THE GEOMETRY OF THE DUAL OF A VECTOR BUNDLE

Radu Miron, Stere Ianuș and Mihai Anastasiei

The differential geometry of the total space of a vector bundle has benefited by many interesting papers since the paper [9] by R. Miron has appeared. That paper has led to a deep study of some remarkable geometric structures. The main results from the geometry of the total space of vector bundle as well as some applications of it to General Relativity were published in a recent monograph (R. Miron, M. Anastasiei [14]).

Related to this geometry the geometry of the Lagrange spaces \( L^n = (M, L) \) as well as the geometry of the generalized Lagrange spaces \( M^n = (M, g_{ij}(x, y)) \), (see [11], [6], [1], [16]) has been extensively developed.

Important applications of the theory of the spaces \( M^n \) in studying the effects of the gravitational field were pointed out by A. K. Aringazin and G. S. Asanov [4].

Let \( \xi = (E, \pi, M) \) be a vector bundle and \( \xi^* = (E^*, \pi^*, M) \) its dual. In this paper we study the differential geometry of the manifold \( E^* \) generalizing the results from the geometry of the total space \( T^*M \) of the cotangent bundle \( (T^*M, \tau^*, M) \) of a manifold, [15], [2], or of a Hamilton space, [12], [13].

Our theory is of interest for the Hamiltonian theory of physical fields.

It is known that the main properties of \( T^*M \) are analogous to those of the total space \( TM \) of the tangent bundle \( (TM, \tau, M) \). But there exist properties which are specific for \( T^*M \). For instance, E. Calabi has remarked that on the total space of cotangent bundle of a complex projective space there exists a Kähler metric whose Ricci tensor identically vanishes.

The paper is organized as follows. In §1 the basic notations as well as the concept of nonlinear connection on \( E^* \) are introduced. In §§ 2–4 \( d \)-tensor fields and \( d \)-connections on \( E^* \) are considered. The main properties of the torsion and curvature of a \( d \)-connection are described, too. The equations of structure of a \( d \)-connections are derived in §5. In §6 \( h \)-metrics, \( v \)-metrics and \( (h, v) \)-metrics on

---

*AMS Subject Classification (1980): Primary 53C05, 53C99*
$E^*$ are introduced and the $d$-connections compatible with them are studied. Also, Hamilton function is introduced and it is shown that it determines a $\nu$-metric on $E^*$. The Legendre transformation as a map $E \to E^*$ is studied in §7.

The terminology and notation are those from the monograph [14].

1. The dual vector bundle

Let $\xi = (E, \pi, M)$ be a real vector bundle, whose base $M$ is an $n$-dimensional manifold, the type fiber $F$ is an $m$-dimensional real linear space and the projection $\pi$ is a differentiable map. We shall denote the dual of $\xi$ by $\xi^* = (E^*, \pi^*, M)$. Its type fiber is $F^*$, the dual of $F$.

A trivialization of $\xi$ induces a trivialization of $\xi^*$. Let $U \subset M$ be the domain of a chart of $M$ and $e \in \pi^{-1}(U) \subset E$. Let us denote by $(x^i, y^a)$ the coordinates of $e$ such that $(x^i)$, $1 \leq i \leq n$, are the coordinates of $\pi(e) = x$ and $(y^a)$, $1 \leq a \leq m$, are the coordinates of the $e$ in the fiber $E_x = \pi^{-1}(x)$. If a change of the bundle chart is performed one obtains (see [14])

$$
\begin{align*}
\overline{\tau}^i &= \tau^i(x^1, \ldots, x^n), & \text{rank} \left| \frac{\partial \tau^i}{\partial x^j} \right| &= n, \\
\overline{\gamma}^a &= M^a_b(x) y^b, & \text{rank} \left| M^a_b(x) \right| &= m.
\end{align*}
$$

Here the Einstein summation convention is used and will always be used in this paper.

Let us consider $u \in \pi^{-1}(U) \subset E^*$ such that $\pi^*(u) = x$ and let $(x^i, \overline{p}^a)$ be the canonical coordinates of $u$. If the local chart is changed these coordinates are transformed as follows:

$$
\begin{align*}
\tau^i &= \tau^i(x^1, \ldots, x^n), & \text{rank} \left| \frac{\partial \tau^i}{\partial x^j} \right| &= n, & \overline{p}^a &= \overline{M}^a_b(\tau) \overline{p}^b,
\end{align*}
$$

where the matrix $\left( \overline{M}^a_b(\tau) \right)$ is the inverse of the matrix $\left( M^a_b(x) \right)$. It follows immediately that locally we have $y^a \overline{p}^a = \tau^i \overline{p}^i$ because $\xi$ and $\xi^*$ are dual.

Let us denote

$$
\partial_i = \frac{\partial}{\partial x^i}, \quad \partial^a = \frac{\partial}{\partial \overline{p}^a}.
$$

These vector fields are transformed as follows:

$$
\begin{align*}
\partial_i &= \frac{\partial \tau^k}{\partial x^i} \partial_k + \frac{\partial \overline{M}^a_b(\tau)}{\partial \tau^i} \frac{\partial \tau^k}{\partial x^j} \overline{p}^a \partial^b \\
\partial^a &= \overline{M}^a_b(\tau) \partial^b.
\end{align*}
$$

By (1.4) we can define a global vector field $\overline{p}$ on $E^*$ such that in a system of local coordinates $\overline{p} = p^a \partial^a$. 

Definition 1.1. The vector field $\bar{p}$ on $E^*$ is called the Liouville vector field.

Let $\pi^* T : TE^* \to TM$ be the tangent map to $\pi^*$. Its kernel, denoted by $VE^*$, will be thought of as a distribution $u \to V_u E^*$ on $E^*$, called the vertical distribution of $\xi^*$. It is easy to see that $\pi^* T(\delta^a) = 0$ for $a = 1, \ldots, m$, hence $(\delta^a)$ is a local basis for the vertical distribution. By the Frobenius theorem this distribution is integrable and its maximal integral submanifolds are exactly the fibers $E^*_x$, $x \in M$.

Definition 1.2. A nonlinear connection on $E^*$ is a differentiable distribution $N^*$ on $E^*$ which is supplementary to the vertical distribution $VE^*$, i.e. $T_u E^* = N^*_u \oplus V_u E^*$ holds for every $u \in E^*$.

Proposition 1.1. If $M$ is a paracompact manifold then there exist nonlinear connections on $E^*$.

Proof. One proceeds as in the case of the bundle (see [14]). Since the submersion $\pi^*$ is differentiable we can associate to any vector field $A \in \mathcal{X}(M)$ a unique vector field $A^h$ on $E^*$ such that for every $u \in E^*$, $A^h_u \in N^*_u$ and $\pi^* T(A^h_u) = A_x$, $\pi^*(u) = x$. The vector field $A^h$ will be called the horizontal lift of $A$ with respect to the nonlinear connection $N$. Setting $\delta_i = (\delta^a)_i$, $i = 1, \ldots, n$, it is obvious that $(\delta_1, \ldots, \delta_n)$ is a local basis for the distribution $N^*$ and that there exists a unique system of functions $N_{ai} : \pi^*-1(U) \to R$ ($1 \leq i \leq n, 1 \leq a \leq m$) such that

$$\delta_i = \partial_i + N_{ai}(x, p)\delta^a.$$ (1.5)

The functions $(N_{ai})$ are called the coefficients of the nonlinear connection $N^*$. Sometimes $N^*$ will be called the horizontal distribution on $E^*$.

As in the case of the nonlinear connections on $E$ (see [14]) on can prove:

Proposition 1.2. If a change of bundle charts is performed the following formulae hold:

$$\delta_i = \frac{\partial x^k}{\partial x^i}\delta_k,$$ (1.6)

$$N_{ai}(x, p) = \overline{M^b_a}(x)\frac{\partial x^k}{\partial x^i}N_{bk}(x, p) + p_b \frac{\partial \overline{M^b_a}}{\partial x^i}.$$ (1.7)

Proposition 1.3. If for a trivialization of $\xi^*$ on the domain of each local chart on $E^*$ a system of functions $(N_{ai})$ which are transformed by (1.7) is given, then there exists an unique nonlinear connection $N^*$ on $E^*$ whose coefficients are the given functions.

It is clear that $(\delta_i, \delta^a)$ is a local basis for $\mathcal{X}(E^*)$, which is adapted to the distribution $N^*$ and to the distribution $VE^*$. If we set

$$\delta p_a = dp_a - N_{ai}(x, p)\, dx^i,$$ (1.8)

then $(dx^i, \delta p_a)$ is the basis dual to $(\delta_i, \delta^a)$.
It is easy to see that
\[(1.9) \qquad \delta p_a = \overline{M}_a^b(x) \delta p_b.\]

Now we shall associate to $N^*$ a 2-form $\rho$ on $M$ which is $VE^*$-valued:
\[(1.10) \qquad \rho(A, B) = [A^h, B^h] - [A, B]^h.\]

It is $VE^*$-valued because $A^h$, $B^h$ and $[A, B]^h$ are $\pi^*$-related to $A, B$ and $[A, B]$ respectively, so that $\rho(A, B)$ is just the vertical component of $[A^h, B^h]$. But we know that $N^*$ is integrable iff the vertical component of $[A^h, B^h]$ vanishes.

So we have:

**Theorem 1.1.** The horizontal distribution $N^*$ is integrable if and only if the 2-form $\rho$ identically vanishes.

Locally, we have:
\[(1.11) \qquad \rho(\partial_i, \partial_j) = [\delta_i, \delta_j] = R_{aij} \hat{\delta}^a,\]
where
\[(1.12) \qquad R_{aij} = \delta_i N_{aj} - \delta_j N_{ai}.\]

We also notice:
\[(1.13) \qquad [\delta_i, \hat{\delta}^a] = -\hat{\delta}^a N_{bi}, \quad [\hat{\delta}^a, \hat{\delta}^b] = 0.\]

2. **$d$-tensor fields on $E^*$**

For every vector field $X$ on $E^*$ we shall denote by $X^H$ and $X^V$ its projections on horizontal and vertical distribution, respectively. So we have
\[(2.1) \qquad X = X^H + X^V,\]
where $X^H_u \in N^*_u$ and $X^V_u \in V_u E^*$ for every $u \in E^*$.

We shall say that $X^H$ is a *horizontal vector field* and $X^V$ is a *vertical vector field*.

If we put
\[(2.2) \qquad X^H = X^i(x, p) \delta_i, \quad X^V = X_a(x, p) \hat{\delta}^a,\]
the following rules of transformation hold:
\[(2.2') \qquad \overline{X}(\overline{x}, \overline{p}) = \frac{\partial \overline{x}^i}{\partial x^j} X^j, \quad \overline{X}_a = \overline{M}_a^b(x) X_b.\]

If $\omega$ is an 1-form on $E^*$, we have the decomposition
\[(2.3) \qquad \omega = \omega^H + \omega^V,\]
where $\omega^H$ and $\omega^V$ are 1-forms on $E^*$ defined by

\begin{align}
(2.3') \quad &\omega^H(X) = \omega(X^H), \quad \omega^H(X^V) = 0, \\
(2.3') \quad &\omega^V(X) = \omega(X^V), \quad \omega^V(X^H) = 0, \quad \forall X \in \mathcal{X}(E^*).
\end{align}

Locally, we have

\begin{equation}
(2.4) \quad \omega^H = \omega^i(x,p)dx^i, \quad \omega^V = \omega^a(x,p)\delta p_a
\end{equation}

and following laws of transformation hold:

\begin{equation}
(2.4') \quad \varpi^i(x,p) = \frac{\partial x^j}{\partial x^i}(x,p)\omega_j(x,p); \quad \varpi^a(x,p) = M^a_b(x)\omega^b(x,p).
\end{equation}

**Definition 2.1.** A tensor field $t \in \tau^*_s(E^*)$ with the property

\begin{align}
(2.5) \quad &t(\hat{\omega}, \ldots, \hat{\omega}, X_1, \ldots, X_s) = t(\hat{\omega}^H, \ldots, \hat{\omega}^V, X^H_1, \ldots, X^H_s),
\end{align}

where $X_1, \ldots, X \in \mathcal{X}(E^*)$ and $\hat{\omega}, \ldots, \hat{\omega} \in \mathcal{A}^*(E^*)$, we shall call distinguished tensor field or $d$-tensor field, on $E^*$. If we put

\begin{equation}
(2.6) \quad t^{i_1 \ldots i_{\alpha_1} \ldots} = t(dx^{i_1}, \ldots, \delta^{j_1}, \ldots, \delta^{m_{\alpha_1}}, \ldots, \delta p_{b_1}, \ldots)
\end{equation}

by (1.6) and (1.9), it follows

As an example we mention that the functions $R_{\alpha i j}$ are the components of a $d$-tensor field. By (1.11) it follows that this $d$-tensor field vanishes iff the horizontal distribution $N^*$ is integrable.

### 3. $d$-connections on $E^*$

When a nonlinear connection $N^*$ on $E^*$ is given, special linear connections on $E^*$ can be considered.

**Definition 3.1.** The distinguished connection or $d$-connection on $E^*$ is a linear connection $D$ on $E^*$ which preserves the distributions $N^*$ and $VE^*$ by parallelism.

Setting

\begin{equation}
(3.1) \quad D_X^h = D_X^u, \quad D_X^v = D_X^v, \quad \forall X \in \mathcal{X}(E^*),
\end{equation}

gives

\begin{equation}
(3.1') \quad D_X = D_X^h + D_X^v, \quad \forall X \in \mathcal{X}(E^*).
\end{equation}

Furthermore, $D^h$ determines an algorithm of an $h$-covariant derivation and $D^v$ determines an algorithm of a $v$-covariant derivation (cf. [14]).
We note the following properties of \( D^h \) and \( D^v \), respectively:

\[
\begin{align}
(D^h_X Y^H)^V &= 0, & (D^h_X Y^V)^H &= 0, \\
(D^h_X Y) = (D^h_X Y^H)^V + (D^h_X Y^V)^V, & D^h_X f &= X^H f,
\end{align}
\]

\[
\begin{align}
(D^v_X Y^H)^V &= 0, & (D^v_X Y^V)^H &= 0, \\
(D^v_X Y) = (D^v_X Y^H)^V + (D^v_X Y^V)^V, & D^v_X f &= X^V f,
\end{align}
\]

where \( f \) is an arbitrary function on \( E^* \).

If \( t \in T^r_t(E^*) \) is a \( d \)-tensor field on \( E^* \) then its \( h \)- and \( v \)-covariant derivatives are given by

\[
\begin{align}
(D^h_X t)(\omega, \ldots, X) &= X^H t(\omega, \ldots, X) - t(D^h_X \omega, \ldots, X) - \cdots - t(\omega, \ldots, D^h_X X), \\
(D^v_X t)(\omega, \ldots, X) &= X^V t(\omega, \ldots, X) - t(D^v_X \omega, \ldots, X) - \cdots - t(\omega, \ldots, D^v_X X),
\end{align}
\]

The torsion \( \Pi \) of a \( d \)-connection \( D \) is completely determined by the following five \( d \)-tensor fields of torsions.

\[
\begin{align}
T^H(x, y) &= [\Pi(X^H, Y^H)]^H, & T^V(x, y) &= [\Pi(X^V, Y^V)]^V, \\
R^H(X, Y) &= -[\Pi(X^H, Y^H)]^V [\Pi(X^V, Y^V)]^H, & P^H(X, Y) &= [\Pi(X^H, Y^V)]^V.
\end{align}
\]

The curvature tensor field \( \mathbb{R} \) of a \( d \)-connection \( D \) satisfies:

\[
[\mathbb{R}(X, Y) Z^H]^V = 0, \quad [\mathbb{R}(X, Y) Z^V]^H = 0.
\]

Hence it is completely determined by the following six \( d \)-tensor fields of curvature:

\[
\begin{align}
R(X, Y) Z &= \mathbb{R}(X^H, Y^H) Z^H, & P(X, Y) Z &= \mathbb{R}(X^V, Y^H) Z^H, \\
S(X, Y) Z &= \mathbb{R}(X^V, Y^V) Z^H, \\
\tilde{R}(X, Y) Z &= \mathbb{R}(X^H, Y^H) Z^V, & \tilde{P}(X, Y) Z &= \mathbb{R}(X^V, Y^H) Z^V,
\end{align}
\]

\[
\tilde{S}(X, Y) Z = \mathbb{R}(X^V, Y^V) Z^V.
\]

Every \( d \)-connection has a remarkable form with respect to the adapted basis, its coefficients having simple laws of transformations and giving a new characterisation of it.

**Theorem 3.1.** A \( d \)-connection \( D \) has, with respect to the adapted basis \((\delta_i, \delta^i)\), the following form:

\[
\begin{align}
D^h_{\delta^k} \delta^j &= H^i_{jk} (x, p) \delta^i, & D^h_{\delta^k} \delta^a &= -\tilde{H}^a_{bk} \delta^b \\
D^v_{\delta^k} \delta^j &= C^j_{i} (x, p) \delta^i, & D^v_{\delta^k} \delta^a &= -\tilde{C}^a_{bc}(x, p) \delta^b.
\end{align}
\]

where the coefficients \( H^i_{jk} \) and \( \tilde{H}^a_{bk} \) have the following laws of transformation

\[
\tilde{H}^i_{jk} = \frac{\partial \delta^{i}}{\partial x^k} \frac{\partial \delta^r}{\partial x^j} \frac{\partial x^s}{\partial x^r} \tilde{H}^s_{rs} + \frac{\partial \delta^{i}}{\partial x^r} \frac{\partial \delta^r}{\partial x^s} \frac{\partial \delta^s}{\partial x^j},
\]

\( \tilde{C}^a_{bc}(x, p) \)
and \( C^j_i \), \( \bar{C}^q_b \) are \( d \)-tensor fields.

**Proof.** Since the \( d \)-connection \( D \) preserves by parallelism the distributions \( N^* \) and \( VE^* \), the formulae (3.8) follow directly from (3.2) and (3.3) by using the basis \( \langle \delta_i, \partial \rangle \). From (3.8) and (1.6), (1.9) one obtains (3.9) and (3.9’) as well as

\[
(3.9') \quad \bar{C}^j_i = \frac{\partial x^j}{\partial x^r} M^r_a \bar{C}^a_i, \quad \bar{C}^q_b = M^d_b M^a_c M^j_i \bar{C}^j_i,
\]

which shows that \( C^j_i \) and \( \bar{C}^q_b \) are \( d \)-tensor fields. QED.

**Theorem 3.2.** If on the domain of each local chart on \( E^* \) are given the functions \( (H_{bk}(x,p), H_{bk}(x,p), C_j^i(x,p), C_b^q(x,p)) \) which transform by (3.9), (3.9’) and (3.9”) when the local chart is changed, then there exists a unique \( d \)-connection \( D \) on \( E^* \) whose local coefficients are given functions and which has the properties:

\[
D^h_{\delta_i} f = \delta_i f, \quad D^\alpha_{\partial_a} f = \partial_a f \quad \forall f \in \mathcal{F}(E^*)
\]

**Proof.** For each local chart we can write (3.8). Then define the covariant derivative with respect to \( X = X^i \delta_i + X_a \partial_a \) by

\[
(3.10) \quad D_X = X^i D_{\delta_i} + X_a D_{\partial_a}.
\]

By standard arguments it follows that \( D \) is a linear connection, globally defined on \( E^* \) having as local coefficients just the given functions. The uniqueness is immediate.

**Theorem 3.3.** If the base manifold of the bundle \( \xi^* \) is paracompact, then there exist \( d \)-connections on \( E^* \).

**Proof.** Let \( N^* \) be a nonlinear connection on \( E^* \) having as local coefficients \( N_{ai}(x,p) \) and let \( \Gamma^i_{hk}(x) \) be a linear connection on \( M \), having as local coefficients \( \Gamma^i_{hk}(x) \). Then the set of functions \( (\Gamma^i_{bk}(x), \partial^a N_{bk}, 0, 0) \) satisfies the hypothesis of the Theorem 3.2 QED.

Next we shall give local expressions for the \( h \)-and \( v \) covariant derivatives of a \( d \)-tensor field.

If a \( d \)-tensor field \( t \) is locally given by

\[
(3.11) \quad t = t_{ij}^{i_1 \cdots i_n} \delta_i \otimes \cdots \otimes dx^j \otimes \partial_{\delta_i} \otimes \cdots,
\]

for \( X = X^H = X^i \delta_i \) we have

\[
(3.12) \quad D^h_X t = X^k t_{ij}^{i_1 \cdots i_n} \delta_i \otimes \cdots \otimes dx^j \otimes \partial_{\delta_i} \otimes \cdots
\]

and for \( X = X^V = X_a \partial_a \) we have

\[
(3.13) \quad D^\alpha_X t = X_a t_{ij}^{i_1 \cdots i_n} \delta_i \otimes \cdots \otimes dx^j \otimes \partial_{\partial_a} \otimes \cdots
\]
where we have set

\[
\begin{aligned}
t_{j...a...|k}^{i...b...} &= \delta_k t_{j...a...}^{i...b...} + H_{jk}^i t_{...a...}^{i...b...} + \cdots + \overline{H}_{ak}^b t_{i...a...}^{b...c...} \\
&- H_{jk}^{i...b...} \cdot \cdots - \overline{H}_{ak}^{i...b...} \\
t_{j...a...|i}^{a...} &= \bar{\partial}^a t_{j...a...}^{i...} + C_{jk}^{i...a...} + C_{ik}^{a...} - C_{id}^{a...}. 
\end{aligned}
\]

For instance, the \(h\)- and \(v\)-covariant derivatives of a horizontal vector field \(X = X^i \delta_i\) are given by

\[
X^i_{|k} = \delta_k X^i + H_{jk}^i X^j, \quad X^{i|a} = \bar{\partial}^a X^i + C_{ja}^{i|a} X^j
\]

and for a vertical vector field \(\bar{X} = X_a \bar{\partial}^a\) these derivatives are given by:

\[
X_{a|k} = \delta_k X_a - \overline{H}_{ak}^b X_b, \quad X_{a|^b} = \bar{\partial}^b X_a - \overline{C}_{a}^{cb} X_c.
\]

Also we have

**Proposition 3.1.** \(h\)- and \(v\)-covariant derivatives of the Liouville vector field \(\bar{\varrho} = p_a \bar{\partial}^a\) are

\[
p_{a|b} = D_{ak}, \quad p_{a|^b} = \delta_{a}^{b} - \overline{C}_{a}^{cb},
\]

where

\[
D_{ak} = N_{ak} - \overline{H}_{ak}^b p_b
\]

and \(\circ\) means the contraction by \(p_a\).

It is obvious that \(D_{ak}\) are the local components of a \(d\)-tensor field. This will be called the deflection tensor field of the \(d\)-connection \(D\).

### 4. Curvatures and torsion of a \(d\)-connection

The \(d\)-tensor fields of torsion and curvature of a \(d\)-connection \(D\) on \(E^*\) given by (3.5) and (3.7), respectively, have interesting forms in the adapted basis \((\delta_i, \bar{\partial}^a)\). Putting:

\[
T^H(\delta_k, \delta_j) = T^i_{jk} \delta_i, \quad T^v(\bar{\partial}^c, \bar{\partial}^b) = S_{ab}^{c} \bar{\partial}^a, \quad R^i(\delta_i, \delta_k) = \overline{R}_{ajk} \bar{\partial}^a
\]

\[
T^H(\bar{\partial}^b, \delta_j) = \overline{C}_{a}^{ib} \delta_i, \quad P^1(\bar{\partial}^b, \delta_j) = P_{aj}^{b} \bar{\partial}^a
\]

and taking into account (3.5) and (3.8) one obtains:

**Proposition 4.1.** In the adapted basis \((\delta_i, \bar{\partial}^a)\) the \(d\)-tensor fields of torsion (3.5) have the coefficients:

\[
T^i_{jk} = H_{jk}^i - H_{kj}^i, \quad S_{a}^{bc} = -(\overline{C}_{a}^{bc} - \overline{C}_{a}^{cb}),
\]

\[
P_{aj}^{b} = - (\bar{\partial}^b N_{aj} - \overline{H}_{aj}^c), \quad \overline{R}_{ajk} = R_{ajk}, \quad \overline{C}_{a}^{ib} = C_{j}^{ib}.
\]
**Proposition 4.2.** The d-connection $D$ is without torsion iff the d-tensor fields $T^i_{jk}, S^i_{a b c}, P^i_{a b}, R_{a i j}, C^i_{j}$ vanish.

Now, putting

$$R(\delta_h, \delta_k)\delta_j = R_j^i\delta_{h i}\delta_1, \quad \bar{R}(\delta_h, \delta_k)\bar{\delta}_j = -\bar{R}^i_{a b k}\bar{\delta}_j,$$

$$S(\bar{\delta}^i, \bar{\delta}^j)\delta_j = S^i_{a b c}\delta_1, \quad \bar{S}(\bar{\delta}^i, \bar{\delta}^j)\bar{\delta}_j = -\bar{S}^i_{a b c}\bar{\delta}_1,$$

$$P(\bar{\delta}^i, \delta_k)\delta_j = P^i_{j k}\delta_1, \quad \bar{P}(\bar{\delta}^i, \delta_k)\bar{\delta}_j = -\bar{P}^i_{j k}\bar{\delta}_1,$$

a straightforward calculation leads to:

**Proposition 4.3.** The d-tensor fields of curvature (3.7) have in the adapted basis $(\delta_i, \bar{\delta}^i)$ the following coefficients:

\[
\begin{align*}
R^i_{j k h} &= \delta_h H^i_{j k} - \delta_k H^i_{j h} + H^i_{j k} H^h_{r k} - H^i_{j h} H^h_{r k} + C^i_{j k} R^h_{k k}, \\
\bar{R}^i_{a b k} &= \delta_h \bar{H}^i_{a k} - \delta_k \bar{H}^i_{a h} + \bar{H}^i_{a k} \bar{H}^h_{c k} - \bar{H}^i_{a h} \bar{H}^h_{c k} + \bar{C}^i_{a b c}, \\
P^i_{j k} &= \delta_h C^{i c}_{j k} + H^i_{j k} C^c_{r k} - C^c_{j k} H^i_{r k} + C^i_{j k} (\delta^c N_k), \\
\bar{P}^i_{a b} &= \delta_h \bar{C}^{i c}_{a b} + H^i_{a b} \bar{C}^c_{d k} - \bar{C}^c_{a b} H^i_{d k} + \bar{C}^{i c}_{a b} (\bar{\delta}^c N_k). \\
S^i_{a b c} &= \delta^c C^{i b}_{a c} - \delta^b C^{i c}_{a c} + C^i_{j k} C^j_{a c} - C^i_{j k} C^j_{k c}, \\
\bar{S}^i_{a b c} &= \bar{\delta}^c \bar{C}^{i b}_{a c} - \bar{\delta}^b \bar{C}^{i c}_{a c} + \bar{C}^{i b}_{a c} \bar{C}^j_{d k} - \bar{C}^{i c}_{a b} \bar{C}^j_{d k}.
\end{align*}
\]

We notice the following more interesting forms of $P$ and $\bar{P}$:

\[
\begin{align*}
P^i_{j k} &= \delta^c H^i_{j k} - C^{i c}_{j k} P^c_{a b}, \\
\bar{P}^i_{a b} &= \bar{\delta}^c \bar{H}^i_{a b} - \bar{C}^{i b}_{a c} \bar{P}^{c k}.
\end{align*}
\]

The Ricci identity

$$[D_X, D_Y]Z = \mathbb{R}(X, Y)Z + D_{[X, Y]}Z, \quad \forall X, Y, Z \in \mathcal{X}(E^*),$$

written in the adapted basis, leads to:

**Proposition 4.4.** If $X^H = X^i \delta_i$ is a horizontal vector field, then the following Ricci identities hold:

\[
\begin{align*}
X^i_{|k}|_h - X^i_{|k}|_h &= X^j R^i_{j k h} - T^i_{k h} X^i_{|k}, \\
X^i_{|k} - X^i_{|c} &= X^j P^i_{j k} + C^i_{j k} X^i_{|k} - P^c_{a k} X^i_{|a}, \\
X^i_{|b} - X^i_{|a} &= X^j S^i_{j b c} - S^a_{b c} X^i_{|a}.
\end{align*}
\]

**Proposition 4.5.** If $X^V = \bar{X}^a \bar{\delta}^a$ is a vertical vector field, then the following Ricci identities hold good:

\[
\begin{align*}
X^a_{|k} - X^a_{|k} &= -X^d \bar{R}^a_{d k h} - T^a_{k h} X^a_{|r} - R^a_{d h k} X^a_{|d}, \\
X^a_{|b} - X^a_{|k} &= -X^d \bar{P}^a_{d b k} - C^a_{k b} X^a_{|r} - P^d_{b k} X^a_{|d} \\
X^a_{|b} - X^a_{|c} &= -X^d S^a_{b c} - S^a_{b c} X^a_{|d}.
\end{align*}
\]
As an application of these propositions, using (3.18) one obtains:

**Theorem 4.1.** For any $d$-connection $D$ the following identities hold good:

$$
D_{ak}^{|h} - D_{ah}^{|k} = -R_a^o h_k - T^r_{kh} D_{ar} - R_{ah} (\delta^d_a - \bar{C}^d_a),
$$

$$
D_{ak}^{|b} + C_{a}^{ob} |_{h} = -P_{ak}^{ob} - C_{b}^{rb} D_{ar} - P_{ah}^{ob} (\delta^d_a - \bar{C}^d_a),
$$

$$
-\bar{C}^a_{ab} - C^a_{ac} b = -S_{a}^{abc} - S_{d}^{abc} (\delta^d_a - \bar{C}^d_a).
$$

A $d$-connection for which $C^a_{ab} = 0$, $D_{ak} = 0$ is said to be of Cartan type. Using Theorem 4.1 one obtains:

**Proposition 4.6.** A $d$-connection of Cartan type has the properties:

$$
R_a^o h_k + R_{ah} = 0, \quad P_{ak}^{ob} + P_{ah}^{ob} = 0, \quad S_{a}^{abc} + S_{d}^{abc} = 0.
$$

5. The equations of structure of a $d$-connection

Let $c: (a,b) \rightarrow E^*$ be a curve of class $C^\infty$ on $E^*$. If $X \in \mathcal{X}(E^*)$ then its covariant derivative along $c$, with respect to the $d$-connection $D$ is $D_c X$, which will be also denoted by $DX/dt$.

The curve $c$ is given locally by

$$
\dot{x}^i = x^i(t), \quad p_a = p_a(t), \quad t \in (a,b) \subset \mathbb{R},
$$

where rank $||dx^i/ dt|| = 1$ and rank $||p_a(t)|| = 1$.

The tangent vector $c$ is represented in the adapted basis as

$$
\dot{c} = \frac{dx^i}{dt} \delta_i + \frac{\delta p_a}{dt} \delta_a,
$$

so that we have

$$
\frac{DX}{dt} = \frac{dx^i}{dt} D^b_i X + \frac{\delta p_a}{dt} D^b_i X.
$$

The covariant differential of $X$, with respect to $D$, is $(DX/ dt) dt$. Hence by (5.3) one obtains:

**Proposition 5.1.** The covariant differential $DX$ of a vector field $X$ on $E$ is expressed locally in the adapted basis $(\delta_i, \partial^a)$ as follows:

$$
DX = (D^b_i X) dx^i + (D^a_{\partial^a} X) \delta p_a.
$$

If $X = X^H = X^i \delta_i$, we have

$$
DX^H = (DX^i) \delta_i
$$
where

\( (5.4') \)

\[ DX^i = X_a^i \, dx^k + X^j|a \delta p_a. \]

If we put

\( (5.5) \)

\[ \omega_j^i = H_j^i \, dx^k + C_j^a \delta p_a \]

we obtain

\( (5.5') \)

\[ DX^i = dX^i + \omega_j^i \, X^j. \]

The 1-forms \( \omega_j^i \) will be called the \( h \)-forms of the \( d \)-connection \( D \). In the same way, for \( X = X^V = X_a \theta^a \), putting

\( (5.6) \)

\[ DX^V = DX_a \theta^a, \]

one obtains from (5.4)

\( (5.6') \)

\[ DX_a = X_a^i \, dx^k + X_a^j \delta p_b \]

and putting

\( (5.7) \)

\[ \bar{\omega}_a^b = \bar{H}_a^b \, dx^k + \bar{C}_a^{bc} \delta p_c \]

one obtains

\( (5.8) \)

\[ DX_a = dX_a - \bar{\omega}_a^b \, X_b. \]

The 1-forms \( \bar{\omega}_a^b \) will be called the \( v \)-forms of the \( d \)-connection \( D \).

The differential of a function \( f \in \mathcal{F}(E^*) \) has the form

\( (5.9) \)

\[ df = \delta_k f \, dx^k + \theta^a f \delta p_a. \]

The exterior differential of the 1-forms \( \delta p_a \), according to (1.8) has the following form:

\( (5.10) \)

\[ d(\delta p_a) = -\frac{1}{2} R_{aij} \, dx^i \wedge dx^j - (\bar{\omega}^b \, N_{ai}) \delta p_b \wedge dx^i. \]

Taking into account previous formulae one obtains:

**Theorem 5.1.** The equations of structure of a \( d \)-connection \( D \) on \( E \) are

\( (5.11) \)

\[ D x^k \wedge \omega_h^i = \Omega_h^i, \quad d(\delta p_a) + \delta p_b \wedge \bar{\omega}_a^b = -\bar{\Omega}_a, \]

\( (5.12) \)

\[ d \omega_j^i - \omega_j^h \wedge \omega_h^i = -\Omega_j^i, \quad d \bar{\omega}_b^a - \bar{\omega}_c^a \wedge \bar{\omega}_b^c = -\bar{\Omega}_b, \]

where the 2-forms of torsion \( \Omega^i, \bar{\Omega}_a \) are given by

\( (5.13) \)

\[ \Omega^i = (1/2) T_{jk}^i \, dx^j \wedge dx^k + C_j^a \, dx^j \wedge \delta p_a, \]

\[ \bar{\Omega}_a = (1/2) R_{aij} \, dx^i \wedge dx^j + P_{ai}^b \, dx^i \wedge \delta p_b + (1/2) S_{a}^{bc} \delta p_b \wedge \delta p_c, \]
and the 2-forms of curvature $\Omega_j^k$, $\Omega_b^a$ are given by
\begin{equation}
\Omega_j^k = (1/2)\gamma_{jk}^i dx^i \wedge dx^k + P_{jk}^a dx^a \wedge \delta p_a + (1/2)S_{jk}^{ac} \delta p_c \wedge \delta p_d,
\end{equation}
\begin{equation}
\Omega_b^a = (1/2)\gamma_{ab}^h dx^h \wedge dx^a + P_{ab}^c dx^c \wedge \delta p_c + (1/2)S_{ab}^{cd} \delta p_c \wedge \delta p_d.
\end{equation}

The equations of structure (5.11) and (5.12) allow us to deduce the Bianchi identities (fifteen in number) which are satisfied by any $d$-connection $D$.

These equations also allow us to obtain geometrical meaning for $d$-tensor fields of torsion and curvature.

6. $v$-and $h$-metrical structures on $E^*$

Let us consider a Hamilton function $H$ on the total space $E^*$ of the vector bundle $\xi^*$ i.e. a function
\begin{equation}
H: E^* \rightarrow R
\end{equation}
which is of the class $C^\infty$ on $E^* \setminus \{0\}$ and continuous on the null section. For the case when $\xi^*$ is the cotangent bundle we refer to [12], [13].

The function $H$ defines a $d$-tensor field of type $(2,0)$, symmetric, whose local components are given by
\begin{equation}
g^{ab}(x,p) = (1/2)\delta^a \delta^b H.
\end{equation}
It is said that a Hamilton function $H$ is regular if
\begin{equation}
\text{rank} \|g^{ab}(x,p)\| = m
\end{equation}
on every domain of a local chart on $E^*$.

We shall assume there is given in advance a nonlinear connection $N^*$ on $E^*$.

Definition 6.1. The $v$-metric on $E^*$ is a $d$-tensor field $G^V$ of the type $(2,0)$ with the properties:
1° $G^V$ is vertical i.e. $G^V(X,Y) = G^V(\alpha X, Y^V), \forall X, Y \in \mathcal{X}(E^*)$.
2° $G^V$ is symmetric.
3° The rank of $G^V$ is equal to $\dim E_x$.

If we set
\begin{equation}
g^{ab}(x,p) = G^V(\delta^a, \delta^b)
\end{equation}
it gives the following local form for $G^V$:
\begin{equation}
G^V = g^{ab}(x,p) \delta p_a \otimes \delta p_b
\end{equation}
and, furthermore
\begin{equation}
g^{ab}(x,p) = g^{ba}(x,p), \text{ rank} \|g^{ab}(x,p)\| = m.
\end{equation}
We shall set \( \| g_{ab}(x, p) \| = \| g^{ab}(x, p) \|^{-1} \).

By (6.2) and (6.2') a regular Hamiltonian function \( H \) defines a \( v \)-metric on \( E^* \). Conversely, we have:

**Proposition 6.1.** A \( v \)-metric \( G^V \) is provided by a regular Hamilton function iff the \( d \)-tensor field whose local components are \( \partial^a g^{bc}(x, p) \) is totally symmetric.

**Proof.** A straightforward calculation using (6.2).

**Definition 6.2.** A \( d \)-connection \( D \) on \( E^* \) is compatible with the \( v \)-metric \( G \) if

\[
D_X G^V = 0, \quad \forall X \in \mathcal{X}(E^*).
\]

We remark that (6.6) can be expressed locally as

\[
g^{ab|k} = 0, \quad g^{a|bc} = 0.
\]

**Theorem 6.1.** If \( \left( \vec{H}^i_{jk}, \vec{H}^a_{bk}, 0, 0 \right) \) are the local coefficients of a fixed \( d \)-connection on \( E^* \), then the \( d \)-connection whose local coefficients are \( \left( \vec{H}^i_{jk}, \vec{H}^a_{bk}, 0, \vec{C}^a_{bc} \right) \), where

\[
\begin{align*}
\vec{H}^a_{bk} &= \vec{H}^a_{bk} - (1/2) g_{bc} g^{ac} |^k, \\
\vec{C}^a_{bc} &= -(1/2) g_{ac} (\partial^b g^{de} + \partial^d g^{be} - \partial^e g^{bc})
\end{align*}
\]

is compatible with the \( v \)-metric \( G \).

**Proof.** One verifies (6.6') for the described \( d \)-connection, taking into account

\[
g^{ac|k} = \delta_k g^{ak} + g^{dc} \vec{H}^a_{dk} + g^{ad} \vec{C}^a_{dk}.
\]

**Definition 6.3.** The \( h \)-metric on \( E^* \) is a \( d \)-tensor field \( G^H \) of type \((0,2)\) having the properties:

1° \( G^H \) is horizontal i.e. \( G^H(X, Y) = G^H(X^H, Y^H), \quad \forall X, Y \in \mathcal{X}(E^*) \),

2° \( G^H \) is symmetric.

3° The rank of \( G^H \) is equal to \( n \) in every point of \( E^* \).

Locally we have

\[
G^H = g_{ij}(x, p) dx^i \otimes dx^j,
\]

where we have set

\[
g_{ij}(x, p) = G^H (\delta_i, \delta_j).
\]
Definition 6.4. A $d$-connection $D$ on $E^*$ is compatible with $G^H$ if it satisfies

$$D_X G^H = 0, \quad \forall X \in \mathcal{X}(E^*).$$

(6.11)

Locally, (6.11) can be written as follows:

$$g_{ij\ell k} = 0, \quad g_{ij}^\alpha = 0.$$

(6.12)

Theorem 6.2. The $d$-connection whose local coefficients are $(H^i_{jk}, \delta^b N_{ab}, C^i_{\alpha} \beta, 0)$, where

$$
\begin{align*}
H^i_{jk} &= (1/2)g^{ih}(\delta_h g_{jk} + \delta_j g_{kh} - \delta_k g_{jh}) \\
C^i_{\alpha} \beta &= (1/2)g^{ih} \delta^\gamma g_{hj}
\end{align*}
$$

is compatible with the $h$-metric $G^H$.

Proof. One verifies (6.12) by a straightforward calculation.

Proposition 6.2. If $G^H$ is an $h$-metric and $G^V$ is a $v$-metric on $E^*$ then the tensor field $G$ of the type $(0,2)$ defined by

$$G = G^H + G^V.$$

(6.14)

is a pseudo-Riemannian metric on $E^*$ with respect to which the distributions $N^*$ and $VE^*$ are orthogonal.

Proof. $G$ is symmetric because $G^H$ and $G^V$ are symmetric. Locally $G$ is given by a matrix

$$
\begin{pmatrix}
g_{ij}(x,p) & 0 \\
0 & g^{ab}(x,p)
\end{pmatrix}
$$

(6.15)

which is nondegenerate because $G^H$ and $G^V$ are so. The signature of $G$ is constant. So $G$ is a pseudo-Riemannian metric on $E^*$. By (6.14) the distributions $N^*$ and $VE^*$ are orthogonal with respect to it. QED.

Definition 6.5. A pseudo-Riemannian metric $G$ given by (6.14) will be called an $(h,v)$-metric on $E^*$.

Remark 6.1. If $G$ is a positive definite metric on $E^*$, then the metric induced by it on $VE^*$ is positive definite, too. Let $N^*$ be the distribution which is orthogonal to $VE^*$ with respect to $G$. Then $G$ restricted to $VE^*$ and $N^*$ gives a $v$-metric $G^V$ and $h$-metric $G^H$, respectively, such that (6.14) holds good.

If $G$ is a pseudo-Riemannian metric and the induced metric $G^V$ on $VE^*$ is pseudo-Riemannian, then $N^*$ can still be defined so that $G^H$ is pseudo-Riemannian and satisfies (6.14). Using the adapted basis $(\delta_i, \delta^a)$ an $(h,v)$-metric $G$ can be written as follows:

$$G = g_{ij} \, dx^i \otimes dx^j + g^{ab} \delta p_a \otimes \delta p_b.$$

(6.16)
Definition 6.6. A $d$-connection $D$ is said to be compatible with an $(h, v)$-metric $G$ if we have

$$D_X G = 0, \quad \forall X \in \mathcal{X}(E^*).$$

The condition (6.17), by virtue of (6.16), is equivalent to:

$$g_{ij|k} = 0, \quad g_{ij}^c = 0, \quad g^{ab|k} = 0, \quad g^{ab|^c} = 0.$$

Theorem 6.3. If $\overset{\wedge}{D}$ given locally by $(\overset{\wedge}{H}^j_{ik}, \overset{\wedge}{H}^a_{bk}, \overset{\wedge}{C}^i_{jd}, \overset{\wedge}{C}^a_{bc})$ is a fixed $d$-connection on $E$, then the $d$-connection $D$ with the coefficients

$$H^i_{jk} = \frac{1}{2} g^{ik} (\delta_k g_{jh} + \delta_j g_{hk} - \delta_h g_{jk}), \quad \overset{\wedge}{H}^a_{bk} = H^a_{bk} - (1/2) g_{bc} g_{ik}^a,$$

$$C^i_{jd} = \frac{1}{2} g^{ih} \delta_j g_{hk} , \quad \overset{\wedge}{C}^a_{bc} = - (1/2) g_{ad} (\delta^a g^{dc} + \delta^c g^{ad} - \delta^d g^{bc})$$

is compatible with the $(h, v)$-metric $G$.

7. Legendre morphisms

Let us consider again the vector bundle $\xi = (E, \pi, M)$. A Lagrangian on $E$ is a map $L : E \to R$ which is differentiable on $E \setminus \{0\}$ and continuous on null section. $L$ is called a regular Lagrangian if with respect to any system of local coordinates $(x^i, y^a)$ on $E$, the $d$-tensor field $h$ defined by

$$h_{ab}(x, y) = \frac{\partial^2 L}{\partial y^a \partial y^b}, \quad \text{where} \quad L = (1/2) L,$$

is nondegenerate on $E \setminus 0$.

The vertical derivative of $L$, denoted by $d_V L$, is

$$(d_V L)_e = d(L|_{E_{\pi(e)}})|_e, \quad \forall e \in E.$$

Considering the dual vector bundle $\xi^*$ let us remark that $VE^*$ can be identified with the bundle $(E \times_M E^*, \pi_1, E)$, where

$$E \times_M E^* = \{(e, u) \in E \times E^*, \pi(e) = \pi^*(u)\}$$

and $\pi_1 : E \times_M E^* \to E$ is a projection.

It is obvious that $(d_V L)_e$ belongs to $E \times_M E^*$.

Following Liberman and Mark, [7], we set:

Definition 7.1. Let $\xi = (E, \pi, M)$ be a vector bundle endowed with a Lagrangian $L$. The Legendre morphism associated to $L$ is a morphism $\Phi : E \to E^*$ defined by

$$\Phi = \pi_2 \circ d_V L,$$
where $\pi_2: E \times_M E^* \to E^*$ is a projection.
Locally, we obtain

\begin{align}
\delta_v \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial y^a} dy^a, \\
\Phi(x, y) &= \left( x^i, p_a = \frac{\partial \mathcal{L}}{\partial y^a} \right).
\end{align}

**Proposition 7.1.** If $L$ is a regular Lagrangian, then the Legendre morphism associated to it is a local diffeomorphism $\Phi: E \setminus \{0\} \to E^* \setminus \{0\}$.

*Proof.* The Jacobi matrix of $\Phi$ in every point of $E \setminus \{0\}$ is \( \frac{\partial \Phi}{\partial(x, y)} \) which is nonsingular, because $L$ is regular. QED.

When the Legendre morphism $\Phi$ is a global diffeomorphism it is called Legendre transformation. In such case $L$ is called hyperregular Lagrangian.

**Proposition 7.2.** Let $L$ be a hyperregular Lagrangian on $E$ and $Z$ the Liouville field on $E$. Then the map $H = 2\mathcal{H}$ where

\begin{equation}
\mathcal{H} = (i(Z) d\mathcal{L} - \mathcal{L}) \circ \Phi^{-1}
\end{equation}

is a Hamilton function on $E^*$.

*Proof.* See [7].

Locally, the map $\bar{\mathcal{L}} = i(Z) d\mathcal{L} - \mathcal{L}$ is written

\begin{equation}
\bar{\mathcal{L}} = y^a \frac{\partial \mathcal{L}}{\partial y^a} - \mathcal{L}(x, y).
\end{equation}

Next we have

\begin{align}
\delta_v \bar{\mathcal{L}} &= y^a dv \left( \frac{\partial \mathcal{L}}{\partial y^a} \right) \\
\delta_v \mathcal{H} &= y^a dp_a,
\end{align}

from which one obtains

\begin{equation}
y^a = \frac{\partial \mathcal{H}}{\partial p_a}.
\end{equation}

Therefore $\Phi^{-1}$ is locally as follows

\begin{equation}
\Phi^{-1}: (x^i, p_a) \to \left( x^i, y^a = \frac{\partial \mathcal{H}}{\partial p_a} \right).
\end{equation}

If we assume that $L$ is only regular, the Legendre morphism can be inverted only locally and by (7.7) and (7.8) we can write

\begin{equation}
\mathcal{H}(x, p) = p_a y^a - \mathcal{L}(x, y),
\end{equation}
where \( y^a = y^a(x, p) \), for \( a = 1, \ldots, m \).

From the above considerations we get

**Proposition 7.3.** Let \( L \) be a regular Lagrangian on \( E \setminus \{0\} \) and \( U \) an open subset of \( E \setminus \{0\} \) on which the Legendre morphism is a diffeomorphism. Then on \( V = \Phi(U) \subset E^* \setminus \{0\} \) a regular Hamilton function \( H \) is obtained and \( \Phi \) carries the \( v \)-metric tensor defined by \( L \) on \( U \) to the \( v \)-metric tensor defined by \( H \) on \( V \).

**Proposition 7A.** The Legendre transformation associated to a hyperregular Lagrangian \( L \) applies the \( v \)-metric \( h \) defined by \( L \) on \( E \) to the \( v \)-metric \( g \) defined by the Hamilton function \( H \) induced on \( E^* \).

**Proposition 7.5.** If \( L \) is a regular Lagrangian, then locally we have

\[
\frac{\partial H}{\partial x^i} = -\frac{\partial L}{\partial x^i}.
\]

Now we are interested in the effects of \( \Phi \) on a nonlinear connection.

**Theorem 7.1.** If \( L \) is a hyperregular Lagrangian, then the Legendre transformation \( \Phi \) associated to it carries a nonlinear connection \( N \) on \( E \) to a nonlinear connection \( N^* \) on \( E^* \). If \( N^a_i \) are the local coefficients of \( N \) and \( N_{ai} \) are the local coefficients of \( N^* \) on \( E^* \), then we have

\[
N_{ai}(x, p) = -(N^b_i + \hat{\partial}^b \partial_i H) h_{ba},
\]

where \( H \) is the Hamilton function induced on \( E^* \) and \( h_{ab} \) are the coefficients of the \( v \)-metric induced by \( L \) on \( E \).

**Proof.** Taking into account (7.6) one can see that the differential \( d\Phi \) acts on the canonical basis as follows

\[
d\Phi(\partial_i) = \partial_i + \frac{\partial L}{\partial y^a \partial x^i} \dot{y}^a = \partial_i - (\hat{\partial}^b \partial_i H) h_{bc} \dot{\gamma}^c
\]

\[
d\phi(\hat{\partial}_a) = h_{ab} \dot{\gamma}^b,
\]

so that on \( (\delta_i) \) \( i = 1, \ldots, n \), \( d\Phi \) acts as

\[
d\Phi(\delta_i) = d\Phi(\delta_i - N^a_i \hat{\partial}_a) = \partial_i - (N^b_i + \hat{\partial}^b \partial_i H) h_{ba} \dot{y}^a.
\]

Therefore the distribution \( N \) is mapped by \( \Phi \) to the distribution \( N^* \) and (7.15) holds good. QED.

**Proposition 7.6.** Let \( L \) be a hyperregular Lagrangian and \( \Phi \) the Legendre transformation associated to it. If \( R^a_{ij} \) and \( R_{aij} \) are the integrability tensors of the nonlinear connection \( N \) and \( N^* \), respectively, then

\[
R_{aij} \circ \Phi^{-1} = h_{ab} R^b_{ij}, \text{ holds good}.
\]
Proof. We have \([\delta_1, \delta_2] = R^{a}_{ij} \hat{\partial}_a (\hat{\partial}_a = \partial / \partial y^a)\). Since \(\Phi\) is a diffeomorphism, 
\[d\Phi(\delta_1), d\Phi(\delta_2)] = R^{a}_{ij} d\Phi(\hat{\partial}_a) = R^{a}_{ij} h_{ab} \hat{\partial}_b.\] On the other hand \([d\Phi(\delta_1), d\Phi(\delta_2)] = (R_{ij} \hat{\partial}_j) \circ \Phi^{-1}\). QED.

**Corollary 7.6.** The distribution \(N\) is integrable if and only if the induced distribution \(N^*\) is integrable.

**REFERENCES**


