)^{-1} |x_k| \sum_{j=1}^{k-1} N^{1/p_j}
\]

\[
\leq \sup_{k \geq 2} \left[ |a_k| \left( \sum_{j=1}^{k-1} N^{1/p_j} \right)^{-1} \right] \sum_{k=2}^{\infty} |x_k| \sum_{j=1}^{k-1} N^{1/p_j} < \infty.
\]

Conversely let \( a \not\in D^{(2)}_{\infty}(p) \). Then for all integers \( N > 1 \), we have

\[
\sup_{k \geq 2} |a_k| \left( \sum_{j=1}^{k-1} N^{1/p_j} \right)^{-1} = \infty.
\]

Hence there is a strictly increasing sequence \( (k(m)) \) of integers \( k(m) \geq 2 \) such that

\[
|a_{k(m)}| \left( \sum_{j=1}^{k(m)-1} m^{1/p_j} \right)^{-1} > m^2 \quad (m = 2, 3, \ldots).
\]

We define the sequence \( x \) by

\[
x_k := \begin{cases} |a_{k(m)}|^{-1} & (k = k(m)) \\ 0 & (k \neq k(m)) \quad (m = 2, 3, \ldots). \end{cases}
\]
Then for all integers $N \geq 2$, we have
\[
\sum_{k=1}^{\infty} |x_k| \sum_{j=1}^{k-1} N^{1/p_j} \leq \sum_{m=2}^{\infty} |a_{k(m)}|^{-1} \sum_{j=1}^{k(m)-1} N^{1/p_j} \leq \\
\sum_{m=2}^{N-1} |a_{k(m)}|^{-1} \sum_{j=1}^{k(m)-1} N^{1/p_j} + \sum_{m=N}^{\infty} |a_{k(m)}|^{-1} \sum_{j=1}^{k(m)-1} m^{1/p_j} \leq \\
\sum_{m=2}^{N-1} |a_{k(m)}|^{-1} \sum_{j=1}^{k(m)-1} N^{1/p_j} + \sum_{m=N}^{\infty} m^{-2} < \infty,
\]
hence $x \in (\Delta \ell_\infty(p))^{|1|}$, and
\[
\sum_{k=1}^{\infty} |a_k x_k| = \sum_{N=2}^{\infty} 1 = \infty,
\]
hence $a \notin (\Delta \ell_\infty(p))^{|1|}$.

(c) Let $a \in D_0^{(1)}(p)$. Since $|a_k| \leq |a_k| N^{1/p_1} \sum_{j=1}^{k-1} N^{-1/p_j}$ $(k = 2, 3, \ldots)$, we have $\sum_{k=1}^{\infty} |a_k| < \infty$. Let $x \in \Delta \ell_0(p)$. Then there is an integer $k_0$ such that $\sup_{k > k_0} |\Delta x_k|^p \leq N^{-1}$, where $N$ is the number in $D_0^{(1)}(p)$. We put $M := \max_{1 \leq k \leq k_0} |\Delta x_k|^p$, $m := \min_{1 \leq k \leq k_0} p_k$, $L := (M + 1)N$ and define the sequence $y$ by $y_k := x_k L^{-1/m}$ $(k = 1, 2, \ldots)$. Then it is easy to see that $\sup_k |\Delta y_k|^p \leq N^{-1}$, and as in (2.1) with $N$ replaced by $N^{-1}$, we have
\[
\sum_{k=1}^{\infty} |a_k y_k| = L^{1/m} \sum_{k=1}^{\infty} |a_k y_k| < \infty.
\]
Conversely, let $a \notin D_0^{(1)}(p)$. Then we can determine a strictly increasing sequence $(k(s))$ of integers such that $k(1) := 1$ and
\[
M_s := \sum_{k=k(s)}^{k(s+1)-1} |a_k| \sum_{j=1}^{k-1} (s + 1)^{-1/p_j} > 1 \quad (s = 1, 2, \ldots).
\]
We define the sequence $x$ by
\[
x_k := \sum_{l=1}^{s-1} \sum_{j=k(l)}^{(l+1)-1} (l + 1)^{-1/p_j} + \sum_{j=k(s)}^{k-1} (s + 1)^{-1/p_j} \\
(k(s) \leq k \leq k(s + 1) - 1; \ s = 1, 2, \ldots).
\]
Then it is easy to see that $|\Delta x_k|^p = 1/(s+1)$ $(k(s) \leq k \leq k(s+1)-1; \ s = 1, 2, \ldots)$ hence $x \in \Delta \ell_0(p)$, and $\sum_{k=1}^{\infty} |a_k x_k| \geq \sum_{s=1}^{\infty} 1 = \infty$, i.e. $a \notin (\Delta \ell_0(p))^{|1|}$. 

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(d) For \( N = 2, 3, \ldots \), we put
\[
E_N := \left\{ a \in \omega : \sum_{k=1}^{\infty} |a_k| \sum_{j=1}^{k-1} N^{-1/p_j} < \infty \right\}
\]
\[
F_N := \left\{ a \in \omega : \sup_{k \geq 2} |a_k| \left[ \sum_{j=1}^{k-1} N^{-1/p_j} \right]^{-1} < \infty \right\}.
\]
By a well known result (cf. [3, Lemma 4 (iv)]), we have to show \( F_N = E_N^{[|]} \) \((N = 2, 3, \ldots)\). The proof of this is standard and therefore omitted.

Now we shall give some new results:

**Theorem 2.2.** For every strictly positive sequence \( p = (p_k) \), we have

(a) \( (\Delta c(p))^{[|]} = D_0^{(1)} \cap \left\{ a \in \omega : \sum_{k=1}^{\infty} |a_k| k < \infty \right\} \) and

(b) \( (\Delta l_{\infty}(p))^\dagger = D_{\infty}(p) \)
\[
:= \bigcap_{N=2}^{\infty} \left\{ a \in \omega : \sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-1} N^{1/p_j} \text{ converges and } \sum_{k=1}^{\infty} N^{1/p_k} |R_k| < \infty \right\},
\]
where \( R_k := \sum_{\alpha=\infty}^{\alpha_1} a_\nu \) \((k = 1, 2, \ldots)\).

**Proof:** (a) Let \( a \in D^{(1)}(p) \) and \( x \in \Delta c(p) \). Then there is a complex number \( l \) such that \( |\Delta x_k - l|^{p_k} \to 0 \) \((k \to \infty)\). We define \( y \) by \( y_k := x_k + lk \) \((k = 1, 2, \ldots)\). Then \( y \in \Delta c_0(p) \) and
\[
\sum_{k=1}^{\infty} |a_k x_k| \leq \sum_{k=1}^{\infty} |a_k| \left| \sum_{j=1}^{k-1} \Delta y_j \right| + |l| \sum_{k=1}^{\infty} |a_k| k < \infty
\]
by Theorem 2.1.(c) and since \( a \in D^{(1)}(p) \). Now let \( a \in (\Delta c(p))^{[|]} \subset (\Delta c_0(p))^{[|]} = D_0^{(1)}(p) \) by Theorem 2.1.(c). Since the sequence \( x \) defined by \( x_k := k \) \((k = 1, 2, \ldots)\) is in \( \Delta c(p) \) we have \( \sum_{k=1}^{\infty} |a_k| k < \infty \).

(b) Let \( a \in D_{\infty}(p) \) and \( x \in \Delta l_{\infty}(p) \). Then there is an integer \( N > \max \left\{ 1, \sup_k |\Delta x_k|^{p_k} \right\} \). We have
\[
\sum_{k=1}^{n} a_k x_k = - \sum_{j=1}^{n-1} \Delta x_j R_j + R_n \sum_{j=1}^{n-1} \Delta x_j + x_1 \sum_{k=1}^{n} a_k \quad (n = 1, 2, \ldots) \quad (2.2)
\]
Obviously the last term on the right in (2.2) is convergent. Since \( \sum_{j=1}^{\infty} |\Delta x_j| \times |R_j| \leq \sum_{j=1}^{\infty} N^{1/p_j} |R_j| < \infty \), the first term on the right in (2.2) is absolutely convergent. Finally by Corollary 2 in [4], the convergence of \( \sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-1} N^{1/p_j} \) implies \( \lim_{n \to \infty} R_n \sum_{j=1}^{k-1} N^{1/p_j} = 0 \). Conversely let \( a \in (\Delta l_{\infty}(p))^\dagger \). Since \( e := (1, 1, \ldots) \in \Delta l_{\infty}(p) \) and \( x = \left[ \sum_{j=1}^{k-1} N^{1/p_j} \right] \in \Delta l_{\infty}(p) \), we conclude the convergence
of $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} a_k \sum_{j=1}^{k-1} N^{1/p_j}$ respectively. Applying Corollary 2 in [4] again we have

$$\lim_{n \to \infty} R_n \sum_{j=1}^{k-1} N^{1/p_j} = 0.$$ 

From (2.2), we obtain the convergence of $\sum_{k=1}^{\infty} \Delta x_k R_k$ for all $x \in \Delta l_\infty(p)$. Since $x \in \Delta l_\infty(p)$ if and only if $y := \Delta x \in l_\infty(p)$, this implies $R \in l_\infty(p)$, hence $\sum_{k=1}^{\infty} N^{1/p_k} |R_k| < \infty$ for all $N > 1$ by a well known theorem (cf. [2, Th. 2]).

3. Some matrix transformations

For any complex matrix $A = (a_{nk})$, we shall write $A_n := (a_{nk})_k$ for the sequence in the $n$-th row of $A$. Given $A$ we define the matrix $B$ by

$$b_{nk} := a_{nk} - a_{n+1,k} \quad (n, k = 1, 2, \ldots).$$

Let $X, Y$ be two subsets of $\omega$. By $(X, Y)$ we denote the class of all matrices $A$ such that the series $A_n x := \sum_{k=1}^{\infty} a_{nk} x_k$ converge for all $x \in X \ (n = 1, 2, \ldots)$ and the sequence $A x := (A_n x)$ is in $Y$ for all $x \in X$.

The following is obvious and therefore stated without proof:

**Lemma 3.1.** Let $X, Y$ be linear sequence spaces. We put $\Delta Y := \{y \in \omega : \Delta y \in Y\}$. Then $A \in (X, \Delta Y)$ if and only if $B \in (X, Y)$ and $A_1 \in X^1$.

Lemma 3.1 and well known results together yield for instance the characterization of the following classes for strictly positive sequences $q \in l_\infty : (l(p), \Delta l_\infty(q))$, $(l(p), \Delta c_0(q))$, $(l(p), \Delta c(q))$, (cf. [5, Th. 5 (i), (ii) and (iii)] if $0 < p_k \leq 1$ ($k = 1, 2, \ldots$), [5, Th. 8 and Th. 9] if $1 < p_k \leq H < \infty$ ($k = 1, 2, \ldots$)). Now we give a characterization for the class $(\Delta l_\infty(p), l_\infty)$:

**Theorem 3.1.** For every strictly positive sequence $p$, we have $A \in (\Delta l_\infty(p), l_\infty)$ if and only if the following three conditions hold:

(i) $M_1(N) := \sup_n \left( \sum_{k=1}^{\infty} a_{nk} \sum_{j=1}^{k-1} N^{1/p_j} \right) < \infty$ for all $N > 1$,

(ii) $M_2(N) := \sup_n \left[ \sum_{\nu=1}^{\infty} N^{1/p_\nu} \left| \sum_{k=\nu+1}^{\infty} a_{nk} \right| \right] < \infty$ for all $N > 1$,

(iii) $M_3 := \sup_n \left( \sum_{k=1}^{\infty} |a_{nk}| \right) < \infty$.

**Proof:** Let conditions (i), (ii) and (iii) be satisfied. Then $A_n \in (\Delta l_\infty(p))^\dagger \ (n = 1, 2, \ldots)$ by Theorem 2.2.(b). Hence the series $A_n x$ converge for all $x \in$
\[ \Delta l_\infty(p) \text{ \( (n = 1, 2, \ldots). \) Further as in the proof of Theorem 2.2(b), we have for} \]
\[ x \in \Delta l_\infty(p) \text{ such that} \sup_k |\Delta x_k|^p < N: \]
\[ \left( \sum_{k=1}^\infty a_n x_k \right) \leq \sum_{k=1}^\infty N^{1/p_k} \left( \sum_{k=1}^\infty a_n \right) + |x_1| \sum_{k=1}^\infty a_n \leq M_2(N) + |x_1|M_3 \]
\[ (n = 1, 2, \ldots), \]

hence \( Ax \in l_\infty. \)

Conversely let \( A \in (\Delta l_\infty(p), l_\infty). \) The necessity of conditions (i) and (iii) follows from the fact that \( (x_k) := \left[ \sum_{j=1}^{k-1} N^{1/p_j} \right] \) and \( e \) are in \( \Delta l_\infty(p). \) In order to show the necessity of condition (ii), we assume that \( M_2(N) = \infty \) for some \( N > 1. \)

Then for the matrix \( C \) defined by
\[ c_{\nu \nu} := \sum_{k=\nu+1}^\infty a_n \quad (n, \nu = 1, 2, \ldots), \]
we have \( C \notin (l_\infty(p), l_\infty) \). (cf. [2, Th. 3]) Hence there is a sequence \( x \in l_\infty(p) \) such that \( \sup_{\nu} |x_{\nu}|^{p_\nu} = 1 \) and \( \sum_{\nu=1}^\infty c_{\nu \nu} x_{\nu} \neq O(1). \) We define the sequence \( y \) by \( y_{\nu} := -\sum_{j=1}^{\nu-1} x_j + x_1 \ (\nu = 1, 2, \ldots). \) Then \( y \in \Delta l_\infty(p) \) and \( \sum_{\nu=1}^\infty a_{\nu \nu} y_{\nu} = \sum_{\nu=1}^\infty c_{\nu \nu} x_{\nu} + x_1 \sum_{\nu=1}^\infty a_{\nu \nu} \neq O(1), \) a contradiction to the assumption \( A \in (\Delta l_\infty(p), l_\infty). \) Therefore we must have \( M_2(N) < \infty \) for all \( N > 1. \)

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