RELATIONS WITH I-STRUCTURE
IN CATEGORIES WITH PULLBACKS

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Abstract. Theory of relations in both set-theoretical and in categorical approach, rarely
is concerned with a possible existing structure between objects on which relations are defined.
The aim of this paper is to give one model of relations having in mind a specific structure, the
so-called I-structure, between objects in the domain of considered relations and to consider some
properties of such category of relations.

Introduction

An n-ary relation $R$ on sets $A_1, A_2, \ldots, A_n$ is usually defined as a subset of
$A_1 \times A_2 \times \cdots \times A_n$. In categories with pullbacks relations are defined by certain
collections of morphisms. In both, common set-theoretical and in more general
categorical approach possible existing relations between objects on which relations
are defined, are rarely considered. The aim of this paper is to define one kind of
(abstract) relational structure and to consider corresponding relations in a category
$K$ with pullbacks.

A relational structure $I$ is defined as a kind of a free graph — category
with arrows corresponding to existing connections between objects on which relations
are considered. Relations with such kind of structure are taken as objects
of a specific subcategory of the comma category $(K \downarrow D)$ where $D$ is a functor
("domain-functor"), $D : I \to K$. Objects of that subcategory $R_K (I, D)$ are nat-
ural transformations, namely those functors $R : I \to K$ for which there exists
a natural transformation (extension) $e : R \to D$. Accordingly, some properties
and operations are considered. Among other results, let us emphasize one that
gives necessary and sufficient conditions for respecting certain limits by extensions.
Those conditions enable us to recognize when a relation with I-structure decom-
posed by “projections” may be recomposed (by functional joins) into the primary
one.

AMS Subject Classification (1980): Primary 18B99, Secondary 08A08, 18D99
Research supported by Republička zajednica za naučni rad SR BiH.
Many authors considered relations for different categories, among others Y. Kawahara [7] and L. Coppey and R. Davar-Panah [6]. Binary relations, defined by pairs of morphisms in categories with pullbacks, have been studied in Kawahara's paper and decompositions and categories of relations are considered in [6]. Some views on different relational models are given by this author in [2] and [3]. The idea of considering abstract relational structure has come from paper by J. Rissanen [10]. Necessary categorical preliminaries may be found in both S. Mac Lane's [8] and E. Manes' book [9].

1. Relational structure

1.1. Let $(E, \leq)$ be an order. A trivial relational structure $T$ is a category defined as follows. For an element $X$ of $E$ let $X$ be an object of $T$; for $X$ and $Y$ objects of $T$, let the set of arrows $T(X, Y)$ consist of one arrow $X \rightarrow Y$, whenever $Y \leq X$, otherwise let $T(X, Y) = \emptyset$.

1.2. A trivial relational structure $T$ is a well-defined category. It defines a graph $G' = \bigcup T$, with the same objects as $\text{Ob}(T)$ (as knots) forgetting which arrows are composition and which are identities. This graph is dual to one usually induced and directed by the given order.

Example 1. Let $M = (X, Y, Z)$ and $E = P(M)$. Consider $(P(M), \subseteq)$. A trivial relational structure contains among others the following arrows $M \rightarrow \{X\}$, $\{X, Y\} \rightarrow \{Y\}$, $X \rightarrow \emptyset$, etc. and one possible interpretation of an arrow in $T$ is “... has more information than...”

1.3. Let $(E, \leq, \sup)$ be a sup-complete semi-lattice and $G' = \bigcup T$. Let $N$ denote a new collection of arrows between some knots of the graph $G'$ (not adding any new knots) and let $G = G' \cup N$. A relational structure $I := I(G)$ induced by a graph $G$ is a category constructed in the following three steps:

(i) Enlarge the graph $G$ by one new arrow $A \rightarrow XY := \sup\{X, Y\}$ whenever $G$ already contains two arrows $A \rightarrow X$ and $A \rightarrow Y$, $X \neq Y$, and apply this rule as long as new arrows may be produced. (Identify $XX$ with $X$ for $X \in E$.)

(ii) Construct a category whose objects are those of $G$ and whose arrows are finite strings $A_1 \rightarrow A_2 \rightarrow \ldots \rightarrow A_n$ composed of $n-1$ arrows $f_i : A_i \rightarrow A_{i+1}$ of $G$, and regard that string as an arrow $A_1 \rightarrow A_n$. The composition of these arrows is defined by juxtaposition of strings (therefore, associative) and the identity arrows are strings $A_n$ of length 1.

(iii) If $X$ and $Y$ are objects of $I$, identify all arrows that belong to $\text{hom}(X, Y)$. Then $\text{hom}(X, Y)$ is either empty or consists of only one arrow.

Example 2. $(P(M), \subseteq, \cup)$, $M = \{X, Y, Z\}$, nontrivial arrows $\{Y\} \rightarrow \{X\}$, $\{Z\} \rightarrow \{X\}$. A relational structure $I$ consists of all (trivial) $T$-arrows, nontrivial arrows, $\{Y\} \rightarrow \{X\}$, $\{Z\} \rightarrow \{X\}$ and new-constructed arrows: $\{Y\} \rightarrow \{X, Y\}$, $\{Y, Z\} \rightarrow \{X\}$. 
1.4. Proposition. A relational structure $I := I(G)$ is a well defined category and

(a) A trivial relational structure $T$ is a subcategory of a corresponding relational structure $I$;
(b) $I$ has finite products, $X \times Y = \sup \{X, Y\}$ (Note: $XX = X$)
(c) $I$ has finite pullbacks: for a pair of arrows $X \rightarrow A, Y \rightarrow A$ pullback is $XY \rightarrow A$; and
(d) $I$ has an initial object (namely $\sup E = 1$).
(e) If $(E, \leq)$ is a complete lattice, $I$ has initial and terminal objects.

1.5. Let $(F, U) : \textbf{Graph} \rightarrow \textbf{Kat}$ be a pair of adjoint functors between the category of (small) graphs and the category of (small) categories. $FG$ is a free category constructed over a graph $G$ of $\textbf{Graph}$ and $UK$ is a graph-like category under the forgetful functor $U$.

Proposition. There exists a quotient category $FG/\sim$ and a functor $Q, Q : FG \rightarrow FG/\sim$ such that:

(a) If $f_1 : X \rightarrow Y$ and $f_2 : X \rightarrow Y$ are arrows of $FG$, then $Qf_1 = Qf_2$;
(b) If $H : FG \rightarrow K$ is any functor, satisfying $Hf_1 = Hf_2$, for all $f_1, f_2 : X \rightarrow Y$, then there exists a unique functor $H' : FG/\sim \rightarrow K$ such that $H'Q = H$;
(c) There is an isomorphism between categories $FG/\sim$ and $I(G)$.

Proof. Functor $Q$ is a bijection on objects and it maps all arrows from $X$ to $Y$ to a unique arrow $X \rightarrow Y$ of $FG/\sim$, so that for any $X, Y$ objects of $FG/\sim$ the set of arrows with domain $X$ and codomain $Y$ contains at most one element. The isomorphism between $FG/\sim$ and $I(G)$ exists by the construction of $I(G)$.

1.6. A morphism $M : I_1 \rightarrow I_2$, of relational structures is a covariant functor which respects (preserves) products. A composition of morphisms of relational structures is the usual composition of covariant functors.

1.7. Proposition. Relational structures together with morphisms of relational structures form a category.

1.8. Proposition. Arrows of a relational structure $I$ possess the following properties:

(a) $X \rightarrow Y, X \rightarrow Z$ if and only if $X \rightarrow YZ$,
(b) If $X \rightarrow Y$ and $V \rightarrow W$ then $XV \rightarrow YW$,
(c) If $X \rightarrow Y$ and $YV \rightarrow Z$ then $XV \rightarrow Z$,
(d) $X \rightarrow Y$ if and only if $X \rightarrow XY$.

The proof is a consequence of well known lattice properties and the construction of products (universal) in relational structure $I$. 
1.9. Let $X$ be an arbitrary object of $I$ and let $z(X)$ denote a collection of $I$-objects defined by the following:

(i) $X$ is an object of $z(X)$,

(ii) If $Y$ is an object of $z(X)$ and there exists in $I$ an arrow $Y \to V$, then $V$ is an object of $z(X)$-collection, and

(iii) all objects of $z(X)$ are given by (i) and (ii).

Consider $z(X)$. If a relational structure has products, one may define a functor (endofunctor) $Cl : I \to I$, where $Cl(X)$ is the product of all objects of $z(X)$, for any object $X$ in $I$, and on arrows $Cl(f) : Cl(X) \to Cl(Y)$ is induced by $z(X) \to z(Y)$, for any arrow $f : X \to Y$ in $I$.

Let $h : Cl \to 1$ and $k : Cl \to Cl^2$ be two natural transformations defined on components by $h_X : Cl(X) \to X$ and $k_X : Cl \to Cl(Cl(X))$.

1.10. Proposition. $(Cl, h, k)$ is a comonad in a relational structure $I$ with products. The corresponding $Cl$-coalgebras are those objects of $I$ for which $Cl(X) = X$.

2. Relations of the given relational structure

Any relational structure $I$ defines, in an abstract manner, relations that may be defined starting form $I$ and corresponding to $I$-objects suitable domains and subobjects of products of domains, having in mind already existing arrows between objects.

2.1. Let $K$ be a category (base-category) with the following pairs of adjoint functors: $(\Delta, T_K) : K \to 1$ and $(\Delta, P^\nu) : K \to K^{++}$ (i.e. $K$ possesses a terminal object and pullbacks) and let $D : I \to K$ be a covariant functor that respects products for different knots from $I$ (domain-functor). A comma category $(K^I \downarrow D)$ defined for the following pair of functors $(id : K^I \to K^I, D : 1 \to K^I)$ has as objects all those functors $S : I \to K$ for which there exists a natural transformation $e^S : S \to D$ and as morphisms all arrows (natural transformations) $\alpha : S \to S'$ such that $e^{S'} \alpha = e^S$ where $S' : I \to K$.

2.2. Arrows of $DT$ may be considered as “projections” and therefore the existence of the following morphisms in $K$ is obvious:

(i) If $f : X \to U$ and $g : Y \to V$ are arrows of $I$ there exists a unique arrow $r : DX \times DY \to DU \times DV$ in category $K$ such that $(p_U, p_V)r = (Df p_X, Dg p_Y)$, where $p_X : DX \times DY \to DX$, $p_Y : DX \times DY \to DY$, $p_U : DU \times DV \to DU$, $p_V : DU \times DV \to DV$

(ii) If $X \to Y$ and $YV \to Z$ are $I$-arrows then there exists a unique arrow $s : DX \times DV \to DZ$ such that $s = DgD(f, 1_V) \simeq Dg(Df, D1_V)$.

2.3. Natural transformations $e : R \to D$ of $(K^I \downarrow D)$ with all components $e_X : RX \to DX$ (mono) subobjects are called extensions.
For \( X \in \text{Ob}(I) \), the following diagram is commutative
\[
\begin{array}{ccc}
E & \xrightarrow{RE} & DE \\
\downarrow & & \downarrow p_X \\
X & \xrightarrow{e_X} & DX
\end{array}
\]
Since \( e : R \to D \) is an extension, for a collection \( \{X_i \mid i = 1, 2, \ldots \} \) of \( I \)-objects, there is a morphism
\[
(Dp)e_{\sup X_i} = e_X, Rp : R(\sup X_i) \to DX,
\]
where \( p : \sup X_i \to X_i \).

**2.4.** A functor \( R : I \to K \) is a relation with \( I \)-structure (\( I \)-relation) whenever there exists in \((K^I \downarrow D)\) an extension \( e : R \to D, D \) being a domain functor. Let \( R, S : I \to K \) be two \( I \)-relations. A morphism between two \( I \)-relations \( R \) and \( S \) is a natural transformation \( t : R \to S \) such that \( te^R = e^S \) where \( e^R : R \to D \) and \( e^S : S \to D \) are extensions.

**Example 3.** (a) One simple interpretation of the Example 2 is the following:
Let \( DX = \) {addresses}; \( DY = \) {cities}; \( DZ = \) {phone numbers} and consider \( R(XYZ) \) as a relation (in \( DX \times DY \times DZ \)) for a restricted number of cities (for example, in one state) and the corresponding addresses and phone numbers.

(b) Consider Example 1 with nontrivial arrows \( X \to Y, Y \to Z, Z \to X \) and let \( D : I \to \text{Set} \) (preserving products for different knots) be given by \( DX = DY = DZ = [0,1] \) and let \( R : I \to \text{Set} \) be defined by \( R(XYZ) = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1\} \). Relation \( e : R \to D \) is determined by an embedding \( e_{XYZ} : R(XYZ) \to D(XYZ) \) and the corresponding projections: \( e_X, e_Y, e_Z, e_{XY}, e_{XZ}, e_{YZ}, e_{XYZ} \).

**2.5.** An \( I \)-morphism \( f : X \to Y \) is embedded into a relation \( R \) if and only if \( R(f)R(t_X) = R(t_Y) \) where \( t_X : 1 \to X, t_Y : 1 \to Y, (1 = \sup E) \) are \( T \)-arrows.

**2.6.** **Proposition.** \( I \)-relations and morphisms between them, in base category \( K \) and with the domain functor \( D : I \to K \), form a subcategory \( \text{Rel} := \text{Rel}_K(I,D) \) of the comma category \((K^I \downarrow D)\) and the following properties are valid:

(i) For any \( I \)-relation \( R, e : R \to D \) and for any \( I \)-object \( X, e_X R t_X = D t_X e_1 \) where \( R(t_X) : R_1 \to RX, D t_X : D 1 \to DX \) and \( 1 = \sup E \).

(ii) Let \( R \) be an \( \text{Rel-object}, e : R \to D \). Any \( I \)-morphism \( f : X \to Y \) is embedded into \( R \) and \((Df)e_X = e_Y R f \).

(iii) Extensions of \( T \)-arrows are (mono-) restrictions of projections.

**Proof.** (iii) If \( t : X \to Y \) is a \( T \)-arrow, then \( \sup\{X,Y\} = X \) and hence \( t : \sup\{X,Y\} \to Y \). Further, \( DX \times DY \cong DX Y \xrightarrow{D t} DY \) is a projection and \( R t \) is a restriction of the projection \( Dt \).
2.7. Proposition. For any $I$-relation $R$ from the category $\text{Rel}$,
(a) A morphism $(e_X, e_Y) : RX \times RY \rightarrow DX \times DY$ is an embedding (monomorphism).
(b) There exists a unique monomorphism $m : R(XY) \rightarrow RX \times RY$ such that $i(e_X, e_Y)m = e_{XY}$ (where $e_{XY} : R(XY) \rightarrow D(XY)$, $i : DX \times DY \simeq D(XY)$).

Proof. (a) By a standard categorical argument.
(b) Since $XY$ is a product in $I$ and $e : R \rightarrow D$ is a natural transformation, $e_X t_X = p_X e_{XY}$ and $e_Y t_Y = p_Y e_{XY}$ where $e_{XY} : R(XY) \rightarrow D(XY) \simeq DX \times DY$ is an $XY$-component of $e$. Since $DX \times DY$ is a product, $(e_X, e_Y)$ is a unique morphism such that $i(e_X, e_Y)m = e_{XY}$. Also, since $e_{XY}$ is monic, $m$ is monic. If $m$ is not unique, let $m, m_1 : R(XY) \Rightarrow RX \times RY, m \neq m_1$. Then, $i(e_X, e_Y)m = e_{XY} = i(e_X, e_Y)m_1$ and since $i(e_X, e_Y)$ is monic, $m = m_1$.

2.8. Lemma. Let $\alpha : R \rightarrow S$ be a morphism between two $I$-relations. An $I$-morphism $f : X \rightarrow Y$ is embedded in both $R$ and $S$, and the following connections are valid: $S(f)\alpha_X = \alpha_Y R(f)$, $(e^S)X \alpha_X = (e^R)_Y$ and $(e^S)_Y \alpha_Y = (e^R)_Y$.

2.9. Proposition. A relation $R$ from the category $\text{Rel}_K(I, D)$ is determined in a unique way by a graph-morphism $h : G \rightarrow UK$.

Proof. An adjoint pair of functors $(F, U) : \text{Graph} \rightarrow \text{Kat}$ extends a morphism $h : G \rightarrow UK$ to a unique functor $H : FG \rightarrow UK$ and then by Proposition 1.4. it extends a functor $H$ to a unique $H' : FG/_{-} \rightarrow K$ so that $H'Q = H$.

3. Operations

3.1. Projections of a relation $R$ with an extension $e : R \rightarrow D$ are $e$-images of the corresponding projections in a relational structure.

Clearly, for a trivial arrow $t : XY \rightarrow X$ the commutativity $(Dt)e_{XY} = e_X (Rt)$ illustrates the presence of one possible projection.

3.2. The product of two $I$-relations $R$ and $S$, with extensions $e^R$ and $e^S$, denoted by $e^R \times e^S : R \times S \rightarrow D$, is defined by components $$(e^R \times e^S)(X) := ((e^R)_X, (e^S)_X) : RX \times SX \rightarrow DX.$$ 

3.3. For any given pair of arrows from the relational structure $I, f : Y \rightarrow X, g : Z \rightarrow X$, the functional join of $Rf$ and $Rg$ is a pullback of a pair of morphisms $(Rf, Rg)$ in the base category $K$. It is denoted by $R(Y) \circ R(Z)$.

3.4. A functional join of relations $R$ and $S$ (with extensions $e^R$ and $e^S$) is a pullback of a pair of morphisms (components) $(e^R)_X, (e^S)_X$ defined by $(R \circ S)(X) := RX \circ DX SX$ and denoted by $e^R \circ e^S : R \circ S \rightarrow D$. Actually, if $RX$ and $SX$ are treated as subobjects of $DX$, $(R \circ S)X$ is the intersection of the subobjects $(e^R)_X$ and $(e^S)_X$ in the partially ordered set of all subobjects of $DX$. 
3.5. In the category \textbf{Set}, a composition of relations \( R(XY) \) and \( R(XZ) \) is defined in the usual way, by \( R(XY) \circ R(XZ) := \{(x,y,z) \mid (x,y) \in R(XY), (x,z) \in R(XZ)\} \).

3.6. \textbf{Proposition.} In the category of sets, a composition of relations \( R(XY) \) and \( R(XZ) \) is exactly the functional join of a pair of projections \( p_X : R(XY) \rightarrow RX, q_X : R(XZ) \rightarrow RX \).

3.7. \textbf{Lemma.} Let \( f : X \rightarrow Y \) be an arrow of a relational structure \( I \) and \( R \) an \( I \)-relation. Then, \( RX \circ_X RY \simeq RX Y \).

\textit{Proof.} It is enough to prove that a morphism \( r : RX Y \rightarrow RX \circ_X RY \), defined by the universal construction of a pullback for a pair of arrows \( (R \text{id}_X, Rf) \) is an isomorphism. Obviously, \( p_Y r = id_{RY} \) where \( p_Y : RX \circ_X RY \rightarrow RY \). Further, \( p_Y \) is a monomorphism. For let \( r_1, r_2 : N \rightarrow RX \circ_X RY \) be a pair of different arrows in the base category \( K \), with \( p_Y r_1 = p_Y r_2 \). Hence, \( R(f)p_Y r_1 = R(f)p_Y r_2 \) and since \( R(f)p_Y = p_X \), it would be \( p_X r_1 = p_X r_2 \). Now, a pair of arrows \( p_X r_1 = p_X r_2 : N \rightarrow RX \), \( p_Y r_1 = p_Y r_2 : N \rightarrow RY \), together with \( \text{id}_{RX} \) and \( R(f) \) forms a commutative square. By the universal property of pullback square, there exists a unique arrow \( r_1 = r_2 : N \rightarrow RX \circ_X RY \). Therefore, \( p_Y \) is a monomorphism and hence \( r \) is an isomorphism.

3.8. \textbf{Corollary.} Let \( I \) be a relational structure with terminal object 0 and let \( R \) be an \( I \)-relation. Then,

\( (a) \ RX \circ_X RX \simeq RX \), \( (b) \ RX \circ_X R0 \simeq R0 \), \( (c) \ R1 \circ_X RY \simeq RY \).  

The following proposition describes (existing) \( K \)-arrow between some objects

\( \rightarrow R(YZ), RX \circ_X RZ, RY \times RZ \).

3.9. \textbf{Proposition.} There are unique \( K \)-monomorphisms \( g : R(YZ) \rightarrow RX \circ_X RZ, k : R(YZ) \rightarrow RY \times RZ, \) and \( h : RX \circ_X RZ \rightarrow RY \times RZ \) such that \( hg = k \).

\textit{Proof.} Consider commutative diagrams (3.9) and the corresponding universal arrows:

\[
\begin{array}{ccc}
R(YZ) & \xrightarrow{g} & RX \circ_X RZ \\
\downarrow k & & \downarrow h \\
RY & \xrightarrow{r_1} & RY \times RZ & \xrightarrow{r_2} & RZ \\
\end{array}
\]

3.10. \textbf{Proposition.} Let \( Y \rightarrow A \leftarrow Z \) and \( a : A \rightarrow B \) be arrows of a relational structure \( I \). Then, the chain of arrows \( 1 \rightarrow A \rightarrow B \rightarrow 0 \) induces, for
any I-relation $R$, a chain of $K$-arrows

$$R_1 \xrightarrow{R(t)} R(YZ) \xrightarrow{g_A} RY \circ_A RZ \xrightarrow{m} RY \circ_B RZ \xrightarrow{} RY \times RZ,$$

with the following equalities: $m g_A = g_B$, $g_B h_B = k = g_A h_A$, $g_A R(t) = t_A$, $g_B R(t) = t_B$, where $t_A : R_1 \rightarrow RY \circ_A RZ$, $t_B : R_1 \rightarrow RY \circ_B RZ$.

3.11. Theorem. For any I-relation in base category $K$ and with the domain functor $D$, $R(XYZ) \simeq R(XY) \circ_X R(XZ)$ if and only if there exists in a relational structure $I$ either an arrow $X \rightarrow Y$ or $X \rightarrow Z$.

Proof. Without loss of generality, suppose $X \rightarrow Y$ is an I-arrow. This arrow yields a unique arrow $X \rightarrow XY$ and hence $X \simeq XY$. Then, since $R$ is a functor, and by 3.7. $R(XY) \circ_X R(XZ) \simeq R(X) \circ_X R(XZ) \simeq R(XZ)$. On the other hand, the arrow $X \rightarrow XY$ yields an (unique) arrow $XZ \rightarrow XY$ and therefore, $R(XYZ) \simeq R(XZ)$. Hence $R(XYZ) \simeq R(XY) \circ_X R(XZ)$.

The converse is obvious by the following example: Let $R$ be an I-relation in $\text{Rel}_G(I, D)$ where $G$ is a graph with three objects and no arrows and let $R(XYZ) = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. Then $R(XY) \circ_X R(XZ) \neq R(XYZ)$.

3.12. Corollary. A functional join operation (whenever defined) has the following properties:

(a) $RX \circ_Y RY \simeq RX \times RY$,
(b) $RX \circ_X RX \simeq RX$,
(c) $RX \circ_A RY \simeq RY \circ_A RX$,
(d) $(RX \circ_A RY) \circ_A RZ \simeq RX \circ_A (RY \circ_A RZ)$,
(e) $(RX \circ_A RY) \circ_B RZ \simeq RX \circ_A (RY \circ_B RZ)$,
(f) $(RX \circ_A RY) \circ_Y RZ \simeq RX \circ_A RZ$,
(g) $(RX \times RY) \circ_A RZ \simeq (RX \circ_A RZ) \times RY$.

4. Decompositions of relations

Corollary 3.12 suggests a generalization of a functional join operation to a successive join operation and, as its special case, multiple functional join.

4.1. For any given collection $W = \{(f_i, g_i) : \text{dom} g_i = \text{dom} f_{i+1}, \text{cod} g_i = \text{cod} f_i, i = 1, 2, \ldots \}$ of $I$-arrows, $R$-successive functional join is a limit for the diagram scheme $W_R := \{(Rf_i, Rg_i) \mid (f_i, g_i) \in W\}$. It is a $K$-object $o W_R$ together with a sequence of $K$-arrows $r_i : o W_R \rightarrow R(\text{dom} f_i)$, $i = 1, 2, \ldots$ with the corresponding universal property: First, for any $i = 1, 2, \ldots R(g_i) r_{i+1} = R(f_i) r_i$, and second, for given collection of arrows $m_i : M \rightarrow R(\text{dom} f_i), i = 1, 2, \ldots$ for which $R(f_{i+1}) m_{i+1} = R(f_i) m_i$ there exists a unique arrow $t : M \rightarrow o W_R$ such that $r_i t = m_i$ for $i = 1, 2, \ldots$.

For a given collection of $I$-arrows $f_i : X_i \rightarrow A_i, i = 1, 2, \ldots$ a multiple functional join is an $R$-successive join for a collection $W = \{(f_i, f_{i+1}) \mid i = 1, 2, \ldots \}$. 
4.2. An extension \( e : R \rightarrow D \) from a category of relations \( \text{Rel}_K(I, D) \) preserves (respects) the limit of a functor \( V : J \rightarrow I \) whenever the following conditions are satisfied:

(L1) If \( u : \Delta \lim \ V \rightarrow V \) is the limit of a functor \( V \), then \( Ru : \Delta \lim \ V \rightarrow RV \)

is the limit of the composition \( RV : J \rightarrow K \),

(L2) There exists a mono-natural transformation \( \Delta : \lim RV \rightarrow DV \), such that

(L3) \( (eV)(Ru) = (Du)(\Delta e_{\lim V}) \).

4.3. Let \( Q(x, y) \) denote any diagram of the form \( x \rightarrow \cdots \leftarrow y \), and let \( V : Q(x, y) \rightarrow I \) be a functor from the diagram category \( Q(x, y) \) into a relational structure \( I \) such that the middle object in \( VQ \) cannot be the initial object.

**Proposition.** An extension \( e : R \rightarrow D \) preserves the limit of a functor \( V : Q(x, y) \rightarrow I \) (i.e. binary functional join) whenever there exists in \( I \) either \( V(x) \rightarrow V(x) \) or \( V(y) \rightarrow V(y) \).

The proof follows immediately from 3.11.

4.4. **Corollary.** Let \( M' \) and \( M'' \) be such finite collections of \( I \)-arrows for which an extension \( e : R \rightarrow D \) preserves a successive functional join. If there exists a functor \( V_i : Q(x, y) \rightarrow I \) such that \( V_i(x) = \bigcup \{ \text{dom } f \mid f \in M' \} \) and \( V_i(y) = \bigcup \{ \text{dom } f \mid f \in M'' \} \) and \( e \) preserves the limit of a functor \( V_i \) then, \( e : R \rightarrow D \) preserves a successive functional join for a collection of \( I \)-arrows \( M = M' \cup M''. \)

**Proof.** By induction, from 4.3.

4.5. **Proposition.** An extension \( e : R \rightarrow D \) preserves limit of \( V : G \rightarrow I \) for those and only those subfamilies of objects in \( I \) satisfying the following conditions:

(i) For any subdiagram \( Q(x, y) \) from \( G \) for which \( \inf \{ V(x), V(y) \} \) \( \neq 0 \) there exist a \( \lim \{ V \} \), and

(ii) there is no finite subdiagram \( Z \) of \( G \) with \( \text{Ob}(VZ) = \{ Z_1, Z_2, Y_1, Y_2, \ldots, Y_n \} \) for which

(a) an extension \( e \) preserves the limit of \( V \) \( \mid_{Q(z_j, y_j)} \) for each \( i, j = 1, 2, \ldots, n \), but

(b) \( e \) preserves neither the limit of \( V \) \( \mid_{Q(y_i, z_k)} \) nor \( V \) \( \mid_{Q(z_i, z_m)} \) for \( i, j = 1, 2, \ldots, n \) and \( k, m = 1, 2 \).

To prove this Proposition one needs two following lemmas.

4.6. **Lemma.** Let \( G_* \) be a diagram satisfying condition (i) of Proposition 4.5 and let \( Z \) be a finite subdiagram of \( G_* \) of the form (ii) in 4.5. Then, there is no extension \( e : R \rightarrow D \) preserving the limit of \( V : G_* \rightarrow I \).

**Proof.** It is enough to prove that for each \( e : R \rightarrow D \), \( \lim_{\text{RV}}(G_*) \neq R \lim_{\text{V}}(G_*) \). An image \( V(G_*) \) generates a subcategory of the relational structure
I. Let $C = \text{Cl}(Y_1, \ldots, Y_m)$. If $Z_1$ is an object in $C$, there are arrows $K \rightarrow Y_1 \rightarrow C \rightarrow Z_1$ and hence $K \rightarrow Z_1$, and therefore $e$ preserves the limit of $V |_{Q(y_1, z_1)}$ which contradicts the assumption of this lemma. Similarly, for $Q(y_n, z_2)$. Therefore, neither $Z_1$ nor $Z_2$ does belong to the considered closure. On the other hand, since neither $Q(z_1, z_2)$ nor $Q(z_2, z_1)$ are subdiagrams of $G_*$, for which $e$ preserves $\lim V |_{Q}$ only $I$-arrows between $Z_1$ and $Z_2$ are $Z_1 \rightarrow 1$ and $Z_2 \rightarrow 1$, and hence $R(Z_1) \ast R(Z_2) \simeq R(Z_1) \times R(Z_2)$. Now, $\lim R(V) \simeq R(Z_1) \times R(Z_2) \ast R(Y_1 Y_2 \ldots Y_n)$, but $R(\lim V) \simeq R(Z_1 Z_2 Y_1 Y_2 \ldots Y_n)$ and obviously, $R(\lim V) \neq R(\lim V)$.

4.7. Lemma. Under assumptions of the Proposition 4.5, for any two $I$-objects $A$ and $B$ from $V(G)$ there exists an object $C$ from the relational structure $I$ and subcollections $A = A_0, A_1, \ldots, A_n = C$ and $C = B_0, B_1, \ldots, B_m = B$, such that $e$ preserves $\lim V |_{Q}$ for $Q(a_i, a_{i-1})$, and $Q(b_j, b_{j-1})$, $(i = 0, 1, \ldots, n-1$ and $j = 0, 1, \ldots, m-1$) where $Q(a_i, a_{i-1})$, and $Q(b_j, b_{j-1})$ are subdiagrams of $G$.

The proof goes by induction on the number of objects between $a$ and $b$, i.e. on the least number of objects $a = y_0, y_1, \ldots, y_k = b$ such that $Q(y_i, y_j)$ is subdiagram of $G$ for all $i, j = 1, 2, \ldots, k$ (and $e$ preserves $\lim V |_{Q}$).

\[
\begin{array}{c}
\Delta \text{D} \lim V \\
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and therefore, the proof follows by induction on the number $i$. Since $e : R \rightarrow D$ is a mono-natural transformation, for each $I$-object $B$, $e_B$ is monic in $K$. Since $\lim V$ is also an $I$-object, there exists a monomorphism $e'_{\lim V} : R(\lim V) \rightarrow DV$ and hence (L2) of 4.2 is satisfied.

Under conditions given in 4.5, (L1) and (L2) of 4.2 are valid and for any $I^G$-morphism $\Delta W \rightarrow V$ and $K^I$-morphism $\Delta S \rightarrow RV$ the diagrams labeled by (4.5) are commutative. Therefore, the condition (L3) of 4.2 is satisfied $(eV)(Ru) = (Du)\Delta e'_{\lim V} = \lim eV$.

Conversely, if an extension $e : R \rightarrow D$ of relation $R$ from Rel respects either successive or multiple pullback for a subdiagram $G_x$ of the relational structure $I$, we shall show that conditions (i) and (ii) are satisfied. If condition (i) doesn't hold, extension $e$ doesn't respect pullbacks by the Corollary 4.5. In case (ii) doesn't hold, extension $e$ doesn't preserve pullbacks by the Lemma 4.7.

Remark. A simple graph-like version of the question considered in this paragraph may be found in [1].

REFERENCES