ON SOME FORMULAS INVOLVING $!n$ AND THE VERIFICATION OF THE $!n$-HYPOTHESIS BY USE OF COMPUTERS

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Abstract. Kurepa's hypothesis for $!n$ is verified for all $n < 311009$. Some new equivalents to the hypothesis are also found.

0. Introduction

Kurepa introduced in [2] the notion of the left factorial in the following way:

$$!n = \sum_{i=0}^{n-1} i!, \quad n \in \mathbb{N},$$

where $\mathbb{N}$ is the set of nonnegative integers. In the same paper Kurepa asked if

$$\forall n \in \mathbb{N} \ (|n, n|) = 2 \quad \text{(KH)}$$

Here $(a, b)$ denotes the greatest common divisor of integers $a$ and $b$. This conjecture is still an open problem in number theory. There are several results which may speak in the behalf of the hypothesis. For example, Kurepa showed in [3] that there are infinitely many $n \in \mathbb{N}$ for which KH is true. Also, the conjecture is verified by use of computers (Slavić for $n \leq 1000$, and Wagstaff extended the calculation for $n \leq 50000$). Finally there are several statements equivalent to the hypothesis ([2], [5], [6]). One of the most interesting is the following statement, which also belongs to Kurepa (see [2]):

$$\forall n > 2 \ |n| \neq 0 \mod n$$

We shall call this statement also KH. This formulation of KH appears as the open problem B44 in [1]. The aim of this paper is to state some new propositions equivalent to KH, and to exhibit a method for the verification of KH for reasonable large $n$. Implementing this method on computers we verified KH for all $n < 311009$.

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1. Some new equivalents

In [2] it was shown that KH can be reduced to primes, i.e. that KH is equivalent to

$$\forall p \in P \ (p > 2 \Rightarrow (p, p) = 1)$$

(PH)

where $P$ is the set of all primes. If $p$ is a prime and PH is true for all primes $\leq p$ then KH is true for all integers $n \leq p$. This statement easily follows from the fact that all primes which divide $n!$ are $\leq n$. This observation is the key point in our approach to the verification of KH by means of computers.

If $p$ is a prime, let $GF(p)$ be the Galois field of $p$ elements.

**Lemma 1.1.** If $p$ is a prime $\geq 3$ then in $GF(p)$ we have

$$!p = \sum_{k=0}^{p-1} (-1)^{k+1} / k!$$

(1-1)

$$!p = \sum_{k=0}^{p-1} (-1)^{k}(k + 1)(k + 2) \ldots (p - 1)$$

(1-2)

**Proof.** All operations in this proof are the operations of the field $GF(p)$. By Wilson Theorem we have

$$(p-1)! = -1.$$  

(1-3)

Further, for $0 \leq k \leq p - 1$,

$$(p-1)! = (p - k - 1)! (p - k) (p - (k - 1)) \ldots (p - 1).$$  

(1-4)

Observing that in $GF(p) \ p = 0$, we have

$$(p-1)! = (p - k - 1)! k! (-1)^{k}.$$  

(1-5)

Therefore,

$$!p = \sum_{k=0}^{p-1} k!$$

$$= \sum_{k=0}^{p-1} (-1)^{k+1} / (p - k - 1)!, \quad \text{using the substitution } j = p - k,$$

$$= \sum_{j=1}^{p} (-1)^{p-j+1} / (j - 1)!$$

$$= \sum_{j=0}^{p-1} (-1)^{j+1} / j!.$$
Thus (1-1) holds. Further, by this identity we have

\[
\sum_{k=0}^{p-1} \frac{(-1)^{k+1}}{k!} = \sum_{k=0}^{p-1} \frac{(-1)^{k+1}}{(p-1)!} (k+1)(k+2) \ldots (p-1), \quad \text{by Wilson Theorem},
\]

\[
= \sum_{k=0}^{p-1} (-1)^k (k+1)(k+2) \ldots (p-1)
\]

\[
= 1 \cdot 2 \cdot 3 \ldots (p-1) - 2 \cdot 3 \ldots (p-1) + \ldots + (-1)^{p-2}(p-1) + (-1)^{p-1},
\]

and this proves (1-2). \hfill \Box

Corollary 1.2. KH is equivalent to:
For all primes \( p \), GF\((p)\) verifies \( \sum_{k=0}^{p-1} (-1)^{k+1}/k! \neq 0 \).

Corollary 1.3. KH is equivalent to any of the following statements
1. For all primes \( p \), GF\((p)\) verifies \( \sum_{k=0}^{p-1} (-1)^k (k+1)(k+2) \ldots (p-1) \neq 0 \).
2. For all primes \( p \), \( \sum_{k=0}^{p-1} (-1)^k (k+1)(k+2) \ldots (p-1) \neq 0 \mod p \).

Corollary 1.4. KH is equivalent to any of the following statements
1. For all primes \( p \), GF\((p)\) verifies \( \sum_{k=0}^{p-1} \binom{p-1}{k} (-1)^k (k+1)(k+2) \ldots (p-1) \neq 0 \).
2. For all primes \( p \), \( \sum_{k=0}^{p-1} \binom{p-1}{k} (-1)^k (k+1)(k+2) \ldots (p-1) \neq 0 \mod p \).

The last corollary follows from the Corollary 1.3. and the well-known relation \( \binom{p-1}{k} = (-1)^k \mod p \) for all primes \( p \) (which is in fact an immediate consequence of (1-5)).

2. Recurrent formulas

First we introduce some notation. If \( m, n \) are integers, then by \( r = \text{rest}(m,n) \) we shall denote the remainder obtained from division of \( m \) by \( n \). Further, by \( \mathbb{Z}_n \) we denote the ring \( (\mathbb{Z}_n,+n,\cdot n,0,1) \), where \( \mathbb{Z}_n = \{0,1,2,\ldots,n-1\} \). Let \( r_n \) be the sequence defined by \( r_n = \text{rest}(n, n) \), \( n \in \mathbb{N} \). In this section we shall derive some recurrent relations in \( \mathbb{Z}_n \) which shall enable us to compute \( r_n \).

Lemma 2.1. If \( q \) is a prime and \( s_i \) is a sequence defined in GF\((q)\) by regressive induction

\[
s_{q-1} = 0, \quad s_i = 1 + i s_{i+1}, \quad i = q-2, q-3, \ldots, 1
\]

then \( r_q = s_1 \).
Proof. We have

\[ q = 1 + 1 \cdot (1 + 2(1 + 3(1 + \ldots + (1 + (q - 3)(1 + (q - 2)q))\ldots) \]

Therefore, taking \( u_i = 1 + i(1 + (i + 1)(1 + \ldots + (1 + (q - 2)q))\ldots), i = q - 2, \ldots, 1, \)
\( u_{q-1} = q, \) we obtain \( u_i = 1 + i u_{i+1}, i = q - 2, q - 3, \ldots, 1, \) and \( q = u_i \). If we take \( s_i = \text{rest}(u_i,q), \) then \( s_i \) satisfies (2-1) in \( GF(q). \)

\[ \text{Lemma 2.2. If } q \text{ is a prime, and } t_i \text{ is a sequence defined in } GF(q) \text{ by induction in the following way:} \]

\[ t_1 = 0 \]
\[ t_i = (-1)^i + it_{i-1}, \quad i = 2, 3, \ldots, q - 1 \]

then \( r_q = t_{q-1}. \)

Proof. By Lemma 1.1 we have in \( GF(q) \)

\[ q = \sum_{k=0}^{q-1} (-1)^k(k + 1)(k + 2)\ldots(q - 1) \]
\[ = 1 \cdot 2 \cdot 3 \ldots (q - 1) - 2 \cdot 3 \ldots (q - 1) + \ldots (-1)^{q-2} + (-1)^{q-1} \]
\[ = ((1 - 1)2 + 1)3 - 1)4 + 1)\ldots)(q - 2) + (-1)^{q-2}(q - 1) + (-1)^{q-1} \]

Thus taking

\[ t_i = (\ldots((1 - 1)2 + 1)3 - 1)4 + 1)\ldots)i + (-1)^i, \quad i = 1, 2, \ldots, q - 2, \]

we obtain (2-2). \hfill \Box

\[ \text{Lemma 2.3. If } q \text{ is a prime, and } v_i \text{ is a sequence defined in } GF(q) \text{ by} \]

\[ v_1 = 0 \]
\[ v_j = 1 - jv_{j+1}, \quad j = 1, 2, \ldots, q - 2, \]

then \( r_q = v_{q-1}. \)

Proof. For the sequence \( s_i \) in Lemma 2.1. we have

\[ s_{q-i} = 1 + (q - i)s_{q-i+1}, \]

so for \( v_i = s_{q-i} \) we have (observe that \( q = 0 \) in \( GF(q) \):)

\[ v_1 = 0 \]
\[ v_j = 1 - jv_{j+1}, \quad j = 1, 2, 3, \ldots q - 1, \]
\[ v_{q-1} = s_1 = r_q. \]
Now we shall consider some other relations for \( r_n \).

**Lemma 2.4.** \( m \mid n \Rightarrow r_n = r_m \mod m \).

**Proof.** Suppose \( m \mid n \). First we have \( \lnot n = r_n \mod n \) so as \( m \mid n \), \( \lnot n = r_n \mod m \). As \( \text{rest}(\lnot n, m) = \text{rest}(\lnot m, m) = r_m \), and \( \lnot n = \text{rest}(\lnot n, m) \mod m \) it follows \( r_n = r_m \mod m \).

**Corollary 2.5.** If \( m \mid n \) and \( r_n \leq m \) then \( r_n = r_m \).

Suppose \( n = n_1 n_2 \cdots n_k \) where \( (n_i, n_j) = 1 \) for \( 1 \leq i < j \leq k \). Then by Lemma 2.4, \( r_n = r_{n_i} \mod n_i, 1 \leq i \leq k \). Therefore, as \( r_n < n \), by Chinese Reminder Theorem \( r_n \) is the least nonnegative solution of the sistem of congruences \( x = r_{n_i} \mod n_i, 1 \leq i \leq k \).

Further, if \( r_n = 0 \) and \( p \) is a prime dividing \( n \) then by Lemma 2.4 \( r_p = 0 \). This yields a proof of the equivalence \( \text{KH} \Leftrightarrow \text{PH} \).

By the above remarks, the number \( r_n \) is uniquely determined by the prime decomposition \( n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \). Namely, in this case there is an isomorphism

\[
\theta : \mathbb{Z}_{p_1^{a_1}} \times \mathbb{Z}_{p_2^{a_2}} \times \cdots \times \mathbb{Z}_{p_k^{a_k}} \rightarrow \mathbb{Z}_n,
\]

where for \( x \in \mathbb{Z}_n, x_i \in \mathbb{Z}_{p_i^{a_i}}, 1 \leq i \leq k \),

\[
x = \theta(x_1, x_2, \ldots, x_k) \text{ iff for all } 1 \leq i \leq k, \quad x = x_i \mod p_i^{a_i}.
\]

Thus \( r_n \) can be computed by

\[
r_n = \theta(r_{p_1^{a_1}}, \ldots, r_{p_k^{a_k}}).
\]

3. **The Second Kurepa hypothesis**

In [2] Kurepa also conjectured the following hypothesis:

**(KH2)** The relation \( m^2 \mid n \), except \( 2^2 \mid 3 \), does not have solutions in integers greater than 1.

Obviously KH2 is equivalent to to the statement that \( \lnot n \) is square-free. Thus we shall consider KH2 only for primes \( m \). If \( p \) is a prime and \( n \geq p \) then \( p^2 \mid p \) implies \( p \mid p \), therefore we have immediately:

**Proposition 3.1.** KH implies that for any \( m \) there are at most finitely many \( n \) such that \( m^2 \mid n \).

Let KH\( (m) \) denote the Kurepa hypothesis for all integers below \( m \) i.e. KH\( (m) \) is the formula

\[
\forall n \leq m \ (n > 2 \Rightarrow \lnot n \neq 0 \mod n)
\]
Therefore, $KH \Leftrightarrow \forall m \; KH(m)$. Further, let $KH2(m)$ denote the formula
\[ \forall k < m \forall n < k \; (1 < k \land 3 < n \implies \text{rest}(\lfloor n/k^2 \rfloor) \neq 0). \]

By above remarks we have immediately the following statement:

**Proposition 3.2.** $KH \land \forall m \; KH2(m) \Rightarrow KH2$.

As an refinement of the last proposition we have the following. Assuming $KH$ is true it suffices to check if for $r_n^2 = \text{rest}(\lfloor n/p^2 \rfloor)$
\[ r_n^2 \neq 0, \quad \text{for } n < p \]  
(3-1)
to conclude that there is no $n$ such that $p^2 \mid n$. Also, if (3-1) is verified for all primes $p \leq m_0$ then for all $m \leq m_0$ there is no $n$ such that $m^2 \mid n$. Kurepa stated in [2] that, except $2^2 \mid 3$, for $m = 2, 3, 4, 5, 6, 7, 8$ for no $n$, $m^2 \mid n$. Using the above observation we approved this statement for first 200 primes, so $KH2(m)$ is true for $m \leq 1223$. In this computation we can use similar recurrent formulas as in Lemma 2.1. Namely it is easy to see that for the sequence $s_i$ defined in $\mathbb{Z}_{p^2}$ by regressive induction
\[ s_{n-1} = n \]
\[ s_i = 1 + i \cdot s_{i+1}, \quad i = n - 2, n - 3, \ldots, 1, \]  
(3-2)
we have $r_n^2 = s_1$.

4. Program implementation

Using formulas developed in previous sections, we implemented computer programs for verifying KH and KH2. All programs are written in FORTRAN 77 for transputer station based on three transputers T800. Some of the routines were implemented in parallel version of FORTRAN 77 (3L Parallel FORTRAN) using the advantage of parallel computing abilities of processors T800.

4.1. **Computer verification of KH.** In fact we were verifying $KH(m)$ for $m \leq m_0$, where $m_0$ is the given bound. The first part of the program is used to generate the sequence of first $i_0$ primes $p_i$, $1 \leq i \leq i_0$. In the main part of the program $KH(p_i)$ is checked for $i \leq i_0$. As for $i_0 = 26880$, for which $p_{i_0} = 311009$, for no $i \leq i_0$ $r_{p_i} = 0$, it follows (see the beginning of Section 1) that $KH(n)$ is true for all $n \leq 311009$; thus in our case $m_0 = 311009$.

In the computation of $r_p$ we were using recurrent formulas (2-1), therefore the program code of this part of the program looks as

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This facility belongs to the Mathematical Faculty of the Faculties of Sciences, University of Belgrade.
DO 1001 J=IP,L
Q=P(J)
KNQD=Q
QD=Q
DO 1002 IL=Q-2,1,-1
1002 KNQD=DEPENDENT(DBLE(1+IL*KNQD),QD)
WRITE(2,405) J,Q,KNQD
IF (KNQD.EQ.0) THEN
STOP
ENDIF
1001 CONTINUE
C P(J) is the sequence of primes
C KNQD and QD are DOUBLE PRECISION variables
C The value of KNQD after the loop 1002 is rQ

Some refinements are used in the program. In order to make possible computation in several sessions, the program is written so that KH(p) can be verified in an interval \( i_1 \leq i \leq i_2 \) (in the above program \( i_1 = 1P \), and \( i_2 = L \)). If \( i \leq 4792 \) then \( p_i \leq 46337 \) so \( 1+IL*KNQD \) in the above listing does not exceed the INTEGER bound in FORTRAN 77, and DBLE can be omitted in the line 1001 of the listing, and variables KNQD, QD can be replaced by the INTEGER variable Q. Therefore, for small values of \( p_i \) (\( p_i \leq 46337 \)) only INTEGER variables can be used, thus the program can be speeded up for several times (and then it takes only several minutes to check KH(46337)).

Further, the whole program is written as a transputer application divided into \( k \) tasks (in our case \( k = 3 \)) so that each processor of the transputer station can execute in parallel a task. The task \( T_j \) is used to verify KH\((p_i)\) for \( i = j \mod k \), \( i \leq i_0 \). From the program code we see that the number of arithmetical operations needed for the computation of \( r_p \) is 4\( p \). Thus the total number of arithmetical operations used in the verification of KH\((x)\) is

\[
A = \sum_{p \leq x} 4p
\]  

(4-2)

where \( p \) in the sum runs over primes. Using Stieltjes integral, we obtain

\[
A = 4 \int_2^x x \pi(x) dx - 4 \int_2^x \pi(x) dx \sim 4x^2 \pi(x) \sim \frac{4x^2}{\ln x}
\]

where \( \pi(x) \) is the number of primes \( \leq x \). Hence, we have the following asymptotic formula

\[
A \sim \frac{4x^2}{\ln x}
\]

(4-3)

If \( k \) transputers are used (we call a such configuration a \( k \)-farm), and if \( \tau \) is an average execution time interval of an arithmetical operation, then

\[
A_k(x) = \frac{4x^2\tau}{k \cdot \ln x}
\]

(4-4)
On some formulas involving \( \ln \) and the verification ...

is the total time used for the above computation. In order to find the efficiency of
\( k \)-farms in respect to one-transputer station, we compare \( A_k(x) \) and \( A_1(y) \) for the
same time interval, i.e.
\[
\frac{4x^2 \tau}{k \cdot \ln x} = \frac{4y^2 \tau}{\ln y}.
\]

Therefore, \( x = \sqrt{k} \sqrt{\ln x/\ln y} \cdot y \). From this equation it follows that the
efficiency of a \( k \)-farm in the verification of \( \text{KH}(y) \) is \( e(y) = \sqrt{k} \sqrt{\ln x/\ln y} > \sqrt{k} \).
For fixed \( k \) it is easy to see that \( \lim_{y \to \infty} \sqrt{\ln x/\ln y} = 1 \), so for large intervals
the efficiency of a \( k \)-farm is asymptotically equal to \( \sqrt{k} \). This means that if one
transputer in a given time interval verifies \( \text{KH}(x) \), then a \( k \)-farm in the same time
interval verifies \( \text{KH}(x \sqrt{k}) \).

Prime factorization table of \( n \)

<table>
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<tr>
<th>( n )</th>
<th>!n</th>
<th>Factorization</th>
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</thead>
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<td>1</td>
<td>1</td>
<td>1</td>
</tr>
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<td>2</td>
<td>2</td>
</tr>
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4.2. Computer verification of \( \text{KH}_2 \). According to Section 3 it suffices to
verify \( \text{KH}_2(m) \). Using formulas (3-2) we checked \( \text{KH}_2(m) \) for \( m \leq 1223 \), so \( m^2 \! | \! n \)
has no solutions for \( m \leq 1223 \) and arbitrary \( n \).
We also computed the prime decomposition of $\ln$ for $n \leq 24$, see the table. As it is seen, $\ln$ is squarefree for $n \leq 24$. For the prime decomposition of these numbers we used the tabulation of $\ln$ in [7], and algorithms presented in [4]. These algorithms are implemented in FORTRAN 77 by M. Dražić.

REFERENCES


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