ON SOME NEUTRIX PRODUCTS OF DISTRIBUTIONS

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Abstract. The neutrix product \[ x^\lambda L(x) \] of the distributions \( x^\lambda L(x) \) and \( x^\mu \) or \( \delta^{(m)} \) is analysed and explicitly calculated, where \( \lambda, \mu \notin \mathbb{Z}, m \in \mathbb{N}_0 \) and \( L \) is a slowly varying function at both zero and infinity [7].

The neutrix product of distributions is defined with a fixed infinitely differentiable function \( \rho : \mathbb{R} \rightarrow [0, \infty) \), which has the following properties:

(i) \( \rho(x) = 0, |x| \geq 1 \),
(ii) \( \rho(x) = \rho(-x), x \in \mathbb{R} \),
(iii) \( \int_{-1}^{1} \rho(x) \, dx = 1 \).

The sequence \( \delta_n(x) = n \cdot \rho(nx), n \in \mathbb{N}, x \in \mathbb{R} \), is a so-called “delta sequence” i.e. it is a sequence of functions from the space \( \mathcal{D} \) which tends to the measure \( \delta \) in the topology of \( \mathcal{D}' \). Further on, for arbitrary \( g \in \mathcal{D}' \) we put

\[ g_n(x) = g * \delta_n(x), \quad n \in \mathbb{N}, x \in \mathbb{R}. \] (1)

Then the sequence of infinitely differentiable functions \( \{g_n\} \) tends to \( g \) in the topology of \( \mathcal{D}' \).

This leads to the following definition of the product of two distributions on an open interval \( (a, b) \):

**Definition 1.** Let \( f \) and \( g \) be distributions and let \( g_n \) be as in (1). We say that the product \( f \circ g \) exists and is equal to the distribution \( h \) on \( (a, b) \) if for each \( \varphi \in \mathcal{D}(a, b) \)

\[ \lim_{n \to \infty} \langle f \circ g_n, \varphi \rangle = \lim_{n \to \infty} \langle f, g_n \cdot \varphi \rangle =: \langle h, \varphi \rangle. \]

It turns out that this definition gives an extension of the product of continuous functions (observed as regular distributions). However, the neutrix product of distributions, see [2], is even more general. In order to define it we need

\*AMS Subject Classification (1980): Primary 46F 10*
Definition 2. A neutriz $N$ is a commutative additive group of functions $\nu : N' \to N''$ (where the domain $N'$ is a set and the range $N''$ is a commutative additive group) with the property that if $\nu$ is in $N$ and $\nu(\xi) = \gamma$ for all $\xi$ in $N'$, then $\gamma = 0$. The functions in $N$ are said to be negligible. Now suppose that $N'$ is contained in a topological space with a limit point $b$ which is not in $N'$ and let $N$ be a commutative additive group of functions $\nu : N' \to N''$ with the property that if $N$ contains a function of $\xi$ which tends to a finite limit $\gamma$ as $\xi$ tends to $b$, then $\gamma = 0$. It follows that $N$ is a neutriz. If now $f : N' \to N''$ and there exists a constant $\beta$ such that $f(\xi) - \beta$ is negligible in $N$, then $\beta$ is called the neutriz limit of $f(\xi)$ as $\xi$ tends to $b$ and we write $\lim_{\xi \to b} f(\xi) = \beta$, where $\beta$ is always unique if it exists.

Now let $N' = \mathbb{N}$ and $N'' = \mathbb{R}$ and $N$ be the neutriz whose negligible functions are all linear sums of functions that tend to zero and all functions of the form

$$n^\lambda, n^\lambda \ln^{m-1} n, \ln^m n$$

for all real $\lambda \neq 0$ and $m \in \mathbb{N}$. Then we have

Definition 3. Let $f$, $g$ and $g_n$ be as in Definition 1. We say that the neutriz product $f \cdot g$ exists and is equal to $h$ on the open interval $(a, b)$ if

$$\lim_{n \to \infty} \langle f \cdot g_n, \varphi \rangle = \lim_{n \to \infty} \langle f, g_n \cdot \varphi \rangle =: \langle h, \varphi \rangle$$

for each $\varphi \in \mathcal{D}(a, b)$.

It is important to note that if the product of two distributions exists by Definition 1 then so does the neutriz product and they are equal, see [2]. However, the converse does not hold as the following example shows:

Example 1. Let $f = g = \delta$. Then for arbitrary $\varphi \in \mathcal{D}$ we have $\langle \delta \cdot \delta_n, \varphi \rangle = \delta_n(0)\varphi(0) = n\varphi(0)\varphi(0)$. It follows that the product $\delta \cdot \delta$ does not exist by Definition 1 but it does by Definition 3 and then $\delta \cdot \delta = 0$.

The neutriz product (Definition 3) has some “expected” properties of a product. For instance, if $h = f \cdot g$ exists for $f, g, \in \mathcal{D}'$, then

$$\text{supp } h \subseteq \text{supp } f \cap \text{supp } g, \quad \text{and}$$

$$\text{sing supp } h \subseteq \text{sing supp } f \cup \text{sing supp } g.$$ 

Further, if $f \cdot g$ and $f' \cdot g$ (or $f \cdot g'$) exist, then $f \cdot g'$ (or $f' \cdot g$) exists too and the Leibniz rule holds: $\langle f \cdot g', \varphi \rangle = \langle f', g \varphi \rangle + \langle f \varphi, g' \rangle$. However, the neutriz product is not commutative as the following example shows.

Example 2. Let $f = \delta$ and $g = x^{-1}$. Then $(x^{-1})_n = x^{-1} \ast \delta_n$ is an odd function and so $(x^{-1})_n(0) = 0$. Thus for arbitrary $\varphi \in \mathcal{D}$ we have

$$\langle \delta \cdot (x^{-1})_n, \varphi \rangle = (x^{-1})_n(0)\varphi(0) = 0$$

implying $\delta \cdot x^{-1} \cdot \varphi = 0$, but

$$\langle x^{-1} \cdot \delta_n, \varphi \rangle = \int_0^\infty x^{-1} \left[ \delta_n(x)\varphi(x) - \delta_n(-x)\varphi(-x) \right] dx$$
\[
= \int_0^\infty x^{-1} \delta_n(x) [\varphi(x) - \varphi(-x)] \, dx \\
= \varphi'(0) + O(1/n)
\]

implying \(x^{-1} \cdot \delta = -\delta'.\)

On using Definition 3, one can find several important (neutrix) products of distributions, like \(x_-^\lambda \ln^j x \cdot x^\mu_+\) for different values of \(\lambda, \mu\) and \(j\) (see [2], [3], [4]). However, the more general cases, like \(x_-^\lambda L(x) \cdot x^\mu_+\) or \(x_-^\lambda L(x) \cdot \delta^{(m)}(x)\) cannot be obtained with the neutrix used in the mentioned papers. Here and also throughout this paper \(L : (0, \infty) \to (0, \infty)\) is a given locally integrable function which satisfies the following conditions

\[
\lim_{x \to 0^+} \frac{L(kx)}{L(x)} = 1 \quad \text{for any } k > 0, \quad (4.1)
\]

\[
\lim_{x \to \infty} \frac{L(kx)}{L(x)} = 1 \quad \text{for any } k > 0. \quad (4.2)
\]

A positive locally integrable function satisfying (4.1) (resp. (4.2)) is called slowly varying at zero (resp. slowly varying at infinity). The first example of a function satisfying the relations (4) is the logarithm.

The distribution \(x_-^\lambda L(x)\) in \(S'_+\) (tempered distributions with supports in \([0, \infty)\)), is defined for different values of the real parameter \(\lambda\) by:

\[
\langle x_-^\lambda L(x), \varphi(x) \rangle = \int_0^\infty x^\lambda L(x) \varphi(x) \, dx \quad \text{if } \lambda > -1, \quad (5.1)
\]

\[
\langle x_-^\lambda L(x), \varphi(x) \rangle = \int_0^\infty x^\lambda L(x) \left[ \varphi(x) - \sum_{j=0}^{l-1} \frac{x^j}{j!} \varphi^{(j)}(0) \right] \, dx \quad (5.2)
\]

if \(-l + 1 < \lambda < -l\) and \(l \in \mathbb{N},\)

\[
\langle x_-^\lambda L(x), \varphi(x) \rangle = \int_1^\infty x^\lambda L(x) \left[ \varphi(x) - \sum_{j=0}^{\lambda-1} \frac{x^j}{j!} \varphi^{(j)}(0) \right] \, dx + \int_1^\infty x^\lambda L(x) \left[ \varphi(x) - \sum_{j=0}^{\lambda-2} \frac{x^j}{j!} \varphi^{(j)}(0) \right] \, dx \quad (5.3)
\]

if \(\lambda \in \mathbb{Z} = \{-1, -2, \ldots\}\) and \(\varphi \in \mathcal{S}.
\]

(By definition if \(\lambda = -1\), then the last summation is omitted.)

It is worth noting that if \(L(x) = 1\) on \((0, \infty)\), then the distribution \(x_-^\lambda\) defined by relations (5) coincides with the distribution \(x_-^\lambda\) defined in [6] (see also [2]):

\[
x_-^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + l + 1)} D^l x_-^{\lambda+l} \quad (6)
\]

if \(-l + 1 < \lambda < -l\), \(l \in \mathbb{N}\). The distribution \(x_-^\mu\) is defined in an analogous way to (6); its support is \((-\infty, 0].\)
The aim of this paper is to analyse and explicitly calculate several neutrix products involving slowly varying functions. For that reason we replace the neutrix \( N \) from Definition 3 with the one whose negligible functions are all linear sums of functions that tend to zero and all functions of the form

\[
n^{\lambda}, n^{\lambda}L(1/n), L(1/n)
\]

for all real \( \lambda \neq 0 \) (compare with (2)). Naturally, if \( L(1/n) \) tends to a non zero limit as \( n \to \infty \), then \( L(1/n) \) is omitted in (7).

Before we turn to the announced neutrix products, we cite two statements that we need later on.

**Theorem 1.** Let \( L \) be a slowly varying function at zero and let \( f \) be a locally integrable function on the interval \([0,b]\) with the property that

\[
\int_0^b x^{-\delta} |f(x)| \, dx < \infty \quad \text{for some } \delta > 0.
\]

Then the integral

\[
\Phi(\varepsilon) = \int_0^b f(x)L(\varepsilon x) \, dx
\]

exists and

\[
\Phi(\varepsilon) \sim L(\varepsilon) \int_0^b f(x) \, dx \quad \text{as } \varepsilon \to 0^+.
\]

**Theorem 2.** Let \( x^{\lambda}_{\pm}L(x) \) be given by (5.2) for \(-(l + 1) < \lambda < -l, l \in \mathbb{N}\) and \( L \) a slowly varying function both at zero and at infinity. Then there exists a locally integrable function \( K : (0, \infty) \to \mathbb{R} \) which is both slowly varying at zero and at infinity and satisfies the following conditions:

\[
D^l(x^{\lambda}_{\pm}K(x)) = x^{\lambda}_{\pm}L(x), \quad K_1(x) \sim ((\lambda + 1) \cdot \ldots \cdot (\lambda + l))^{-1} L(x)
\]

as \( x \to 0^+ \) and as \( x \to +\infty \).

Theorem 1 is an easy consequence of Théorème 2 from [1, p. 82], while Theorem 2 was proved in [8, p. 180, Lemma 2].

Because of (3.1) and (3.2) it is clear that

\[
x^{\lambda}_{\pm}L(x) \cdot x^{\mu}_{\pm} = \sum_{j=0}^{n} a_j D^j \delta(x)
\]

for some constants \( a_j \) and some \( n \in \mathbb{N}_0 \) whatever \( \lambda \) and \( \mu \) are, provided that the left hand side exists. The case \( L(x) \equiv 1 \) was analysed in [3]:

**Theorem 3.** The neutrix product of \( x^{\lambda}_{\pm} \) and \( x^{\mu}_{\pm} \) exists and \( x^{\lambda}_{\pm} \cdot x^{\mu}_{\pm} = 0 \), provided that \( \lambda + \mu \notin \mathbb{Z}_- = \{-1, -2, \ldots\} \).
We first of all prove the following generalization of (8):

THEOREM 4. The neutrix product of \( x_+^\lambda L(x) \) and \( x^-_\mu \) exists and

\[
x_+^\lambda L(x) \cdot x^-_\mu = 0
\]

(9)

provided that \( \lambda, \mu, \lambda + \mu \notin \mathbb{Z}_- \), i.e. all \( a_j \) from (8) are zero.

Proof. We follow the lines of the proof of Theorem 6 from [3], giving the modifications necessary because of the slowly varying function \( L \).

Suppose first of all that \( \lambda, \mu, \lambda + \mu > -1 \). Then the left hand side of (9) exists even in the sense of Definition 1, since by (5.1) and (1) we have for \( \varphi \in S \)

\[
\langle x_+^\lambda L(x), (x^-_\mu)_n \varphi(x) \rangle \\
= \int_0^\infty x^\lambda L(x) \varphi(x) \left( \int_0^{1/n} (t-x)^\mu \delta_n(t) \, dt \right) \, dx \\
= \int_0^{1/n} x^\lambda L(x) \varphi(x) \left( \int_x^{1/n} (t-x)^\mu \delta_n(t) \, dt \right) \, dx \\
= \int_0^{1/n} \delta_n(t) \left( \int_0^t x^\lambda L(x) (t-x)^\mu \varphi(x) \, dx \right) \, dt \\
= n^{-\lambda-\mu-2} \int_0^{1} s^{\lambda+\mu+1} \delta_n \left( \frac{s}{n} \right) \int_0^{1} v^\lambda (1-v)^\mu L \left( \frac{sv}{n} \right) \varphi \left( \frac{sv}{n} \right) \, dvds.
\]

The function \( L \) is slowly varying at zero and so we can find a positive number \( \varepsilon > 0 \) such that

\[
|L(x)| \leq C x^{-\varepsilon} \quad \text{for } 0 < x < 1 \text{ and } \lambda + \mu + 1 - \varepsilon > 0;
\]

(10)

the constant \( C \) depends on \( \lambda, \mu \) and \( \varepsilon \). Further on, the function \( \varphi \) is fast decreasing, hence bounded and so

\[
|\langle x_+^\lambda L(x), (x^-_\mu)_n \varphi(x) \rangle| \leq C' n^{-\lambda-\mu+1+\varepsilon} \int_0^{1} s^{\lambda+\mu+1} \rho(s) \left( \int_0^{1} v^\lambda (1-v)^\mu \, dv \right) \, ds
\]

for some constant \( C' > 0 \) and \( \varepsilon > 0 \) from (10).

Now let \( \lambda > -1, -(m+1) < \mu < -m, m \in \mathbb{N} \), and \( \lambda + \mu \notin \mathbb{Z} \). As in [3], we have for \( j \in \mathbb{N}_0 \)

\[
\frac{\Gamma(\mu + m + 1)}{\Gamma(\mu + 1)} \int_{-\infty}^{\infty} x_+^\lambda L(x) (x^-_\mu)_n x^j \, dx \\
= n^{-\lambda-\mu-j-1} \int_0^{1} s^{\lambda+\mu+j+m+1} \rho^{(m)}(s) \int_0^{1} v^\lambda (1-v)^\mu+m L \left( \frac{sv}{n} \right) \, dvds.
\]

On using Theorem 1, we see that the right hand side behaves as

\[
L \left( \frac{1}{n} \right) n^{-\lambda-\mu-1} B(\lambda + j + 1, \mu + m + 1) \int_0^{1} s^{\lambda+\mu+j+m+1} \rho^{(m)}(s) \, ds.
\]
Hence by the suppositions on $\lambda, \mu, j$ and the function $L$ we have
\[
N \lim_{n \to \infty} \int_{-\infty}^{\infty} x_+^\lambda L(x)(x_+^\mu)_n x^j \, dx = 0. \tag{11}
\]
We note that if $j \in \mathbb{N}$ is chosen so that $j > -(\lambda + \mu + 1)$ then (11) holds even in the usual sense. The remainder of the proof for the case $\lambda > -1, \mu, \lambda + \mu \notin \mathbb{Z}$ is essentially as in [3, pp. 324–325] and so we omit it here.

Let us now suppose that (9) is proved for any $\lambda$ such that $\lambda > -l, \lambda \notin \mathbb{Z}$, any $\mu$ such that $\mu, \lambda + \mu \notin \mathbb{Z}$, and any slowly varying function $L$ at both zero and at infinity. On using Theorem 2 for given $\lambda - 1$ and $L$, we can find a function $K$ which is slowly varying at both zero and at infinity and satisfies
\[
D^{l+1}(x_+^{\lambda+l} K(x)) = x_+^{\lambda-1} L(x). \tag{12}
\]

The Leibniz rule gives
\[
D^{l+1}(x_+^{\lambda+l} K(x) \cdot x_+^\mu) = D^{l+1}(x_+^{\lambda+l} K(x)) \cdot x_+^\mu + \sum_{j=0}^{l} \left( \begin{array}{c} l+1 \\ j \end{array} \right) D^j(x_+^{\lambda+l} K(x)) \cdot D^{l+1-j}(x_+^\mu),
\]
or on using (12)
\[
x_+^{\lambda-1} L(x) \cdot x_+^\mu = \sum_{j=0}^{l} C_j D^j(x_+^{\lambda+l} K(x)) \cdot x_+^{\mu-1+l+j}
\]
for some constants $C_j$, provided that we can show the existence of the right-hand side. In fact we will show that each term in the last sum is zero. It is obviously enough to show that
\[
D^j(x_+^{\lambda+l} K(x)) \cdot x_+^{\mu'} = 0 \tag{13}
\]
for $j = 0, 1, \ldots, l$ and $\lambda, \mu', \lambda + \mu' \notin \mathbb{Z}$. This has been proved already for $j = 0$. If (13) is true for some $j \in \{0, 1, \ldots, l-1\}$, then
\[
D^{j+1}(x_+^{\lambda+l} K(x)) \cdot x_+^\mu = D(D^j(x_+^{\lambda+l} K(x)) \cdot x_+^\mu') + \mu' D^j(x_+^{\lambda+l} K(x)) \cdot x_+^{\mu'-1} = 0,
\]
i.e. it is true for $j + 1$ as well. We have thus proved (9) for $\lambda, \mu, \lambda + \mu \notin \mathbb{Z}$.

We now prove

**Theorem 5.** The neutrix product of $x_+^\lambda L(x)$ and $\delta^{(m)}(x)$ exists and
\[
x_+^\lambda L(x) \cdot \delta^{(m)}(x) = 0 \tag{14}
\]
for $m \in \mathbb{N}_0$ and $\lambda \neq 0, \pm 1, \ldots, \pm m, -m-1, -m-2, \ldots$.

**Proof.** Assume first that $\lambda > -1$ and $\lambda \neq 0, 1, \ldots, m$. Then for $j = 0, 1, \ldots$ we have
\[
\int_{-\infty}^{\infty} x_+^{\lambda} L(x) \delta_n^{(m)}(x) x^j \, dx = n^{m-\lambda-j} \int_0^1 t^{j+\lambda} L \left( \frac{t}{\mu} \right) \rho(t) \, dt \\
\sim n^{m-\lambda-j} L \left( \frac{1}{n} \right) \int_0^1 t^{j+\lambda} \rho(t) \, dt \quad \text{as } n \to \infty.
\]

Hence the functions \( \int_{-\infty}^{\infty} x_+^{\lambda} L(x) \delta_n^{(m)}(x) x^j \, dx \) are negligible for \( j = 0, 1, \ldots \) and \( j \neq m - \lambda \). Then we have

\[
\left| \int_{-\infty}^{\infty} x_+^{\lambda} L(x) \delta_n^{(m)}(x) x^{m+1} \, dx \right| \leq C L \left( \frac{1}{n} \right) n^{-\lambda-1} \int_0^1 t^{\lambda+m+1} |\rho(t)| \, dt
\]

for some constant \( C > 0 \). Since \( L \) is slowly varying at zero, the right hand side of the last inequality tends to zero as \( n \) tends to infinity. Using Taylor's theorem for a test function \( \varphi \in \mathcal{S} \), we have

\[
\varphi(x) = \sum_{j=0}^{m} \frac{x^j}{j!} \varphi^{(j)}(0) + \frac{x^{m+1}}{(m+1)!} \varphi^{(m+1)}(\xi x)
\]

for some \( \xi = \xi(x) \in [0, 1] \). It follows from (15) and (16) that

\[
N. \lim_{n \to \infty} \langle x_+^{\lambda} L(x), \delta_n^{(m)}(x) \varphi(x) \rangle = 0
\]

i.e. (14) follows for \( \lambda > -1 \) and \( \lambda \neq 0, 1, \ldots, m \).

Now assume that \(-2 < \lambda < -1\). On using Theorem 2 we can find a locally integrable function \( K \) which is slowly varying both at zero and infinity such that

\[
D(x_+^{\lambda+1} K(x)) = x_+^{\lambda} L(x).
\]

Then

\[
0 = D(x_+^{\lambda+1} K(x) \cdot \delta^{(m)}(x)) = x_+^{\lambda} L(x) \cdot \delta^{(m)}(x) + x_+^{\lambda+1} K(x) \cdot \delta^{(m+1)}(x).
\]

It follows from what we have just proved that (14) holds for \(-2 < \lambda < -1\). More generally, it follows by induction that (14) holds for \( m \in \mathbb{N}_0 \) and \( \lambda \neq 0, 1, \ldots, m, -m-1, -m-2, \ldots \).

We are now going to consider the product \( x_+^{\lambda} L(x) \cdot x_-^{m} \) for \( \lambda \neq \mathbb{Z}_- \). For this purpose we note that by definition

\[
D^m \ln x_+ = -(m-1)! x_+^{-m}, \quad m \in \mathbb{N}
\]

and this is in accordance with (5.3) for \( L(x) \equiv 1 \). Further

\[
D^m \ln x_- = -(m-1)! x_-^{-m}, \quad m \in \mathbb{N}.
\]

**Theorem 6.** The product \( x_+^{\lambda} L(x) \cdot x_-^{m} \) exists and \( x_+^{\lambda} L(x) \cdot x_-^{m} = 0 \) for \( m \in \mathbb{N}_0 \) and \( \lambda \neq \mathbb{Z}_- \).

**Proof.** For \( \lambda > -1 \) we have

\[
\langle x_+^{\lambda} L(x), (x_-^{m})_n x^j \rangle = \int_0^{1/n} x_+^{\lambda+j} L(x) \int_x^{1/n} \ln(t-x) \delta_n^{(m)}(t) \, dt \, dx
\]
\[
= \int_0^{1/n} t^{\lambda+j+1}\delta_\alpha(t) \int_0^1 e^{\lambda+j} \ln(t-\nu)L(t\nu) \, d\nu \, dt.
\]

Putting \( t = s/n \) and using the method from the proof of Theorem 4 we find that
the last double integral is negligible. The rest of the proof is as in [3, p. 326], with
the already used modifications.

We will now use the statement and the proof of Theorem 4 for finding the
\( \alpha \)-product of \( x_\lambda^+L(x) \) and \( x_-^\mu \). This product was analysed in [5]
as a natural generalization of the neutrix product defined by a vector.

**Definition 4.** Let \( f \) and \( g \) be distributions and let \( g_n \) be as in (1). We say
that the \( \alpha \)-neutrix product of \( f \) and \( g_n \), denoted by \( f \ast \alpha g_n \), exists and is equal to the
distribution vector \( h = [h_0, h_1, \ldots, h_r, \ldots] \) on the open interval \((a, b)\)
if
\[
N- \lim_{n \to \infty} \langle f, g_n \varphi \rangle = \langle h_0, \varphi \rangle, \quad N- \lim_{n \to \infty} n^{-\alpha-r} \langle f, g_n \varphi \rangle = \langle h_r, \varphi \rangle
\]
for \( r = 1, 2, \ldots \) and all test functions \( \varphi \in D(a, b) \). It is supposed that \(-1 < \alpha \leq 0\).

The following generalization of Theorem 3 was proved in [5].

**Theorem 7.** Let \( \lambda, \mu \) be real numbers such that \( \lambda, \mu, \lambda+\mu \not\in \mathbb{Z} \) and \( \lambda+\mu < -1 \).
Then the \( \alpha \)-product \( x_\lambda^+ \ast \alpha x_-^\mu \) exists and
\[
x_\lambda^+ \ast \alpha x_-^\mu = h(\lambda, \mu) = [0, h_1(\lambda, \mu), \ldots, h_q(\lambda, \mu)],
\]
where \( q = [-\lambda - \mu], \alpha = -\lambda - \mu - q - 1 \),
\[
h_i(\lambda, \mu) = \frac{\beta(\mu + 1, \lambda + q - i + 1)}{(q - i)!} (-1)^{q-i} a_{q-i}(\lambda, \mu) \delta(q-i),
\]
\[
a_i(\lambda, \mu) = \frac{(-1)^p \Gamma(\lambda + \mu + p + i + 2)}{\Gamma(\lambda + \mu + p + 2)} \int_0^1 u^{\lambda+\mu+p+i+1} \rho(u) \, du
\]
for \( i = 1, \ldots, q \) and \( \beta \) and \( \Gamma \) denote the beta and gamma functions respectively.

In order to find the \( \alpha \)-product of \( x_\lambda^+L(x) \) and \( x_-^\mu \), we must slightly change
Definition 4. In fact, replacing (17) by
\[
N- \lim_{n \to \infty} n^{-\alpha-r} L(1/n) \langle f, g_n \varphi \rangle = \langle h_r, \varphi \rangle, \quad r \in \mathbb{N},
\]
we get a product which will be denoted by \( f \ast (\alpha, L) g; \) \( L \) is a slowly varying function
with properties (4.1) and (4.2). Similarly as Theorem 4 from [5], one can prove

**Theorem 8.** Let \( \lambda, \mu \) be real numbers such that \( \lambda, \mu, \lambda+\mu \not\in \mathbb{Z} \) and \( \lambda+\mu < -1 \).
Then the \((\alpha, L)\) product
\[
x_\lambda^+L(x) \ast (\alpha, L) x_-^\mu = h(\lambda, \mu) = [0, h_1(\lambda, \mu), \ldots, h_q(\lambda, \mu)],
\]
where \( \alpha, q \) and \( h_i(\lambda, \mu) \) are as given in Theorem 7.
Thus we can write
\[ (x_+ L(x), (x_-)_n \varphi) = (0, \varphi) + \sum_{r=1}^{q} \frac{n^{\alpha+r}}{L(1/n)}(h, \varphi) + O\left( \frac{n^{\alpha}}{L(1/n)} \right) \]
as \( n \to \infty \).

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