ON A DENSE $G_δ$-DIAGONAL

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Abstract. We study topological spaces the diagonal of which contains a dense set which is a $G_δ$-set in $X \times X$.

We use the usual notation and terminology as in [6], [7], [2]. All spaces are at least $T_2$.

Let us say that $X$ is a space with a dense $G_δ$-diagonal if there exists a $G_δ$-subset $U$ of the space $X \times X$ such that $U \subset Δ_X$ and $\overline{U} = Δ_X$. Here $Δ_X = \{(x,x) : x \in X\}$ is the diagonal in $X \times X$.

This notion was introduced in [11] under the name “weak $G_δ$-diagonal” (see also [12] about related subjects). In the same paper it was proved that if the space $\exp X$ of all closed subsets of $X$ with the Vietoris topology is weakly perfect, then $X$ has a dense $G_δ$-diagonal. A space $X$ is called weakly perfect [11], [13] if every closed subset of $X$ contains a dense set which is a $G_δ$-set in $X$. Note that there are spaces which are weakly perfect but not perfect [9].

Proposition 1. $X$ is a space with a dense $G_δ$-diagonal if and only if there exists a subspace $Y \subset X$ such that $\overline{Y} = X$, $Y$ is a $G_δ$-set in $X$ and $Y$ has a $G_δ$-diagonal.

Proof. $(\implies)$ Let $\{U_n : n \in \mathbb{N}^+\}$ be a family of open subsets in $X \times X$ such that $\bigcap\{U_n : n \in \mathbb{N}^+\} \subset Δ_X$ and $\bigcap\{U_n : n \in \mathbb{N}^+\}$ is dense in $Δ_X$. Put $V_n = \{x \in X : (x,x) \in U_n\}$. Clearly, each $V_n$ is open in $X$ and $Y = \bigcap\{V_n : n \in \mathbb{N}^+\}$ is the subspace we are looking for.

$(\impliedby)$ Let $Y$ be a $G_δ$-subset of $X$. Then $Y \times Y$ is a $G_δ$-subset of $X \times X$. Indeed, let $Y = \bigcap\{V_n : n \in \mathbb{N}^+\}$ where each $V_n$ is open in $X$. We can choose $V_n$ to satisfy the condition: $V_{n+1} \subset V_n$ for all $n \in \mathbb{N}^+$. Then $Y \times Y = \bigcap\{V_n \times V_n : n \in \mathbb{N}^+\}$.

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If $Y$ is dense in $X$ then $\Delta_Y$ is dense in $\Delta_X$. If the diagonal $\Delta_Y$ is a $G_\delta$-subset of $Y \times Y$ then $\Delta_Y$ is a $G_\delta$-subset of $X \times X$ as $Y \times Y$ is a $G_\delta$-subset of $X \times X$.

**Theorem 1.** Let $X$ be a Čech-complete space. Then $X$ has a dense $G_\delta$-diagonal if and only if it contains a dense subspace metrizable by a complete metric.

**Proof.** ($\Leftarrow$) If $Y$ is dense in $X$ and the space $Y$ is metrizable by a complete metric then $Y$ has a $G_\delta$-diagonal and $Y$ is a $G_\delta$-subset of $Y$ (see [6], [7]). Then by Proposition 1, $X$ is a space with a dense $G_\delta$-diagonal. (We didn’t use in this part of the argument Čech-completeness of $X$).

($\Rightarrow$) Assume that $X$ has a dense $G_\delta$-diagonal. By Proposition 1 there exists a $G_\delta$-subset $Y$ of $X$ which is dense in $X$ and is a space with a $G_\delta$-diagonal. As $X$ is Čech-complete and $Y$ is a $G_\delta$ in $X$ the space $Y$ is also Čech-complete. By a result of Šapirovskii (see [15]), there exists a paracompact Čech-complete subspace $Z$ of $Y$ which is dense in $Y$. Then $Z$ is also dense in $X$. The space $Z$ also has a $G_\delta$-diagonal (this property is obviously inherited by arbitrary subspaces). But it is well known that every paracompact Čech-complete space with $G_\delta$-diagonal is metrizable (see [7]). Moreover if a metrizable space is Čech-complete then it is metrizable by a complete metric [6], [7]. It follows that $Z$ is metrizable by a complete metric. The theorem is proved.

**Remark 1.** From the proof of the first part of Theorem 1 and the fact that countable product of complete metric spaces is complete we have: if a space $X$ contains a dense subspace metrizable by a complete metric, then the spaces $X^n$, $n \in \mathbb{N}^+$, and $X^\omega$ have a dense $G_\delta$-diagonal.

**Question 1.** Can a space $X^\omega$ be weakly perfect?

**Corollary 1.** Let $X$ be a Čech-complete space with a dense $G_\delta$-diagonal such that the Souslin number of $X$ is countable. Then $X$ has a countable $\pi$-base. Hence $X$ is separable and every dense subspace of $X$ is separable.

Recall that a $\pi$-base of a space $X$ is a family $\mathcal{V}$ of non-empty open subsets of $X$ such that every open subset $U$ of $X$ contains some $V \in \mathcal{V}$ (see [2], [6], [10]).

**Proof of Corollary 1.** By Theorem 1 there exists a dense metrizable subspace $Y$ of the space $X$. As $X = Y$, the Souslin number of $X$ does not exceed the Souslin number of $X$ (see [2], [10]). Hence $c(Y) \leq \omega$. As $Y$ is metrizable it follows that $Y$ has a countable base $\mathcal{B}$. For each $U \in \mathcal{B}$ fix an open subset $\bar{U}$ of $X$ such that $\bar{U} \cap Y = U$. Then the countable family $\{\bar{U} : U \in \mathcal{B}\}$ of open subsets of $X$ is a $\pi$-base of $X$ — this is shown easily using the fact that $Y$ is dense in $X$.

**Corollary 2.** Let $X$ be a Čech-complete space such that the space $X \times X$ is weakly perfect. Then in every closed subspace of $X$ there exists a dense subspace metrizable by a complete metric.

**Proof.** Let $X_1$ be a closed subspace of $X$. Then $X_1$ is Čech-complete and weakly perfect — both properties are inherited by closed subspaces. Obviously if
the space $X_1 \times X_1$ is weakly perfect, then $X_1$ has a dense $\mathbb{G}_\delta$-diagonal. Hence $X_1$ satisfies the assumptions in Theorem 1 and thus there exists a dense subspace in $X_1$ metrizable by a complete metric.

Recall that spread $s(X)$ of a space $X$ is the supremum of cardinalities of discrete subspaces of $X$.

**Theorem 2.** Let $X$ be a Čech-complete space such that the space $X \times X$ is weakly perfect. Then spread of $X$ is equal to hereditary density of $X$: $s(X) = \text{hd}(X)$. In particular, if all discrete subspaces of $X$ are countable, then $X$ is hereditarily separable.

**Proof.** For metrizable spaces spread is equal to density. We also have $s(Y) \leq s(X)$ for every subspace $Y \subset X$. From Corollary 2 it follows now that density of every closed subspace of $X$ does not exceed spread of $X$. As $X$ is Čech-complete it is a $k$-space and for $k$-spaces the following inequality (of Arhangel’skii-Šapirovskii) holds: tightness is not greater than spread (see [2]). Thus $t(X) \leq s(X)$. Put $s(X) = \tau$ and let $Y$ be any subspace of $X$. Then $t(Y) \leq \tau$ and $d(Y) \leq \tau$ as $Y$ is closed in $X$. Fix a subset $A \subset X$ such that $A = \overline{A}$ and $|A| \leq \tau$. For each $a \in A$ we can fix a subset $B_a \subset Y$ such that $|B_a| \leq \tau$ and $a \in \overline{B_a}$. Then for the set $M = \bigcup \{B_a : a \in A\}$ we have: $|M| \leq \tau \cdot \tau = \tau$, $M \subset Y$ and $\overline{M} = \overline{Y} \supset Y$. Thus $d(Y) \leq \tau = s(X)$, i.e. $\text{hd}(X) \leq s(X)$. It is always true that $s(X) \leq \text{hd}(X)$. Hence $\text{hd}(X) = s(X)$.

**Remark 2.** Our results on weakly perfect $X \times X$ remain true under weaker assumption that every closed subspace $F$ of $\Delta_X$ contains a subset $A$ which is a $G_\delta$-set in $F$ and is dense in $F$.

From Corollary 2 we derive

**Corollary 3.** Let $X$ be a compact non-separable space, the Souslin number of which is countable. Then $X$ does not have a dense $\mathbb{G}_\delta$-diagonal. Hence $X \times X$ is not weakly perfect.

From Theorem 1 we get

**Corollary 4.** If $X$ is a Čech-complete space with a dense $\mathbb{G}_\delta$-diagonal, then $X$ satisfies the first axiom of countability at a dense $G_\delta$-set of points.

**Proof.** There exists a dense subspace $Y$ of $X$ metrizable by a complete metric. Then $Y$ is a $G_\delta$-subset of $X$ and $X$ is first countable at every point of $Y$ (as $X$ is regular and $Y$ is dense in $X$ — see [10]).

Every dyadic compactum which is first countable at a dense set of points is metrizable — this is the well known result of Efimov (see [7]). Now Corollary 4 implies the following assertion:

**Corollary 5.** If a dyadic compactum $X$ has a dense $\mathbb{G}_\delta$-diagonal then $X$ is metrizable.
Let us recall that a space $X$ is called $\aleph_0$-monolithic if closure of every countable subset $A \subset X$ is a space with a countable network [1] (see also [4], [5]). Every compact space with a countable network is metrizable [6], [7]. Applying Corollary 1 we get

**Corollary 6.** If $X$ is an $\aleph_0$-monolithic compact space the Souslin number of which is countable and $X$ has a dense $G_\delta$-diagonal, then $X$ is metrizable.

Of course the last assertion is also true for Čech-complete spaces.

In connection with Corollary 4 we have the following assertion which can be proved in a similar way as one proves the fact that every space with a $G_\delta$-diagonal has countable pseudo-character.

**Proposition 2.** If a space $X$ has a dense $G_\delta$-diagonal, then the set of points of countable pseudocharacter is dense in $X$.

From this proposition and the fact that for every topological group $G$ one has $\psi(G) = \Delta(G)$ [3] we derive

**Corollary 7.** If $G$ is a topological group with a dense $G_\delta$-diagonal, then $G$ has a $G_\delta$-diagonal.

There is an interesting necessary and sufficient condition for a space $X$ to have a dense $G_\delta$-diagonal.

**Proposition 3.** A space $(X, \mathcal{T})$ has a dense $G_\delta$-diagonal if and only if there exist a subset $Y \subset X$ dense in $(X, \mathcal{T})$ and a topology $\mathcal{T}_1$ on $X$ such that $\mathcal{T} \subset \mathcal{T}_1$, the space $(X, \mathcal{T}_1)$ has a $G_\delta$-diagonal and $\mathcal{T}$ is a base of $(X, \mathcal{T}_1)$ at all points $y \in Y$.

**Proof.** ($\Leftarrow$) There exist open sets $U_n, n \in \mathbb{N}^+$, in the product space $(X, \mathcal{T}_1) \times (X, \mathcal{T})$ such that $\bigcap \{U_n : n \in \mathbb{N}^+\} = \Delta_X$. For each $y \in Y$ and each $n \in \mathbb{N}^+$ we can fix a $V(y, n) \in \mathcal{T}$ such that $y \in V(y, n)$ and $V(y, n) \times V(y, n) \subset U_n$. Put $G_n = \bigcup \{V(y, n)^2 : y \in Y\}$ for every $n \in \mathbb{N}^+$. Obviously $\Delta_Y \subset G_n \subset U_n$ and $G_n$ is open in $(X, \mathcal{T}) \times (X, \mathcal{T})$. Hence $\Delta_Y \subset \bigcap \{G_n : n \in \mathbb{N}^+\} \subset \Delta_X$. As $\Delta_Y$ is dense in $\Delta_X$, the set $\bigcap \{G_n : n \in \mathbb{N}^+\}$ is the one we were looking for. Thus $X$ has a dense $G_\delta$-diagonal.

($\Rightarrow$) Let $B$ be a dense subset of $\Delta_X$ which is a $G_\delta$-subset in the space $(X, \mathcal{T}) \times (X, \mathcal{T})$. Fix open sets $U_n$ in $(X, \mathcal{T}) \times (X, \mathcal{T})$ for $n \in \mathbb{N}^+$ such that $\bigcap \{U_n : n \in \mathbb{N}^+\} = B$. Put $Y = \{x \in X : (x, x) \in B\}$ and $B_1 = \mathcal{T} \cup \{\{x\} : x \in X \setminus Y\}$. Then $B_1$ is a base of a topology $\mathcal{T}_1$ on $X$. It is clear that $\mathcal{T} \subset \mathcal{T}_1$ and that $\mathcal{T}$ is a base of the space $(X, \mathcal{T}_1)$ at all points of the set $Y$. It remains to check that the space $(X, \mathcal{T}_1)$ has a $G_\delta$-diagonal.

Let $W_n = U_n \cup \Delta_X$. Then $W_n$ is open in the product space $(X, \mathcal{T}_1) \times (X, \mathcal{T}_1)$ by the definition of $\mathcal{T}_1$. Clearly, $\bigcap \{W_n : n \in \mathbb{N}^+\} = \Delta_X$. Hence $(X, \mathcal{T}_1)$ has a $G_\delta$-diagonal. The proposition is proved.

As every metrizable space has a $G_\delta$-diagonal the following assertion is a direct corollary of Proposition 3.
Theorem 3. A space \((X, \mathcal{T})\) has a dense \(G_\delta\)-diagonal if there exists a metrizable topology \(\mathcal{T}_1\) on \(X\) such that \(\mathcal{T} \subset \mathcal{T}_1\) and the set of all points at which \(\mathcal{T}\) is a base of the topology \(\mathcal{T}_1\) is dense in the space \((X, \mathcal{T})\).

The conditions in Theorem 3 are satisfied by every Eberlein compactum (see T.4.3 in [4]). Thus we have

Corollary 8. Every Eberlein compactum has a dense \(G_\delta\)-diagonal.

One could derive Corollary 8 from Theorem 1 on the following fact — Namio-ka’s theorem (see [2]): in every Eberlein compactum there exists a dense subspace metrizable by a complete metric.

Every Gul’ko compact space [5] also has a dense subspace metrizable by a complete metric (Leiderman-Gruenhage; see [14], [8] or [5]. Thus applying Theorem 1 we get.

Corollary 9. Every Gul’ko compact space has a dense \(G_\delta\)-diagonal.

Remark 3. S. Todorčević has shown that not in each Corson compactum [5] there exists a dense metrizable subspace. It follows from Theorem 1 that not every Corson compactum has a dense \(G_\delta\)-diagonal.

Remark 4. If the set of all isolated points of a space \(X\) is dense in \(X\), then \(X\) has a dense \(G_\delta\)-diagonal. This is evident. Thus if \(X\) is a scattered space then every subspace of \(X\) has a dense \(G_\delta\)-diagonal while \(X\) itself need not have a \(G_\delta\)-diagonal (take a compact non-metrizable scattered space — for example, the space \(T(\omega_1 + 1)\)).

We conclude the paper with several questions on weakly perfect spaces and spaces with a dense \(G_\delta\)-diagonal.

Question 2 [11]. What can we say on density of weakly perfect compact spaces? Is it true that density of each such space is \(\leq \aleph_1\)?

Question 3 [11]. Is it true that for every weakly perfect countably compact space \(X\) spread of \(X\) is countable?

Question 4. Is it true that every symmetrizable space \(X\) has a dense \(G_\delta\)-diagonal? is weakly perfect?

In connection with this question it should be noted that there are symmetrizable spaces without a \(G_\delta\)-diagonal and non-perfect.

Question 5. Let \(X\) be a weakly perfect compact space. Is it true then that \(X\) contains a dense metrizable subspace?

Question 6. Is every weakly perfect compact space of countable Souslin number separable?

Question 7. Let \(X\) be a compact space such that \(X \times X\) is weakly perfect. What about \(X\)? Is \(X\) perfect?
Question 8. When there exists a countable family $\mathcal{U}$ of open sets in $X \times X$ such that $\bigcap \mathcal{U}$ is dense in $\Delta_X$ and for each open neighborhood $V$ of $\Delta_X$ in $X \times X$ one can find $U \in \mathcal{U}$ such that $U \subset V$? Such $\mathcal{U}$ will be called a dense $\Delta$-base of $X$.

Let us note that if $X$ has a dense discrete subspace then $X$ has a countable dense $\Delta$-base.

Question 9. Let $X$ be a compact space with a countable dense $\Delta$-base. Does there exist a dense open metrizable subspace $Y \subset X$? dense separable metrizable subspace $Z \subset X$?

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