COMPLETENESS THEOREM FOR A MONADIC LOGIC
WITH BOTH FIRST-ORDER AND PROBABILITY QUANTIFIERS

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Abstract. We prove a completeness theorem for a logic with both probability and first-order quantifiers in the case when the basic language contains only unary relation symbols.

Let $A \subseteq HC$ be an admissible set which contains infinite ordinals and let $L$ be a nonempty $A$-recursive language which contains only unary relation symbols; $HC$ denotes, as usual, the set of hereditarily countable sets.

Definition 1. The set of formulas of $(L_{\omega P \exists})_{AP}$ is the least set such that:
(i) each atomic formula of first-order logic without equality symbol is a formula of $(L_{\omega P \exists})_{AP}$; (ii) if $\varphi$ is a formula, then $\neg \varphi$ is a formula; (iii) if $\Phi \in A$ is a set of formulas, then $\wedge \Phi$ is a formula; (iv) if $\varphi$ is a finite formula, then $(\exists v_n)\varphi$ is a formula; (v) if $\varphi$ is a formula and $r \in A \cap [0,1]$, then $(Px \geq r)\varphi$ is a formula.

Abbreviations $(Px \leq r)$, $(Px = r)$ and $(\forall v_n)$ are introduced as usual.

Definition 2. A probability structure for $L$ is a structure $(\mathfrak{A}, \mu)$ where $\mathfrak{A}$ is a first-order structure for $L$ (with universe $A$), and $\mu$ is a $\sigma$-additive probability measure on $A$ such that each relation of $\mathfrak{A}$ is $\sigma$-measurable.

We can define in the usual way satisfaction relation in a probability structure; here $\mu^n$ denotes the $n$-fold product of $\mu$’s.

Thus: $(\mathfrak{A}, \mu) \models (Px \geq r)\varphi(x, a)$ iff $\mu^n \{ b \in A^n \mid (\mathfrak{A}, \mu) \models \varphi(b, a) \} \geq r$.

The axioms for $(L_{\omega P \exists})_{AP}$ are the axioms A1–A6 and B1–B6 from [K] with the usual first-order axioms. The rules of inference are the rules R1–R3 from [K] with the usual first-order generalization added.

Soundness theorem. If the set $\Phi$ of sentences of $(L_{\omega P \exists})_{AP}$ has a model, then it is consistent.

Lemma 1. Each $(L_{\omega P \exists})_{AP}$ sentence is $(L_{\omega P \exists})_{AP}$-equivalent to a $\sigma$-Boolean combination of finite sentences.

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\textbf{Proof.} The proof can be obtained in the similar way as the proof of the Normal Form Theorem from [H2]. So we omit it.

The notion of a weak structure \((\mathfrak{A}, \mu_n)_{n \in \omega}\) can be introduced as in [H1].

**Lemma 2.** A sentence of \((L_{\omega P3})_{AP}\) is consistent if and only if it has a weak model in which each theorem of \((L_{\omega P3})_{AP}\) is true.

**Proof.** Hoover’s modification of Henkin’s argument (see [H1]) would work.

**Lemma 3.** Let \((\mathfrak{A}, \mu_n)_{n \in \omega}\) be a weak structure, \(\varphi(x, y)\) a finite \((L_{\omega P3})_{HC}\) \(P\)-formula and \(b \in A^m\). Then there is a quantifier free formula \(\Phi(x)\) such that:

\((\mathfrak{A}, \mu_n)_{n \in \omega} \models (\forall x) (\varphi(x, b) \iff \Phi(x))\).

**Proof.** We use induction on the complexity of \(\varphi\). If \(\varphi\) is atomic the statement is trivial. The inductive step when \(\varphi\) is a propositional combination of formulas of smaller rank is also trivial. Suppose now that \(\varphi\) is of the form \((Pz \geq r)\psi(x, z)\). By the inductive assumption we may assume that \(\psi\) is a quantifier free formula.

Further, suppose that \(x\) is \((x_0, x_1, \ldots, x_n)\) and that all relational symbols which occur in \(\psi\) are \(R_0, R_1, \ldots, R_k\). Now define: \(\Gamma(v) = \{ \bigwedge \{ R_i^{[i]}(v) \mid 0 \leq i \leq k \} \mid f \in 2^{k+1} \}\).

Let \(\Sigma(x)\) be the set of all formulas of the form

\[\bigwedge \{ \Phi_i(x_i) \mid 0 \leq i \leq n \} \Phi_i(x_i) \in \Gamma(x_i)\]

for which there exists \(a_0, a_1, \ldots, a_n \in \mathcal{A}\) with: \((\mathfrak{A}, \mu_n)_{n \in \omega} \models \Phi_i(a_i)\) for \(0 \leq i \leq n\), and \((\mathfrak{A}, \mu_n)_{n \in \omega} \models (Pz \geq r)\psi(a, z)\). Finally let \(\Phi(x)\) be the formula \(\forall \Sigma(x)\). It is straightforward to check that the following holds:

\((\mathfrak{A}, \mu_n)_{n \in \omega} \models (\forall x) (\varphi(x, b) \iff \Phi(x))\).

The case when \(\varphi\) is of the form \((\exists z)\psi(x, z)\) can be dealt with in the same way as the previous one, so the claim of the lemma is established.

**Corollary 1.** Let \((\mathfrak{A}, \mu_n)_{n \in \omega}\) be a weak probability structure.

(a) If \(B \subseteq A^n\) is definable by a finite formula, with parameters from \(A\), then \(B\) is \(n\mu_1\) measurable; here by \(n\mu_1\) we denote the finitely additive \(n\) product of \(\mu_1\)’s.

(b) If \(B \subseteq A^n\) is definable by a formula, possible infinite with parameters from \(A\), and \(\mu_n\) is \(\sigma\)-additive then \(B\) is \(\mu^n\)-measurable.

Thus, the corollary allows us to identify \((\mathfrak{A}, \mu_1)\) with \((\mathfrak{A}, \mu_n)_{n \in \omega}\) when only finite formulas are considered.

**Corollary 2.** Let \((\mathfrak{A}, \mu_n)_{n \in \omega}\) be a weak probability structure. Then for every finite \((L_{\omega P3})_{HC}\) \(P\)-formula \(\varphi(x, y)\) with parameters from \(A\), the set \(\{ n\mu_1 \{ b \in A^n \mid (\mathfrak{A}, \mu_1) \models \varphi(b, a) \} \mid a \in A^m \}\) is finite.

**Completeness theorem.** A sentence \(\varphi\) of \((L_{\omega P3})_{AP}\) is consistent if and only if it has a probability model.
Proof. The nontrivial part is to prove that \( \neg \varphi \) implies \( \neg \varphi \), so suppose \( \neg \varphi \).

By Lemma 2 there is a weak structure \((A, \mu_n)_{n \in \omega}\) which is a model for \( \neg \varphi \) and every axiom. By Lemma 1 it is enough to find a probability structure \((B, \nu)\) which is a model for all finite \((L_{\omega, \omega})_{\mathcal{AP}}\) sentences which hold in \((A, \mu_n)_{n \in \omega}\). To do that we will use Rasković’s method from [R]. Let \( K = L \cup C \) \((K_{\omega, \omega})_{\mathcal{AP}}\) be the language (logic) introduced in Hoover’s construction [H1], where \( C \) is a countable set of new constant symbols and \( C \in A \).

Now, we introduce a language \( M \) with three sorts of variables. Let \( X, Y, Z, \ldots \) be variables for sets, \( x_0, x_1, \ldots \) variables for urelements and \( r, s, \ldots \) variables for reals from \([0, 1]\). We suppose that predicates of our language are \( E_n(x_0, x_1, \ldots, x_{n-1}, X) \) for \( n \geq 1 \) (with a canonical meaning \((x_0, x_1, \ldots, x_{n-1}) \in X\)) and \( \mu(X, r) \) (with a meaning \( \mu(X) = r \)). For each finite \((K_{\omega, \omega})_{\mathcal{HC}} P\)-formula we have a constant symbol \( A_p \) for a set, for each real number \( r \in [0, 1] \) a constant symbol \( r \), and a set \( D \) of new constant symbols of the cardinality of the continuum. Functional symbols are \( + \) and \( \cdot \) for reals.

Let \( T \) be the first order theory with the following list of axioms:

1. \((\forall x)(\forall y)(E_n(x, y) \land E_m(x, X) \land E_n(x, X)) \iff x = y \).
2. Axioms of extensionality: \((\forall x)(E_n(x, X) \iff E_n(x, Y)) \iff X = Y \).
3. Axioms of satisfaction:
   - \((\forall x)(E_n(x, A_\varphi) \iff \land_{\psi \in \Phi} E_n(x, A_\varphi)) \) for \( \varphi \) is \( \land \Phi, \Phi \) finite;
   - \((\forall x)(E_n(x, A_\varphi) \iff \neg E_n(x, A_\varphi)) \) for \( \varphi \) in \( \neg \Phi \);
   - \((\forall x)(E_n(x, A_\varphi) \iff (\exists y)E_n(x, y, A_\varphi)) \) for \( \varphi \) is \( \exists \Phi \);
   - \((\forall x)(E_n(x, A_\varphi) \iff (\exists x)(\mu(x, X, r^1) \lor \mu(x, X, r^2) \lor \ldots \lor \mu(x, X, r^5) \land (\forall x)(E_n+1(x, y, A_\varphi) \iff E_m(y, X))))) \) for \( \varphi \) is \( (P_x \geq r) \psi \) where \( r^1, r^2, \ldots, r^5 \) are all reals from the set \( \{m^\mu_1(b \in A^n | (A, \mu_1) \models \psi(b, a)) | a \in A^m \} \) \((\ast)\)
4. Axioms of additivity:
   - \((\forall x)(\exists x)\mu(x, r) \)
   - \((\forall x)(\forall y)(\exists z)(E_n(x, X) \land E_n(x, Y)) \iff (\exists z)(\exists z_1)(E_n(x, z) \land (\forall x)(E_n(x, Z) \iff (E_n(x, X) \lor E_n(x, Y)) \land \mu(Z, r + s)))) \) for \( n \in \omega \).
5. Axioms which are transformations of finite axioms of \((K_{\omega, \omega})_{\mathcal{HC}} P\):
   - \((\forall x)(E_n(x, A_\varphi) \iff \varphi(x)) \) where \( \varphi \) is a finite axiom.
6. Sets of axioms which ensures \( \sigma \)-additivity of extended measure:
   - \(\{E_n(d, A_\varphi) \cup \{\neg E_n(d, A_\varphi) \mid m \in \omega \} \)
where \(\{\varphi_m \mid m \in \omega \} \) is a sequence of finite formulas, \(d\) is a tuple of different constant symbols from \( D \) and all such tuples for a different sequences of formulas are pairwise disjoint, \(\{a \in A^n | (A, \mu) \models \varphi_m(a)) \mid m \in \omega \} \) is a monotone increasing sequence of subsets of \( A^n \), \( (A, \mu) \models (\forall x)(\varphi_m(x) \implies \varphi(x)) \) and
\[
\mu(\{a \in A^n | (A, \mu) \models \varphi(a)) \}
\]
\[
> \sup(\{\mu(\{a \in A^n | (A, \mu) \models \varphi_m(a)) \})
\]
(7) Axioms of a field (for real numbers) with a diagram for + and .

Let a standard structure for the first order logic for \( M \) be the structure \( \mathfrak{M} = (M, B, F, E^\mathfrak{M}_n, \mu^\mathfrak{M}_n, +, \cdot, d^\mathfrak{M}_n, A^\mathfrak{M}_n, r) \) for short, \( \mathfrak{M} = (M, B, F, A_\varphi)_{S} \), where \( B \subseteq \bigcup_{n \geq 1} P(M^n) \), \( F = F' \cap [0, 1] \), \( F' \subseteq R \) a field, \( E^\mathfrak{M}_n \subseteq M^n \times B \), \( \mu^\mathfrak{M}_n : B \to F \), \( +, \cdot : F^2 \to F \), \( d^\mathfrak{M}_n \in M \), \( A^\mathfrak{M}_n \in B \) and \( S \subseteq \{ \varphi \in (L_\omega P_\exists)HC \p \mid \varphi \text{ is finite} \} \).

We claim that \( T \) is consistent. To prove the claim it is enough, by compactness, to show that all finite subtheories of \( T \) are consistent.

First, note that a weak structure can be transformed to a standard structure by taking:

\[
A^\mathfrak{M}_n = \{ \mathfrak{a} \in M^n \mid (\mathfrak{A}, \mu) \models \varphi(\mathfrak{a}) \}, \quad B = \{ A_\varphi \mid \varphi \in (K_\omega P_\exists)HC \p \text{ is finite} \},
\]

and arbitrarily interpreting constants from \( D \), we may get a model for a fixed finite subtheory of \( T \).

Let \( T' \) be a finite subtheory of \( T \) and let \( \varphi, \{ \varphi_n \mid n \in \omega \} \) be as in the axiom 6. Pick some \( m \in \omega \) such that \( \neg E_n(d, A_{\varphi_n}) \in T' \) for all \( k \geq m \). By (**) we may choose \( d^\mathfrak{M}_n \in \{ \mathfrak{a} \in M^n \mid (\mathfrak{A}, \mu_1) \models \varphi(\mathfrak{a}) \} \setminus \bigcup_{n < m} \{ \mathfrak{a} \in M^n \mid (\mathfrak{A}, \mu_1) \models \varphi(\mathfrak{a}) \} \). Thus we get a model for \( T' \).

Since every finite subtheory \( T' \subseteq T \) has a model, by compactness, we conclude that \( T \) has a model, say \( \mathfrak{M} \). Now we can transform our model \( \mathfrak{M} \) to a probability structure with a first order part \( \mathfrak{B} \). For a relational symbol \( R \) of the language \( L \) we define relation \( R^\mathfrak{B} = \{ x \in M \mid E^\mathfrak{M}_n(x) \} \), and a finitely additive measure \( \mathfrak{P} \) on the ring \( \{ A_\varphi \mid \varphi \text{ is finite} \} \) with: \( \mathfrak{P}(A_\varphi) = r \) if \( \mu(A_\varphi, r) \) holds in \( \mathfrak{M} = (M, \ldots) \).

Note that axiom 3d ensures \( \mathfrak{P} \) to map \( \{ A_\varphi \mid \varphi \text{ is finite} \} \) into the reals. Axiom 6 allows us to apply Caratheodory’s Theorem to the measure \( \{ A_\varphi \mid \varphi \text{ is finite} \}, \mathfrak{P} \). Thus \( \mathfrak{P} \) can be extended to a \( \sigma \)-additive measure \( \nu \) on the \( \sigma \)-ring which extends \( \{ A_\varphi \mid \varphi \text{ is finite} \} \). Let \( \nu \) be the \( \sigma \)-additive extension of \( \mathfrak{P} \). It is straightforward to check that \( (\mathfrak{B}, \nu) \) is a probability structure which satisfies the same finite \( (L_\omega P_\exists)_A \) sentences as \( (\mathfrak{A}, \mu_1) \) does. That finishes a proof of the theorem.

REFERENCES


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