SYMPLECTIC AND COSYMPLECTIC FOLIATIONS
ON COSYMPLECTIC MANIFOLDS*

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Abstract. We prove that a compact symplectic or cosymplectic foliation on a cosymplectic
manifold is stable. This result extends to the odd-dimensional case the corresponding one for
symplectic foliations on symplectic manifolds. A large family of examples is given.

1. Introduction

As it is* well-known a compact holomorphic foliation of a Kähler manifold is
stable (see [10]). The result holds for compact almost complex (resp., symplectic)
foliations of an almost Kähler (resp., symplectic) manifold (see [6, 7]).

In this paper, we study the stability of foliations on cosymplectic manifolds.
First, we introduce the notion of symplectic and cosymplectic foliations on a cosym-
plectic manifold, accordingly to the dimension of the foliation. Then we prove that
a compact symplectic or cosymplectic foliation on a cosymplectic manifold is stable.
To prove this, we use our previous results for the stability of invariant foliations of
almost contact manifolds [2].

2. Algebraic preliminaries

Let $E$ be a $(2n + 1)$-dimensional vector space over $R$. The space $E$ is called
cosymplectic if there exist a 2-form $\Phi$ and a 1-form $\eta$ such that $\eta \wedge \Phi^\eta \neq 0$. In such
a case we say that the pair $(\Phi, \eta)$ is a cosymplectic structure on $E$ and the triple
$(E, \Phi, \eta)$ is called a cosymplectic vector space.

Let $(E, \Phi, \eta)$ be a cosymplectic vector space. Then there is a unique vector $\xi$
such that
$$\eta(\xi) = 1, \quad \Phi(\xi, v) = 0,$$

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for all vector \( v \in E \). The vector \( \xi \) is called the \textit{canonical vector} of the cosymplectic vector space \((E, \Phi, \eta)\). Note that the vector \( \xi \) is characterized by the following condition:

\[
\omega(\xi) \eta \wedge \Phi^n = \omega \wedge \Phi^n,
\]

for all 1-forms \( \omega \) on \( E \).

Let \( E^\perp_\eta \) be the annihilator space of \( \eta \), i.e.,

\[ E^\perp_\eta = \{ v \in E \mid \eta(v) = 0 \}. \]

It is clear that \( E^\perp_\eta \) is a symplectic vector space with respect to the induced 2-form \( \Phi \).

A 2\( s \)-dimensional subspace \( F \) is called \textit{symplectic} if it is a symplectic subspace of \( E^\perp_\eta \). If \( \dim F = 2s + 1 \), then \( F \) is called \textit{cosymplectic} if the pair \((\Phi, \eta)\) induces a cosymplectic structure on \( F \) with canonical vector \( \xi \).

A \((2n + 1)\)-dimensional vector space \( E \) over \( R \) is said to be \textit{almost contact} if it admits a linear mapping \( \phi : E \to E \), a vector \( \xi \) and a 1-form \( \eta : E \to R \) such that

\[
\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.
\]

A subspace \( F \) of \( E \) is said to be \textit{invariant} if \( \phi(v) \in F \) for all \( v \in F \) (see [11]). We easily see that only two cases occur for any invariant subspace \( F \) of \( E \).

1. If the vector \( \xi \notin F \), then \( F \) has even dimension, \( \phi \) induces an almost complex structure on \( F \) and \( \eta|_F = 0 \).

2. If the vector \( \xi \in F \), then \( F \) has odd dimension and it is an almost contact vector space endowed with the restrictions of \( \phi \) and \( \eta \).

These definitions may be extended fiberwise to vector bundles. Thus, let \( \pi : E \to M \) be a vector bundle over an \( n \)-dimensional manifold \( M \) and with fiber \( R^{2n+1} \). Then \( \pi : E \to M \) is called \textit{cosymplectic} if there exist cross-sections \( \Phi \) and \( \eta \) of \( \Lambda^2 E^* \) and \( \Lambda^1 E^* \), respectively, whose restrictions to the fibers of \( E \) define a cosymplectic structure. Hence there exists a unique cross-section \( \xi \) of \( \pi : E \to M \) such that

\[
\eta(\xi) = 1, \quad \Phi(\xi, X) = 0,
\]

for all sections \( X \) of \( E \).

The section \( \xi \) is called the \textit{canonical section} of the cosymplectic vector bundle \( \pi : E \to M \). Note that for each point \( \xi_x \) is the canonical vector of the cosymplectic structure induced in the fiber \( E_x \).

A vector bundle \( \pi : E \to M \) is called \textit{almost contact} if there exist a vector bundle automorphism \( \phi \), a cross-section \( \xi \) of \( E \) and a cross-section \( \eta \) of \( \Lambda^1 E^* \), whose restrictions to the fibers of \( E \) define an almost contact structure.

In a similar way, we define symplectic and cosymplectic subbundles of a cosymplectic bundle, and invariant subbundles of an almost contact bundle.

Next, let \((E, \Phi, \eta)\) be a cosymplectic vector bundle over \( M \) with canonical section \( \xi \), and \( F \) a symplectic or cosymplectic subbundle. Then we have
PROPOSITION 1. There exists an almost contact structure \((\phi, \xi, \eta)\) and a metric \(g\) in \(E\) such that:

1. \(g_x(\phi_x u, \phi_x v) = g_x(u, v) - \eta_x(u)\eta_x(v),\)
2. \(\Phi_x(u, v) = g_x(u, \phi_x v),\)
3. \(F_x\) is an invariant vector subspace of \(E_x,\)

for all \(x \in M, u, v \in E_x.\)

Proof. Let \(E^+_\eta\) be the symplectic subbundle of \(E\) whose fiber at \(x \in M\) is the space

\[
(E^+_\eta)_x = \{u \in E_x \mid \eta_x(u) = 0\}.
\]

We consider two cases, say \(F\) is a symplectic or cosymplectic subbundle of \(E.\) First, suppose that \(F\) is a symplectic subbundle of \(E.\) Thus, \(F\) is a symplectic subbundle of \(E^+_\eta.\) Then, from Theorem 3.4 of [6], there exists an almost complex structure \(J\) on \(E^+_\eta\) (i.e., \(J\) is an automorphism \(J: E^+_\eta \to E^+_\eta\) of the vector bundle \(E^+_\eta\) with \(J^2 = -I\) and a metric \(h\) in \(E^+_\eta\) such that:

(i) \(h_x(u, v) = h_x(J_xu, J_xv),\)
(ii) \(\Phi_x(u, v) = h_x(u, J_xv),\)
(iii) \(F\) is a complex subbundle of \(E^+_\eta.\)

We set

\[
\phi_x u = J_x(u - \eta_x(u)\xi_x),
\]

and

\[
g_x(u, v) = h_x(u - \eta_x(u)\xi_x, v - \eta_x(v)\xi_x) + \eta_x(u)\eta_x(v),
\]

for all \(x \in M, u, v \in E_x.\) Then it is easy to prove that \((\phi, \xi, \eta)\) is an almost contact structure, \(g\) a metric on \(M\) and (1), (2) and (3) are satisfied.

Now, suppose that \(F\) is a cosymplectic subbundle of \(E.\) Then \(F^+_{\eta}\) is a symplectic subbundle of \(E^+_\eta.\) Thus, by a similar device, we deduce the result. \(\square\)

3. Foliations on cosymplectic manifolds

First, we recall some definitions about foliations on manifolds [5, 9].

Let \(F\) be a foliation of dimension \(p\) on a \(n\)-dimensional manifold \(M.\) We denote by \(TF\) the vector subbundle of \(TM\) which consists of the tangent vectors to \(F,\) and by \(T_xF\) the fiber of \(TF\) over \(x.\) If \(X\) is a vector field tangent to \(F\) (i.e., \(X(x) \in T_xF\) for all \(x \in M\)) then we put \(X \in F.\)

The foliation \(F\) is said to be compact if each leaf of \(F\) is compact. A leaf \(L\) of a compact foliation \(F\) is said to be stable if every neighborhood \(U\) of \(L\) contains an invariant neighborhood \(V\) of \(L,\) i.e., \(V\) is a collection of leaves. \(F\) is said to be stable if every leaf of \(F\) is stable.

Let \(M\) be a cosymplectic manifold with structure \((\Phi, \eta),\) i.e., \(\eta \wedge \Phi^n \neq 0, d\eta = 0, d\Phi = 0.\) Then \((TM, \Phi, \eta)\) is a cosymplectic vector bundle. A foliation \(F\) of
dimension $p = 2s$ (resp. $p = 2s + 1$) is said to be symplectic (resp. cosymplectic) if the vector subbundle $TF$ of $TM$ is symplectic (resp. cosymplectic).

Let us recall that an almost contact metric manifold $(M, \phi, \eta, \xi, g)$ is called almost cosymplectic (in the sense of Blair [1]) if $d\Phi = 0, d\eta = 0$, where $\Phi$ is the fundamental 2-form of $M$, i.e., $\Phi(X, Y) = g(X, \phi Y)$.

Now, let $(M, \Phi, \eta)$ be a cosymplectic manifold with canonical vector field $\xi$, and $F$ a symplectic or cosymplectic foliation. Then, from Proposition 1, we have.

**Proposition 2.** There exists on $M$ an almost contact metric structure $(\phi, \eta, \xi, g)$ with fundamental 2-form $\Phi$ which is almost cosymplectic, and the foliation $F$ is invariant.

Finally, from Proposition 2 and Theorem 1 of [2] we easily deduce our main result.

**Theorem 1.** A compact symplectic or cosymplectic foliation $F$ of a cosymplectic manifold $(M, \Phi, \eta)$ is stable.

4. Examples

Let $S_r$ be the $2r + 1$-dimensional solvable non-nilpotent Lie group of matrices of the form

$$
\begin{pmatrix}
e^z & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & e^{-z} & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & e^z & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & e^{-z} & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & e^z & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & e^{-z} & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1
\end{pmatrix}
$$

where $x_i, y_i, z \in R, 1 \leq i \leq r$. Then $S_r$ may be identified with $R^{2r+1}$ by assigning to each matrix in $S_r$ its global coordinates $(x_1, y_1, \ldots, x_r, y_r, z)$.

There exists a canonical injective Lie group homomorphism $j_r : S_r \rightarrow S_{r+1}$ defined by

$$j_r(x_1, y_1, \ldots, x_r, y_r, z) = (x_1, y_1, \ldots, x_r, y_r, 0, 0, z)$$

Then $S_r$ may be considered as a Lie subgroup of $S_{r+1}$ and we have a chain of Lie groups

$$\{e\} \subset S_1 \subset S_2 \subset \ldots \subset S_r \subset S_{r+1} \subset \ldots$$
Alternatively, $S_r$ can be described as the semidirect group $S_r = R \ltimes \alpha R^{2r}$, where $\phi(z) : R^{2r} \rightarrow R^{2r}$ is given by the matrix

$$
\begin{pmatrix}
e^z & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & e^{-z} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & e^z & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & e^{-z} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & e^z & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & e^{-z}
\end{pmatrix}
$$

A simple computation shows that

$$\{\tilde{\alpha}_i = e^{-z}dx_i, \tilde{\beta}_i = e^zdy_i, \tilde{\gamma} = dz\}$$

is a family of linearly independent left invariant 1-forms on $S_r$. Then we have

$$d\tilde{\alpha}_i = \tilde{\alpha}_i \wedge \tilde{\gamma}, d\tilde{\beta}_i = -\tilde{\beta}_i \wedge \tilde{\gamma}, d\tilde{\gamma} = 0.$$

The corresponding dual basis of left invariant vector fields on $S_r$ is

$$\{\tilde{X}_i = e^z \frac{\partial}{\partial x_i}, \tilde{Y}_i = e^{-z} \frac{\partial}{\partial y_i}, \tilde{Z} = \frac{\partial}{\partial z}\}$$

and we have

$$[\tilde{X}_i, \tilde{Z}] = -\tilde{X}_i, [\tilde{Y}_i, \tilde{Z}] = \tilde{Y}_i,$$

all the other brackets being zero.

Now, let $B \in SL(2, \mathbb{Z})$ be an unimodular matrix with positive real and different eigenvalues $\lambda$ and $\lambda^{-1}$ and $P \in GL(2, \mathbb{R})$ such that

$$PBP^{-1} = \begin{pmatrix} \lambda & 0 \\
0 & \lambda^{-1} \end{pmatrix}$$

Let be $z_0 \in R$ such that $\lambda = e^{\pi z_0}$ and consider the lattice $L_r = P_r(Z^{2r})$, where

$$P_r = \begin{pmatrix} P & 0 & \cdots & 0 \\
0 & P & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & P \end{pmatrix}$$

Then $L_r$ is invariant by $\phi(nz_0) = \phi(z_0)^n, \forall n \in Z$ and $\Gamma_r = (z_0)Z \ltimes L_r$ is a co-compact subgroup of $S_r$, i.e., Solv($r$) = $\Gamma_r \backslash S_r$ is a compact non-nilpotent solvmanifold of dimension $2r+1$. We notice that Solv(1) is the manifold considered in [8] and Solv(1) $\times S^1$ is the manifold considered in [3, 4].
Alternatively, the manifold $\text{Solv}(r)$ may be seen as the total space of a $T^{2r}$-bundle over $S^1$. In fact, let $T^{2r} = R^{2r}/L_r$ the 2r-dimensional torus and $p : Z \rightarrow \text{Diff}(T^{2r})$ the representation defined as follows: $\rho(n)$ represents the transformation of $T^{2r}$ covered by the linear transformation of $R^{2r}$ given by the matrix

$$
\begin{pmatrix}
  e^{z_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\
  0 & e^{-z_1} & 0 & 0 & \cdots & 0 & 0 \\
  0 & 0 & e^{z_2} & 0 & \cdots & 0 & 0 \\
  0 & 0 & 0 & e^{-z_2} & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & \cdots & e^{z_n} & 0 \\
  0 & 0 & 0 & 0 & \cdots & 0 & e^{-z_n}
\end{pmatrix}
$$

This representation determines an action $A : Z \times (T^{2r} \times R) \rightarrow T^{2r} \times R$ defined by

$$A(n, [x_1, y_1, \ldots , x_r, y_r], z) = (\rho(n)([x_1, y_1, \ldots , x_r, y_r]), z + n).$$

Then $p : T^{2r} \times Z R \rightarrow S^1$ is a $T^{2r}$-bundle where the projection $p$ is given by

$$p([x_1, y_1, \ldots , x_r, y_r], z) = [z].$$

Then it is clear that $T^{2r} \times Z R$ may be canonically identified to $\text{Solv}(r)$.

Since $j_r(\Gamma_r) \subset \Gamma_{r+1}$ then $j_r$ induces a canonical embedding

$$J_r : \text{Solv}(r) \rightarrow \text{Solv}(r + 1).$$

If $\pi_r : S_r \rightarrow \text{Solv}(r)$ is the canonical projection, then we have a global basis $\{\alpha_i, \beta_i, \gamma\}$ of 1-forms on $\text{Solv}(r)$ such that

$$\pi_r^* \alpha_i = \bar{\alpha}_i, \quad \pi_r^* \beta_i = \bar{\beta}_i, \quad \pi_r^* \gamma = \bar{\gamma}_i,$$

$$d\alpha_i = \alpha_i \wedge \gamma, \quad d\beta_i = -\beta_i \wedge \gamma, \quad d\gamma = 0,$$

and the corresponding dual basis of vector fields, denoted by $\{X_i, Y_i, Z\}$ verifies

$$[X_i, Z] = -X_i, \quad [Y_i, Z] = Y_i,$$

the other brackets being all zero. Obviously, $\bar{X}_i$, $\bar{Y}_i$, $\bar{Z}$ and $X_i$, $Y_i$, $Z$ are $\pi_r$-related.

Now, for any integer $s$, $1 \leq s < r$, let us consider the left invariant involutive distribution $\bar{F}_s$ on $S_r$ globally spanned by $\{\bar{X}_i, \bar{Y}_i, \bar{Z} \mid 1 \leq i \leq s\}$. Then $\bar{F}_s$ is a subalgebra of the Lie algebra of $S_r$; in fact, $\bar{F}_s$ is the Lie algebra of the Lie subgroup $S_s$. Thus, the leaves of the foliation $\bar{F}_s$ determined by $\bar{F}_s$ are all diffeomorphic to $S_s$. Furthermore, since $\bar{F}_s$ is left invariant, then it descends to a distribution $\bar{F}_s$ on $\text{Solv}(r)$; $\bar{F}_s$ defines a foliation $F_s$ on $\text{Solv}(r)$ whose leaves are all diffeomorphic to $\text{Solv}(s)$.

Consider the cosymplectic structure $(\Phi, \eta)$ on $\text{Solv}(r)$ defined by

$$\Phi = \sum_{i=1}^{r} \alpha_i \wedge \beta_i, \quad \eta = \gamma.$$
A simple computation shows that $F_r$ is a cosymplectic foliation on the cosymplectic manifold $(\text{Solv}(r), \Phi, \eta)$ and, from Theorem 1, it is stable.

Next, let $F$ be the involutive distribution on $\text{Solv}(r)$ globally spanned by \(\{X_i, Y_i \mid 1 \leq i \leq r\}\). Then $F$ is a foliation $F$ on $\text{Solv}(r)$ whose leaves are precisely the fibres of the fibration $p: \text{Solv}(r) \rightarrow S^1$, which are $2r$-dimensional tori. Thus, $F$ is a compact foliation. Furthermore, it is easy to prove that $F$ is a symplectic foliation on the cosymplectic manifold $(\text{Solv}(r), \Phi, \eta)$ and, from Theorem 1, it is stable. (We notice that this last result follows directly since the leaves of $F$ are the fibres of $p$, which is a fibration with compact fibres [9]).

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