THE NEG.-PROPOSITIONAL CALCULUS

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Abstract. Consistency and completeness are proved for an axiomatic system intended to be a formalization of propositional contradictions.

By changing in the truth tables the values of the logical operations of conjunction, disjunction and implication (but not of negation ~), so as to obtain new functions designated by ε, v, and →, respectively:

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<th>ε</th>
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and by taking an interest in formulas with the truth value ⊥, we may construct an algebra of contradictions (identically false formulas) \( \mathfrak{A} = \{ (T, \perp); \varepsilon, v, \rightarrow, \sim \} \), which could serve as the principal model of a propositional calculus called neg-propositional calculus.

This formal system may be axiomatized with the following axiom-schemata:

1. \( (A \rightarrow B) \rightarrow A \)
2. \( ((A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow B)) \rightarrow (C \rightarrow B) \)
3. \( A \rightarrow \sim(A \in A) \)
4. \( ((A \in B) \rightarrow \sim B) \rightarrow \sim A \)
5a. \( \sim A \rightarrow A \in B \)
5b. \( \sim B \rightarrow A \in B \)
6. \( ((\sim C \rightarrow A \lor B) \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C) \)
7a. \( A \lor B \rightarrow \sim A \)
7b. \( A \lor B \rightarrow \sim B \)

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8. \((\neg B \rightarrow (A \rightarrow B)) \rightarrow (\neg A \rightarrow B)\)

and the rule of inference

\[
\begin{array}{c}
\langle I \rangle \\
A \rightarrow B, B \\
\hline
A
\end{array}
\]

We use \(A, B, C, \ldots, A_1, \ldots\), as schematic letters for formulas, and \(\Gamma_1, \Gamma_2\), as letters for lists of formulas.

We can easily prove the following:

**Lemma 1.** (i) \(A \vdash A\); (ii) if \(\Gamma_1 \vdash A\), then \(B, \Gamma_2 \vdash A\); (iii) if \(C, C, \Gamma_1 \vdash A\), then \(C, \Gamma_1, C \vdash A\); (iv) if \(\Gamma_1, C, D, \Gamma_2 \vdash A\), then \(\Gamma_1, D, C, \Gamma_2 \vdash A\); (v) if \(\Gamma_1 \vdash A_1, \ldots, \neg A_n\) and \(A_1, \ldots, A_n \vdash B\), then \(\Gamma_1 \vdash B\), or in particular: if \(\vdash A_1, \ldots, \vdash A_n\), and \(A_1, \ldots, A_n \vdash B\), then \(\Gamma_1 \vdash B\).

**Theorem 1.** If \(B_1, \ldots, B_m \vdash A \rightarrow B_1\), then \(B_1, B_2, \ldots, B_m \vdash A\).

**Proof.** By adding, according to Lemma 1, the formula \(B_1\) to the sequence \(B_2, \ldots, B_m\), by the rule of inference \(\langle I \rangle\) we obtain \(A\).

**Corollary 1.** If \(\vdash A \rightarrow B\), then \(B \vdash A\).

**Corollary 2.** If \(\vdash (\ldots (A \rightarrow B_1) \rightarrow \ldots) \rightarrow B_m\), then \(B_1, \ldots, B_m \vdash A\).

**Theorem 2** (Deduction theorem). If \(\Gamma, B \vdash A\), then \(\Gamma \vdash A \rightarrow B\).

**Proof.** By induction on the length \(k\) of the given deduction.

1° When \(k = 1\), the formula \(A\) is either an axiom or one of formulas \(\Gamma\) or \(B\). In the first case, such a deduction is

1. \(A\) axiom-hypothesis
2. \((A \rightarrow B) \rightarrow A\) axiom-schema 1
3. \(A \rightarrow B\) \(\langle I \rangle\) 1, 2

and analogously in the remaining two cases.

2° Assume the theorem holds for all lengths \(\leq k\), and let the length of the given deduction be \(k + 1\).

Besides the three cases already discussed, an additional case arises here: \(A\) is an immediate consequence, by the rule \(\langle I \rangle\), of two preceding formulas of the sequence.

Let \(B\) be the formulas \(P\) (at the \(r\)-th, \(r \leq k\)), and \(A \rightarrow P\) (at the \(s\)-th, \(s \leq k\)) step. In both cases, by the hypothesis, it is possible to construct two deductions of \(P \rightarrow B\) and \((A \rightarrow P) \rightarrow B\). Let us add to them the following instance of axiom-schema 2:

\(((A \rightarrow B) \rightarrow ((A \rightarrow P) \rightarrow B)) \rightarrow (P \rightarrow B)\)
Then a double application of the rule of inference yields \( A \rightarrow B \).

**Corollary 1.** If \( B \vdash A \), then \( \vdash A \rightarrow B \).

**Corollary 2.** If \( B_1, \ldots, B_{m-1}, B_m \vdash A \), then \( \vdash (\ldots (A \rightarrow B_1) \rightarrow \ldots \rightarrow B_{m-1}) \rightarrow B_m \).

We derive the introduction and the elimination rules for connectives in the theorem after the following lemma:

**Lemma 1.** (a) If \( B \vdash A \) and \( B \vdash \sim A \), then \( \vdash \sim B \); (b) if \( B \vdash A \), then \( \sim A \vdash \sim B \); (c) \( \sim A \vdash \sim B \); (c2) \( A \vdash \sim \sim A \).

**Theorem 3.**

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<th>Introduction</th>
<th>Elimination</th>
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<td>( \rightarrow )</td>
<td>( A \rightarrow B, B \vdash A )</td>
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<tr>
<td>( \vdash )</td>
<td>( \Gamma, B \vdash A ), then ( \Gamma \vdash A \rightarrow B )</td>
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<td>( \varepsilon )</td>
<td>( \sim A, \sim B \vdash A \in B )</td>
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<td>( \vdash )</td>
<td>( A \in B \vdash \sim A )</td>
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<tr>
<td>( B \vdash A \lor B )</td>
<td>( A \in B \vdash \sim B )</td>
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<td>( \sim B \vdash A \lor B )</td>
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<td>( \sim A \vdash A \lor B )</td>
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**Proof.** The rule of \( \rightarrow \)-introduction is just the deduction theorem, and \( \rightarrow \)-elimination is our rule of inference \((I)\). For the remaining rules we use the axiom-schema 4 for \( \varepsilon \)-introduction, 5a and 5b for \( \varepsilon \)-elimination, 7a and 7b for \( \lor \)-introduction, 6 for \( \lor \)-elimination, Lemma 2 (a) for \( \sim \sim \)-introduction, and finally Lemma 2 (c1) for \( \sim \)-elimination.

Can this formal system describe an intuitive domain of bivalent propositions? To answer this question, we define as follows the consistency and completeness of a formal system:

**Definition 1.** (i) The neg.-propositional calculus is **semantically consistent** if every provable formula, interpreted on the model of the calculus — the algebra \( \mathfrak{A} \) — is a contradiction.

(ii) The neg.-propositional calculus is **simply consistent** if there is an \( A \) such that \( \vdash A \) and \( \vdash \sim A \).

(iii) The neg.-propositional calculus is **syntactically consistent** if a formula is unprovable in it.

**Theorem 4.** The neg.-propositional calculus is (semantically, simple, syntactically) consistent.
Proof. On the indicated model, the truth value of all axioms is \( \bot \) and the rule of inference keeps this property (semantical consistency). The value \( \bot \) can not at the same time belong to a formula and to its negation (simple consistency), and so, as an example of an unprovable formula we have \( A \in \sim A \) (syntactical consistency).

Definition 2. (i) The neg-propositional calculus is \textit{semantically complete} if every contradiction is provable in it.

(ii) The neg-propositional calculus is \textit{simply complete} if for every \( A \) either \( \vdash A \) or \( \vdash \sim A \).

(iii) The neg-propositional calculus is \textit{syntactically complete} if it has the following property: if one adds to the axioms an unprovable formula, one can prove every formula.

Completeness in the first sense implies completeness in other two senses. Then we prove the following:

**Theorem 5.** The neg-propositional calculus is semantically complete.

Let us assign some \( n \)-tuple of values \( T, \bot \), to the propositional letters \( A_1, \ldots, A_n \) that occur in the formula \( F \). Then by \( B_1, \ldots, B_n \), we denote a corrected \( n \)-tuple, where \( B_i (1 \leq i \leq n) \) is \( A_i \) or \( \sim A_i \), according as the assigned value is \( \bot \) or \( T \).

**Lemma 3.** \( B_1, \ldots, B_n \vdash F \) if \( \tau(F) = \bot \) and \( B_1, \ldots, B_n \vdash \sim F \) if \( \tau(F) = T \).

\textit{Proof.} Let us denote by \( d \) the degree of the formula \( F \), i.e. the number of connectives in \( F \).

\textit{Basis} (\( d = 0 \)). Since \( F \) is a propositional letter \( A_i \), for \( i \in \{1, \ldots, n\} \), we have \( A_i \vdash A_i \) if \( \tau(A_i) = \bot \), or \( \sim A_i \vdash \sim A_i \) if \( \tau(A_i) = T \).

\textit{Induction step.} If the degree of \( F \) is \( k + 1 \), then \( F \) has one of the forms:

(a) \( \sim A \), (b) \( A \rightarrow B \), (c) \( A \lor B \) or (d) \( A \in B \) with \( A, B \) of degree \( \leq n \).

(a) We must show: (a1) \( B_1, \ldots, B_n \vdash \sim F \) if \( B_1, \ldots, B_n \vdash A \) and (a2) \( B_1, \ldots, B_n \vdash F \) if \( B_1, \ldots, B_n \vdash \sim A \), or (a1) \( A \vdash \sim A \) and (a2) \( \sim A \vdash \sim A \).

The proof is carried out by Lemma 1.

(b) If \( F \) has the form \( A \rightarrow B \), the four subcases arise:

(b1) If \( B_1, \ldots, B_n \vdash \sim A \) and \( B_1, \ldots, B_n \vdash \sim B \), then \( B_1, \ldots, B_n \vdash A \rightarrow B \) (and by Lemma 1, it is sufficient to prove \( \sim A, \sim B \vdash A \rightarrow B \)).

(b2) If \( B_1, \ldots, B_n \vdash \sim A \) and \( B_1, \ldots, B_n \vdash B \), then \( B_1, \ldots, B_n \vdash \sim (A \rightarrow B) \) \( (\sim A, B \vdash \sim (A \rightarrow B)) \).

(b3) If \( B_1, \ldots, B_n \vdash A \) and \( B_1, \ldots, B_n \vdash \sim B \), then \( B_1, \ldots, B_n \vdash A \rightarrow B \) \( (A, \sim B \vdash A \rightarrow B) \).

(b4) If \( B_1, \ldots, B_n \vdash A \) and \( B_1, \ldots, B_n \vdash B \), then \( B_1, \ldots, B_n \vdash A \rightarrow B \) \( (A, B \vdash A \rightarrow B) \).

**Sublemma 1.** \( A, \sim A \vdash B \).
(b1) 1. \(B, \sim B \vdash A\)  
Sublemma 1
2. \(\sim B \vdash A \rightarrow B\)  
\(\rightarrow \) - in. 1
3. \(\sim A, \sim B \vdash A \rightarrow B\)  
Lemma 1, 2

(b2) 1. \(A \rightarrow B, B \vdash A\)  
\(\rightarrow \) - el.
2. \(A \rightarrow B, B \vdash B\)  
Lemma 1
3. \(A \lor B, B \vdash \sim (A \rightarrow B)\)  
v - el. 1, 2
4. \(\sim A \vdash A \lor B\)  
v - in.
5. \(\sim A, B \vdash A \lor B\)  
Lemma 1, 4
6. \(\sim A, B \vdash B\)  
Lemma 1
7. \(\sim A, B \vdash \sim (A \rightarrow B)\)  
Lemma 1, 3, 5, 6

For (b3) and (b4) we proceed similarly.

For (c1), (c2), (c3) we use v - in.

(c4) 1. \(A \rightarrow B, B \vdash A\)  
\(\rightarrow \) - el.
2. \(A \rightarrow B, B \vdash B\)  
Lemma 1
3. \(B, A \lor B \vdash \sim (A \lor B)\)  
v - el. 1, 2
4. \(B, A \rightarrow B \vdash \sim (A \lor B)\)  
Lemma 2, 3
5. \(A, B \vdash B\)  
Lemma 1
6. \(A, B \vdash A \rightarrow B\)  
(b4)
7. \(A, B \vdash \sim (A \lor B)\)  
Lemma 1, 5, 6, 3

For (d1) we use \(\varepsilon \) - in.

**Sublemma 2.** \(\sim A \lor \sim B \vdash \sim (A \in B)\).

(In the proof we use \(\varepsilon \) - el. and v - el.)

(d2) 1. \(\sim B \vdash A \lor B\)  
v - in.
2. \(B \vdash \sim A \lor \sim B\)  
\(A \vdash A, B \vdash B, 1\)
3. \(\sim A \lor \sim B \vdash \sim (A \in B)\)  
Sublemma 2
4. \(B \vdash \sim (A \in B)\)  
Lemma 1, 2, 3
5. \(\sim A, B \vdash \sim (A \in B)\)  
Lemma 1, 4

Case (d3) is similar to (d2). For (d4) we use (d3).

**Lemma 4.** If for any of \(2^n\) possible sequences of values we always have \(B_1, \ldots, B_n \vdash F\), then

\[A_1 \lor A_1, \ldots, A_n \lor A_n \vdash F.\]
Proof. Let us take \( n = 2 \). By hypothesis, \( \sim A_1, \sim A_2 \vdash F; \sim A_1, A_2 \vdash F; \)
\( A_1, \sim A_2 \vdash F \) and \( A_1, A_2 \vdash F \). So, by a triple application of \( \nu \)-elimination (using
Lemma 2), we obtain:

1. \( \sim A_1, \sim A_2 \vdash F \) hyp.
2. \( \sim A_1, A_2 \vdash F \) hyp.
3. \( \sim A_1, \sim F \vdash \sim \sim A_1 \vdash A_1 \) Lemma 2, 1
4. \( \sim A_1, \sim F \vdash \sim A_2 \) Lemma 2, 2
5. \( \sim A_1, A_2 \nu \sim A_2 \vdash F \) \( \nu \)-el. 3, 4
6. \( A_1, \sim A_2 \vdash F \) hyp.
7. \( A_1, A_2 \vdash F \) hyp.
8. \( A_1, \sim F \vdash A_2 \) Lemma 2, 6
9. \( A_1, \sim F \vdash \sim A_2 \) Lemma 2, 7
10. \( A_1, A_2 \nu \sim A_2 \vdash F \) \( \nu \)-el. 8, 9
11. \( A_1 \nu \sim A_1, A_2 \nu \sim A_2 \vdash F \) \( \nu \)-el. 5, 10

Proof of Theorem 5. Let \( F \) be a contradiction and \( A_1, \ldots, A_n \) all of its letters.
By the last lemma:

\[
A_1 \nu \sim A_1, \ldots, A_n \nu \sim A_n \vdash F
\]

and, since we have \( \vdash A \nu \sim A \) by Lemma 1, we should finally have \( \vdash F \).

This proof follows the method exposed in [1].

REFERENCES