ON HYPERCYLINDERS IN CONFORMALLY
SYMMETRIC MANIFOLDS

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Abstract. Hypercylinders in conformally symmetric manifolds are considered. The main
result is the following theorem: Let \((M, g)\) be a hypercylinder in a parabolic essentially conformally
symmetric manifold \((N, \tilde{g})\), \(\dim N \geq 5\) and let \(\tilde{U}\) be the subset of \(N\) consisting of all points of
\(N\) at which the Ricci tensor \(\tilde{S}\) of \((N, \tilde{g})\) is not recurrent. If \(\tilde{U} \cap M\) is a dense subset of \(M\), then
\((M, g)\) is a conformally recurrent manifold.

1. Introduction. Totally umbilical submanifolds in locally symmetric, recurrent, conformally flat, conformally symmetric and conformally recurrent mani-
ifolds were investigated by many authors (e.g. [6], [10], [19], [21], [24]–[27], [29],
[33]). An important part of these investigations treats problems concerning totally
umbilical hypersurfaces in these classes of manifolds (e.g. [7], [8], [20], [28], [30]).
On the other hand, totally umbilical hypersurfaces, as well as hypercylinders,
are special examples of quasi-umbilical hypersurfaces. Certain results on quasi-
umbilical hypersurfaces in locally symmetric, recurrent and conformally flat mani-
ifolds are presented in [3], [34] and [22] respectively. Moreover, hypercylinders in
locally symmetric and conformally flat manifolds were studied in [9] and [37]
(see also [2]) respectively. We shall continue study in this direction considering
hypercylinders in conformally symmetric manifolds.

Let \((N, \tilde{g})\) be an \(n\)-dimensional, \(n \geq 4\), semi-Riemannian manifold with the
metric tensor \(\tilde{g}\) and let \(\nabla\) be the Levi-Civita connection of \((N, \tilde{g})\). Let \((N, \tilde{g})\) be
covered by a system of charts \(\{\tilde{U}; x^r\}\). We denote by \(\tilde{g}_{rs}, \{\tilde{f}_{rs}\}, \tilde{\nabla}_s, \tilde{R}_{rstu}, \tilde{C}_{rstu},
\tilde{S}_{rs}\) and \(\tilde{K}\) the local components of the metric tensor \(\tilde{g}\), the Christoffel symbols, the
operator of covariant differentiation, the Riemann-Christoffel curvature tensor \(\tilde{R}\),
the Weyl conformal curvature tensor \(\tilde{C}\), the Ricci tensor \(\tilde{S}\) and the scalar curvature
\(\tilde{K}\) of \((N, \tilde{g})\) respectively, where \(r, s, t, u, v, w \in \{1, 2, \ldots, n\}\).

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We have
\[
\tilde{C}_{rstu} = \tilde{R}_{rstu} + \frac{\tilde{K}}{(n-1)(n-2)}(\tilde{g}_{ru}\tilde{g}_{st} - \tilde{g}_{rs}\tilde{g}_{tu}) \\
- \frac{1}{n-2}(\tilde{g}_{ru}\tilde{S}_{ls} + \tilde{g}_{ls}\tilde{S}_{ru} - \tilde{g}_{rs}\tilde{S}_{lu} - \tilde{g}_{lu}\tilde{S}_{rs}).
\]
(1.1)

A \((0, k)\)-tensor field \(T\) on \(N\) is said to be recurrent [36] if the condition
\[
T(X_1, \ldots, X_k)\tilde{\nabla}T(Y_1, \ldots, Y_k; Z) = T(Y_1, \ldots, Y_k)\tilde{\nabla}T(X_1, \ldots, X_k; Z)
\]
holds on \(N\), where \(X_1, \ldots, X_k, Y_1, \ldots, Y_k, Z \in \mathfrak{X}(N)\), \(\mathfrak{X}(N)\) being the Lie algebra of vector fields on \(N\). In particular, if \(\tilde{\nabla}T\) vanishes on \(N\), then \(T\) is called parallel. A manifold \((N, \tilde{g})\), \(n \geq 4\), is said to be locally symmetric [31] (resp. conformally symmetric [4]) if its tensor \(\tilde{R}\) (resp. tensor \(\tilde{C}\)) is parallel with respect to \(\tilde{\nabla}\). Further, a manifold \((N, \tilde{g})\), \(n \geq 4\), is said to be recurrent [38] (resp. conformally recurrent [1] or Ricci recurrent [32]) if its tensor \(\tilde{R}\) (resp. tensor \(\tilde{C}\) or tensor \(\tilde{S}\)) is recurrent. A conformally symmetric manifold \((N, \tilde{g})\) which is neither locally symmetric nor conformally flat is called essentially conformally symmetric or shortly e.c.s. manifold. Various examples of e.c.s. manifolds are given in [35], [11] and [18]. All e.c.s. metrics are indefinite ([16, Theorem 2]). Any e.c.s. manifold \((N, \tilde{g})\) satisfies the following equation ([18], [17])
\[
F\tilde{C}(X, Y, Z, W) = \tilde{S}(X, W)\tilde{S}(Y, Z) - \tilde{S}(X, Z)\tilde{S}(Y, W)
\]
for some function \(F\), where \(X, Y, Z, W \in \mathfrak{X}(M)\). \(F\) is called the fundamental function of \((N, \tilde{g})\). All e.c.s. manifolds can be divided into the following five non-empty and mutually disjoint classes (according to the behaviour of the Ricci tensor and the fundamental function \(F\) [12]):

**Class I.** Ricci recurrent ones (they all satisfy \(F = 0\)).

**Class II.** Parabolic e.c.s. manifold [15] (satisfying \(F = 0\) identically but not Ricci recurrent).

**Class III.** Elliptic ones [14] (\(F = \text{constant} \neq 0\), semidefinite everywhere).

**Class IV.** Hyperbolic ones [13] (\(F = \text{constant} \neq 0\), semidefinite nowhere)

**Class V.** Those with \(F\) non-constant.

**Lemma 1** (Theorem 7, 8, 9 and formula (6) of [17] and Theorem 7 of [18]).

Let \((N, \tilde{g})\) be an e.c.s. manifold and let \(\{\tilde{U}; x^r\}\) be a chart on \(N\). Then the following relations are satisfied on \(\tilde{U}\):
\[
\tilde{\nabla}_r \tilde{C}_{rstu} = 0, \\
F \tilde{C}_{rstu} = \tilde{S}_{ur} \tilde{S}_{ls} - \tilde{S}_{ls} \tilde{S}_{ur}, \\
\tilde{\nabla}_r \tilde{S}_{ls} = \tilde{\nabla}_l \tilde{S}_{rs}, \\
\tilde{k} = 0.
\]
(1.2) (1.3) (1.4) (1.5)
2. Hypercylinders. Let $M$ be a hypersurface in an $n$-dimensional, $n \geq 4$, semi-Riemannian manifold $(N, \tilde{g})$ and let the tensor $g$, induced by the metric tensor $\tilde{g}$, be the metric tensor of $M$. Moreover, let $x^r = x^r(y^a)$ be the local expression of $M$ in $N$. Then we have $g_{ab} = \tilde{g}_{rs} B^r_a B^s_b$, where

$$B^r_{a_1 \cdots a_n} = B^r_{a_1} \cdots B^s_{a_n}, \quad B^r_a = \partial_a x^r, \quad \partial_a = \partial / (\partial y^a),$$

and $g_{ab}$ are the local components of the tensor $g$. Further, we denote by \( \{ a \ b \ c \ \} \), $R_{abcd}$, $S_{ad}$, $C_{abcd}$ and $K$ the local components of the Christoffel symbols, the curvature tensor $R$, the Ricci tensor $S$, the Weyl conformal curvature tensor $C$ and the scalar curvature $K$ of $(M, g)$ respectively. Here and below, $a, b, c, d, e, f, h, i, j \in \{1, \ldots, n-1\}$. Let $N^r$ be the local components of a local unit vector field normal to $M$. Then we have the following relations

$$\tilde{g}_{rs} N^r N^s = \varepsilon, \quad \tilde{g}_{rs} N^r B^s_a = 0, \quad g^{ab} B^r_{ab} = \tilde{g}^{rs} - \varepsilon N^r N^s, \quad \varepsilon = \pm 1. \quad (2.1)$$

The hypersurface $(M, g)$ is said to be a cylindrical hypersurface or shortly a hypercylinder (cf. [5, pp. 147–148], [9]) in $(N, \tilde{g})$ if the second fundamental tensor $H$ of $(M, g)$ satisfies on $M$ the condition $H = \beta u \otimes u$, where $\beta$ is a function and $u$ a 1-form on $M$, respectively. Let $p$ be a point of the hypercylinder $(M, g)$. Then the following equality

$$H_{ad} = \beta u_a u_d \quad (2.2)$$

holds on some neighbourhood $U \subset M$ of $p$, where $H_{ad}$ and $u_a$ are the local components of $H$ and $u$ on $U$, respectively. We denote by $\nabla$ the operator of the van der Waerden-Bortolotti covariant derivative. Then, in virtue of (2.2), the Gauss and Weingarten formulas for $(M, g)$ in $(N, \tilde{g})$ take on $U$ the following form

$$\nabla_a B^r_a = \varepsilon H_{ad} N^r = \varepsilon \beta u_a u_d N^r, \quad (2.3)$$

$$\nabla_a N^r = -H_{ad} g^{cd} B^r_c = -\beta u_d u^a B^r_a, \quad u^d = g^{da} u_a. \quad (2.4)$$

respectively. Furthermore, by (2.2), the Gauss and Codazzi equations for $(M, g)$ in $(N, \tilde{g})$ can be expressed on $U$ as follows:

$$R_{abcd} = \tilde{R}_{rads} B^r_{abcd} + \varepsilon (H_{ad} H_{bc} - H_{ac} H_{bd}) = \tilde{R}_{rads} B^r_{abcd}, \quad (2.5)$$

$$V_{bcd} = N^r \tilde{R}_{rads} B^r_{bcd} = \nabla_d H_{bc} - \nabla_c H_{bd} = u_b (\beta u_c - \beta u_d) + \beta (u_d \nabla_c u_b - u_b \nabla_d u_c + u_c (\nabla_d u_b - \nabla_c u_d)), \quad \beta_c = \nabla_c \beta. \quad (2.6)$$

From this, by contraction with $g^{bc}$ and making use of (2.1), we obtain

$$v_d = g^{bc} V_{bcd} = N^r \tilde{S}_{rs} B^r_d$$

$$= g^{bc} u_b u_d \beta - u^h \beta u_d + \beta (u_h \nabla_h u_d - (\nabla_h u^h) u_d + 2 u^h \nabla_d u_h). \quad (2.7)$$
Lemma 2. Let \((M, g)\) be a hypercylinder in a semi-Riemannian manifold \((N, \tilde{g}), n \geq 4\). If \(p\) is a point of \(M\) such that the relations (2.2) and \(\beta \neq 0\) are satisfied at every point of some neighborhood \(U \subset M\) of \(p\) then the equality

\[
V_{bcd} = u_b^a u^h V_{hd} + u_d (u^h V_{bh} + u_c) - u_c (u^h V_{bd} + u_d v_d)
\]

(2.8)

holds on \(U\).

Proof. From (2.5), by making use of (2.3) and (2.6), it follows that

\[
\nabla_v R_{abcd} = B^{rstu}_{vabcd} \tilde{g}_{rstu} + \varepsilon \beta u_c (u_a V_{bcd} - u_b V_{cad} + u_c V_{dab} - u_d V_{abc}).
\]

(2.9)

This, by contraction with \(g^{te}\) and an application of (2.1), (2.7) and the identity

\[
\tilde{g}^{rr} \nabla_v \tilde{R}_{rstu} = \tilde{g}_{r} \tilde{S}_{rs} - \tilde{g}_{ls} \tilde{S}_{us},
\]

(2.10)

yields

\[
\nabla_v S_{ad} = B^{ts}_{cd} \tilde{g}_{vts} - \varepsilon N^s N^t B^{rstu}_{vabcd} \tilde{g}_{v \tilde{R}_{rstu}} + \varepsilon \beta u_d K_{ad},
\]

(2.11)

where

\[
K_{ad} = u_a v_d + u_d v_a + u^h V_{bd} + u^h V_{cb},
\]

(2.12)

\[
k = g^{ad} K_{ad} = 4 u^h v_h.
\]

(2.13)

On the other hand, contracting (2.9) with \(g^{ae}\) and using (2.10) and (2.2), we find

\[
\nabla_d S_{bc} - \nabla_c S_{bd} = B^{rst}_{dab} (\tilde{g}_{u \tilde{S}_{rs}} - \tilde{g}_{l \tilde{S}_{us}}) - \varepsilon N^r N^s B^{rstu}_{vabcd} \tilde{g}_{v \tilde{R}_{rstu}} + \varepsilon \beta (V_{bcd} - u_b u^h V_{cad} - u_c u^h V_{dab} + u_d u^h V_{abc}).
\]

(2.14)

The following equality follows immediately from the second Bianchi identity

\[
N^r N^s B^{st}_{abcd} \nabla_v \tilde{R}_{rstu} = N^r N^s B^{st}_{abcd} \tilde{g}_{v \tilde{R}_{rstu}} - N^v N^r B^{stu}_{cd} \tilde{g}_{uv \tilde{R}_{rs}} - N^v N^r B^{stu}_{cd} \tilde{g}_{uv \tilde{R}_{rs}},
\]

which, in virtue of (2.11) and (2.12), turns into

\[
\varepsilon N^r N^s B^{stu}_{abcd} \tilde{g}_{v \tilde{R}_{rstu}} = B^{stv}_{cd} (\tilde{g}_{v \tilde{S}_{ru}} - \tilde{g}_{r \tilde{S}_{vu}}) + \nabla_c S_{db} - \nabla_d S_{cb} + \varepsilon \beta u_d (u^h V_{cb} + u^h V_{bc}) - u_c (u^h V_{bd} + u^h V_{db} + u_d u^h V_{c}).
\]

The last equation, together with (2.14), completes the proof.

Lemma 3. Let \((M, g)\) be a hypercylinder in a semi-Riemannian manifold \((N, \tilde{g}), n \geq 4\). If \(p\) is a point of \(M\) such that the relations (1.7) and (2.2) are fulfilled at every point of some neighborhood \(U \subset M\) of \(p\) then the equalities

\[
u^h V_{h} = 0,
\]

(2.15)

\[
u^h v_h = 0
\]

(2.16)

hold on \(U\), where

\[
\omega_{a cd} = \beta (u_d (\nabla_d u_c - \nabla_c u_d) + u_c (\nabla_d u_c - \nabla_c u_d) + u_d (\nabla_c u_a - \nabla_a u_c)).
\]

(2.17)
Proof. Transvecting (1.7) with $B^\text{ext}_{abcd}$ and using the Ricci identity, (2.1), (2.5) and (2.6), we find
\[
\nabla_f\nabla_e R_{abcd} - \nabla_e \nabla_f R_{abcd} + \varepsilon(-V_{scd}V_{ae} + V_{acd}V_{se} - V_{sad}V_{cef} + V_{cas}V_{def}) = 0. \tag{2.18}
\]
Next, contracting the above equation with $g^d$ and $g^c$ and applying (2.7) we get (2.15). Finally, (2.16) is an immediate consequence of (2.15) and the following identity
\[
u_a V_{bcd} + u_c V_{bda} + u_d V_{bac} = u_\omega a_{acd}. \tag{2.19}
\]
Our lemma is thus proved.

Remark. In the next sections we shall consider hypercylinders satisfying certain additional conditions. Let $(M, g)$ be a hypercylinder in a manifold $(N, \bar{g})$ and let $p$ be a point of $M$ such that the relation (2.2) is fulfilled at every point of some neighbourhood $U \subset M$ of $p$. We assume that at every point of $U$ one of the following relations is satisfied:
\[
eg\neg g^a u_a u_d = 1, \tag{2.20}
\]
\[
eg\neg g^a u_a u_d = -1, \tag{2.21}
\]
\[
eg\neg g^a u_a u_d = 0. \tag{2.22}
\]
Thus the scalar $g^a u_a u_d$ is a constant on $U$. This fact implies
\[
u^h \nabla_a u_h = 0. \tag{2.23}
\]
Transvecting now (2.6) with $u^b$, $u^c$ and $u^d$ respectively and applying (2.20)–(2.23) we easily get
\[
u^h V_{hcd} = \eta(\beta_d u_c - \beta_c u_d + \beta(\nabla_d u_c - \nabla_c u_d)), \tag{2.24}
\]
\[
u^h u^h V_{hjd} = \eta(\beta_d - \eta(\nabla_h u_d + \beta u^h \nabla_h u_d)), \tag{2.25}
\]
\[
u^h V_{bch} = (u^h \beta_h u_c - \eta \beta_c)u_b + (u_c u^h \nabla_h u_b - \eta \nabla_c u_b + u_d u^h \nabla_h u_c) \tag{2.26}
\]
respectively, where $\eta \in \{-1, 0, 1\}$.

3. Hypercylinders in conformally symmetric manifolds.

Lemma 4. Let $(M, g)$ be a hypercylinder in a conformally symmetric manifold $(N, \bar{g})$, $n \geq 4$ and let $p$ be a point of $M$ such that the conditions (2.2) and $\beta \neq 0$ are satisfied at every point of some neighbourhood $U \subset M$ of $p$. Then we have:

(i) the equality
\[
\nabla e C_{abcd} = \varepsilon \beta_u \left( \frac{k}{n-2} \right) \left( g_{a(c} d_{b)} - g_{a d} g_{c b} \right) + \left( u_b u_c - \frac{\beta_c}{n-3} \right) K_{a d} + \left( u_a u_d - \frac{\beta d}{n-3} \right) K_{b c}
\]
\[
- \left( u_a u_c - \frac{g_{ac}}{n-3} \right) K_{bd} - \left( u_b u_d - \frac{g_{bd}}{n-3} \right) K_{ac}
\]
holds on \( U \).

(ii) If at every point of \( U \) (2.21) is fulfilled then \( \nabla C = 0 \) holds on \( U \).

Proof. (i): Translating (1.2) with \( \varepsilon N^u N^d B_{r,s}^{uv} \) and using (1.1) and (2.1) we get

\[-\varepsilon N^u N^d B_{r,s}^{uv} \nabla_v \tilde{R}_{r,su} = -\frac{B_{cda}^{uv} \nabla_v \tilde{S}_{ur}}{n-2} + \frac{\varepsilon (\nabla_v \tilde{S}_{ur}) N^u N^d B_{c}^{uv} g_{hd}}{n-2} + \frac{B_{c}^{uv} (\nabla_v \tilde{K}) g_{hd}}{(n-1)(n-2)}.
\]

Substituting this in (2.11) find

\[
\frac{B_{cda}^{uv} \nabla_v \tilde{S}_{ur}}{n-2} = \frac{\nabla v S_{ad}}{n-3} - \frac{\varepsilon u_c K_{ad}}{n-3} + \frac{\varepsilon N^u N^d B_{c}^{uv} \nabla_v \tilde{S}_{ur}}{n-2} - \frac{\varepsilon N^u N^d B_{c}^{uv} (\nabla_v \tilde{K}) g_{hd}}{(n-1)(n-2)} \frac{B_{c}^{uv} (\nabla_v \tilde{K}) g_{hd}}{(n-1)(n-2)}.
\]

Contracting (3.2) with \( g^{ad} \) and using (2.1) and (2.13) we obtain

\[
2\varepsilon N^u N^d B_{c}^{uv} \nabla_v \tilde{S}_{ur} = -\frac{\nabla v K}{n-3} + \frac{\varepsilon \beta_{ke} g_{hd}}{n-3} + \frac{2B_{c}^{uv} (\nabla_v \tilde{K})}{n-1}.
\]

Now, by the above equation, (3.2) takes the form

\[
\frac{B_{cda}^{uv} \nabla_v \tilde{S}_{ur}}{n-2} = \frac{\nabla v S_{ad}}{n-3} - \frac{\varepsilon u_c K_{ad}}{n-3} + \frac{\varepsilon N^u N^d B_{c}^{uv} \nabla_v \tilde{S}_{ur}}{n-2} - \frac{\varepsilon N^u N^d B_{c}^{uv} (\nabla_v \tilde{K}) g_{hd}}{(n-1)(n-2)} \frac{B_{c}^{uv} (\nabla_v \tilde{K}) g_{hd}}{(n-1)(n-2)}.
\]

But this, together with (2.9), (1.1) and (2.2) gives

\[
\nabla_v C_{abcd} = \varepsilon \beta_{uc} \left( u_a V_{bd} - u_b V_{ad} + u_c V_{ab} - u_d V_{ca} \right) - \frac{g_{ad} K_{bc} + g_{bc} K_{ad} - g_{a} K_{bd} - g_{b} K_{ad} + k(g_{ad} g_{bc} - g_{ac} g_{bd})}{n-2}.
\]

On the other hand, (2.8), (2.12) and (2.13) yield

\[
u_a V_{bd} - u_b V_{ad} + u_c V_{ab} - u_d V_{ca} = u_a u_d K_{bc} + u_b u_c K_{ad} - u_a u_c K_{bd} - u_b u_d K_{ac}.
\]

Now (3.3) turns into (3.1), completing the proof of (i).

(ii): Identity (2.8), in virtue of (2.24), (2.26), (2.7) and (2.21), reduces to \( V_{bcd} = 0 \). Now (3.3) completes the proof.

From Lemma 4(i) the following proposition follows easily.
Proposition 1. Let \((M, g)\) be a hypercylinder in a conformally symmetric manifold \((N, \tilde{g})\), \(n \geq 5\) and let \(p\) be a point of \(M\) such that the conditions (2.2), (2.20) and \(\beta \neq 0\) are fulfilled at every point of some neighbourhood \(U \subset M\) of \(p\). Then the condition \(\nabla C = 0\) is satisfied on \(U\) if and only if the relation

\[
K_{ad} = \frac{k}{2(n - 2)}(n - 3)u_d u_d + g_{ad}
\]

holds on \(U\).

Lemma 5. Let \((M, g)\) be a hypercylinder in a conformally symmetric manifold \((N, \tilde{g})\), \(n \geq 5\) and let \(p\) be a point of \(M\) such that the relation \(H_{ad} = 0\) holds at \(p\). Then the tensor \(\nabla C\) vanishes at \(p\).

Proof. We note that the equality

\[
\nabla_e (\tilde{C}_{rstu} B^{rstu}_{abcd}) = B^{rstu}_{eabcd} \tilde{V}_e \tilde{C}_{rstu} + \tilde{C}_{rstu} B^{rstu}_{abcd} \nabla_e B^r_a - \tilde{C}_{rstu} B^{rstu}_{abcd} \nabla_e B^r_a
\]

holds on some neighbourhood \(U\) of \(p\). This, by (1.1), (2.5), (1.2) and (2.3) reduces at \(p\) to

\[
\nabla_e R_{abcd} + \left(\nabla_e \hat{K}\right) (g_{bd} g_{ac} - g_{ad} g_{bc})
\]

\[
= \frac{1}{n - 2} (g_{bd} B^{eur}_{abcd} \tilde{V}_e \tilde{S}_{ru} + g_{ad} B^{euv}_{cabc} \tilde{V}_e \tilde{S}_{uv} - g_{bd} B^{euv}_{cabc} \tilde{V}_e \tilde{S}_{ru} - g_{ad} B^{eur}_{abcd} \tilde{V}_e \tilde{S}_{uv})
\]

Next, contracting (3.5) with \(g^{bc}\) we obtain

\[
\frac{B^{eur}_{abcd} \tilde{V}_e \tilde{S}_{ru}}{n - 2} = \nabla_r S_{ad} + \frac{1}{n - 3} \left(\nabla_e \hat{K}\right) g_{ad} - \frac{(\tilde{V}_e \tilde{S}_{ru}) B^{euv}_{cabc} g^{bc} g_{ad}}{(n - 1)(n - 3)}
\]

Substituting this into (3.5) we get our assertion.

Lemma 6. Let \((M, g)\) be a hypercylinder in a conformally symmetric manifold \((N, \tilde{g})\), \(n \geq 5\) and let \(p\) be a point of \(M\) such that the relations (2.2) and \(\beta \neq 0\) are satisfied at every point of some neighbourhood \(U \subset M\) of \(p\). If the equality (2.22) is fulfilled at \(p\) then the tensor \(\nabla C\) vanishes at \(p\).

Proof. Transvecting (2.6) with \(u^b\) and \(u^d\) and using (2.22) we get

\[
u^h V_{bch} = u^h \beta_h u_b u_c + \beta (u_\alpha u^h \nabla_h u_\alpha + u_\beta u^h \nabla_h u_\beta - u_\gamma u^h \nabla_h u_\gamma),
\]

\[
u^h V_{bdc} = \beta (u_\alpha u^h \nabla_d u_\alpha - u_\gamma u^h \nabla_c u_\gamma)
\]

respectively. Moreover, from (2.7), by (2.22), we obtain

\[
u_d = -u^h \beta_h u_d + \beta (2u^h \nabla_d u_\alpha - (\nabla_h u^h) u_d - u^h \nabla_h u_d).
\]

Now we can verify that the identity (2.8), by the above three relations, reduces to \(V_{bcd} = 0\). Finally, (3.3) completes the proof.
4. Hypercylinders in non-Ricci-recurrent parabolic e.c.s. manifolds.

**Lemma 7** [15, Lemmas 1 and 4]. Let \((N, \tilde{g}), n \geq 4\), be a parabolic e.c.s. manifold. If \(p\) is a point of \(N\) such that

\[
(\tilde{S}_{ur} \nabla_v \tilde{S}_{ts} - \tilde{S}_{ts} \nabla_v \tilde{S}_{ur})(p) \neq 0,
\]  

(4.1)

then there exists a neighbourhood \(\tilde{U}\) of \(p\) with two vector fields \(A\) and \(B\) which are unique (up to a sign of \(A\)) determined by the following two conditions

\[
\tilde{S}_{rs} = eA_r A_s, \quad e = \pm 1,
\]  

(4.2)

\[
\nabla_{s}\tilde{S}_{rs} = B_u \tilde{S}_{rs} + B_r \tilde{S}_{su} + B_s \tilde{S}_{ur}.
\]  

(4.3)

where \(A_r, B_r\) are the local components of \(A\) and \(B\) respectively. The vector fields \(A\) and \(B\) satisfy on \(\tilde{U}\) the following relations:

\[
\tilde{g}\nabla^s A_r A_s = 0, \quad \tilde{g}\nabla^s A_r B_s = 0,
\]  

(4.4)

\[
\nabla_{s} A_r = (1/2) A_r B_s + A_s B_r,
\]  

(4.5)

\[
\nabla_{s} B_r = B_r B_s + 3\lambda B_r A_s + \lambda A_r B_s + \sigma A_r A_s,
\]  

(4.6)

where \(\lambda\) and \(\sigma\) are some functions on \(\tilde{U}\). Moreover, we have

\[
\tilde{C}_{rstu} = -\Phi (A_r B_s - A_s B_r)(A_t B_u - A_u B_t)
\]  

(4.7)

for a certain (uniquely determined) function \(\Phi\).

**Lemma 8.** Let \((M, g)\) be a hypercylinder in a parabolic e.c.s. manifold \((N, \tilde{g})\), \(n \geq 4\) and let \(p\) be a point of \(M\) such that the conditions: \((2.2), (2.20), (4.1)\) and

\[
N^r A_r \neq 0
\]  

(4.8)

are fulfilled at every point of some neighbourhood \(U \subset M\) of \(p\). Then the equality

\[
\omega_{abc} = 0
\]  

(4.9)

holds on \(U\).

**Proof.** The equality \((2.16)\), in virtue of \((2.7), (4.2)\) and \((4.8)\), give

\[
u^h A_h \omega_{abc} = 0,
\]  

(4.10)

where

\[
A_h = A_r B^r_h.
\]  

(4.11)

Suppose that at a point \(q\) of \(U\) we have

\[
\omega_{abc}(q) \neq 0.
\]  

(4.12)

Then, by \((4.10)\), the equality

\[
u^h A_h = 0
\]  

(4.13)
holds on some open subset $U' \subset U$. From this we obtain

$$A_h \nabla_c u^h + u^h \nabla_c A_h = 0. \quad (4.14)$$

Using (4.5) and (2.3), we can easily verify that the following equality is fulfilled on $U$

$$\nabla_c A_d = A_d B_c / 2 + A_c B_d + \varepsilon \beta N^r A_r u_a u_c, \quad (4.15)$$

$$B_c = B_r B_c^r. \quad (4.16)$$

Substituting (4.15) into (4.14) and applying (4.13) we get

$$A_h \nabla_c u^h + u^h B_h A_c + \varepsilon \beta N^r A_r u_c = 0. \quad (4.17)$$

The formula (2.17), because of (2.7), (4.2) and (4.8), yields

$$A^h \hat{V}_{he f} = 0, \quad (4.18)$$

where $A^h = g^{ah} A_a$. Thus (4.18), by (2.6) and (4.13), gives

$$A^h (u_c \nabla_d u_h - u_d \nabla_c u_h) = 0,$$

whence, by (4.17), it follows that

$$u^h B_h (u_c A_d - u_d A_c) = 0.$$ 

If $(u_c A_d - u_d A_c)(q) = 0$ then also $A_d(q) = 0$. The last equation, in virtue of the relation

$$g^{ad} A_a A_d + \varepsilon (N^r A_r)^2 = 0, \quad (4.19)$$

which follows immediately from (4.4) and (2.1), gives $(N^r A_r)(q) = 0$. But this contradicts (4.8). If $(u^h B_h)(q) = 0$ then from (4.7), by transvection with $N^r B_{bcd} u^b$ and the use of (1.1), (2.1), (4.2), (4.11), (4.16), (1.5) and (4.13), we obtain

$$(u^h \hat{V}_{hec} + (\varepsilon / (n - 2)) N^r A_r (A_c u_d - A_d u_c))(q) = 0. \quad (4.20)$$

On the other hand, transvection (2.19) with $u^a$ we get

$$u_a u^h \hat{V}_{hcd} + u_c u^h \hat{V}_{lda} + u_d u^h \hat{V}_{bac} = \omega_{acd}.$$ 

This, by (4.20), gives $\omega_{acd}(q) = 0$, a contradiction. Our lemma is thus proved.

**Lemma 9.** Let $(M, g)$ be a hypercylinder in a parabolic e.c.s. manifold $(N, \tilde{g})$ and let $p$ be a point of $M$ such that the conditions (2.2), (2.20), (4.1) and

$$N^r A_r = 0 \quad (4.21)$$

are fulfilled. Then the equality (4.9) holds at $p$.

**Proof.** First of all we note that at $p$ the following relation

$$A_d \neq 0 \quad (4.22)$$
is satisfied. In fact, if we had \( A_d = 0 \) the, by (4.21), we get \( A(p) = 0 \) and \( \widetilde{S}(p) = 0 \), which contradicts (4.1). Transsecting now (1.8) with \( N^w B^{rad}_{abcd} \) and making use of (4.2), (4.22), (4.21), (2.1) and (2.6), we find

\[
A_a V_{bcd} = A_b V_{acd}.
\]

(4.23)

Multiplying (4.23) by \( u_f \) and summing the resulting equality cyclically in \( f, c, d \) and applying (2.19) we obtain \( (A_a u_b - A_b u_a) \omega_{fcd} = 0 \). Assume that \( A_a u_b - A_b u_a \) vanishes at \( p \). Then we have

\[
A_n = u^h A_h u_a.
\]

(4.24)

So, (4.23) turns into \( u^h A_h (u_a V_{bcd} - u_b V_{acd}) = 0 \). Summing this cyclically in \( a, c, d \) and using again (2.19), we get \( u^h A_h \omega_{acd} = 0 \). From (4.24), in virtue of (4.22), it follows that \( u^h A_h \) is non-zero at \( p \). Thus the last equality completes the proof.

**Proposition 2.** Let \((M, g)\) be a hypercylinder in a parabolic e.c.s. manifold \((N, \tilde{g})\), \( n \geq 5 \), and let \( p \) be a point of \( M \) such that the conditions: (2.2), \( \beta \neq 0 \), (2.20) and (4.1) are fulfilled at every point of some neighbourhood \( U \subset M \) of \( p \). Then \( C \) is a recurrent tensor on \( U \).

**Proof.** The identity (2.19), in view of Lemmas 8 and 9 reduces to

\[
u_a V_{bcd} + u_b V_{ada} + u_d V_{abc} = 0.
\]

(4.25)

This, by transsection with \( u^a \), yields

\[
V_{bcd} = u_a u^h V_{bch} - u_c u^h V_{bdh}.
\]

(4.26)

On the other hand, transsecting (4.7) with \( N^r B^{rad}_{b,c} \) and using (1.1), (2.6), (4.2), (2.1), (1.5) and (4.7), we find

\[
V_{bcd} = \frac{e^{N^r A_r}}{n-2} (A_d g_{bc} - A_c g_{bd}) + \phi D_b (A_c B_d - A_d B_c),
\]

(4.27)

where \( D_b = N^r B_r A_b - N^r A_r B_b \). Substituting (4.27) into (4.26) and (4.25) respectively, we obtain

\[
V_{bcd} = \Phi D_b (u_d C_c - u_c C_d)
\]

+ \( (e^{N^r A_r}/(n-2))(u^h A_h (u_d g_{bc} - u_c g_{bd}) - u_b (A_c u_d - A_d u_c)), \)

(4.28)

\[
\Phi D_b (u_a A_c B_d - A_d B_c) + u_c (A_d B_a - A_a B_d) - u_d (A_d B_c - A_c B_d))
\]

\[
\frac{e^{N^r A_r}}{n-2} ((u_a A_d - u_d A_a) g_{bc}
\]

+ \( (u_c A_a - u_a A_c) g_{bd} + (u_d A_c - u_c A_d) g_{bh} = 0 \)

(4.29)

respectively, where \( C_c = u^h B_h A_c - u^h A_h B_c \). From (4.29), by transsection with \( u^a u^h \), we get

\[
\Phi u^h D_b (A_c B_d - A_d B_c) = u_d C_c - u_c C_d.
\]

(4.30)

Further, assuming (4.28) cyclically in \( b, c, d \), we obtain

\[
D_b (u_d C_c - u_c C_d) + D_c (u_b C_d - u_d C_b) + D_d (u_c C_b - u_b C_c) = 0,
\]

(4.31)
which, by multiplication with \( u_f \) and antisymmetrization in \( b, f \), gives
\[
(u_f D_b - u_b D_f)(u_d C_c - u_c C_d) = (C_b u_f - C_f u_b)(u_d C_c - u_c C_d).
\]
But this implies
\[
C_c(u_f D_b - u_b D_f) = (D_c - u^h D_h u_c)(u_f C_b - u_b C_f).
\] (4.31)

If \( C_c(p) = 0 \) then (4.28) turns into
\[
V_{b c d} = \frac{eN^r A_r}{n-2}(u^h A_h(u_d g_{bc} - u_c g_{hd}) - u_b (A_c u_d - A_d u_c)).
\]

Using this we can rewrite (3.3) in the following form
\[
\nabla_e C_{a b c d} = \frac{\varepsilon \beta u_e}{(n-2)(n-3)}(g_{ad} g_{bc} - g_{ac} g_{bd})
\]
\[
+ \frac{2 eN^r A_r u^h A_h}{n-2}(u_a u_d g_{bc} + u_b u_c g_{hd} - u_a u_c g_{hd} - u_b u_d g_{ac})
\]
\[
- \frac{1}{n-3}(g_{ad} K_{bc} + g_{bc} K_{ad} - g_{ac} K_{bd} - g_{bd} K_{ac}).
\] (4.32)

which reduces to \( \nabla C = 0 \). If \( C_c(p) \neq 0 \) then (4.31) and (4.30) yield
\[
u_f D_b - u_b D_f = \tau(A_b B_f - A_f B_b), \quad \tau \in \mathbb{R}.
\] (4.33)

Moreover, using (4.28) and (4.33) we obtain
\[
u_a V_{b c d} - u_b V_{a c d} + u_c V_{a d b} - u_d V_{a b c} = ((eN^r A_r)/(n-2))((u_a A_d + u_d A_a) g_{bc})
\]
\[+(u_c A_b + u_b A_c) g_{ad} - (u_a A_c + u_c A_a) g_{hd} - (u_b A_d + u_d A_b) g_{ac})
\]
\[- 2 \Phi \tau (A_b B_a - A_a B_b)(A_c B_d - A_d B_c).
\]

This, by an application of \( \tilde{C}_{r a d s} R^{s t u}_{a b c d} = -\Phi (A_d B_b - A_b B_d)(A_c B_d - A_d B_c), (1.1), (2.5), (1.5) \) and (4.2), turns into
\[
u_a V_{b c d} - u_b V_{a c d} + u_c V_{a d b} - u_d V_{a b c} = 2 \tau R_{a b c d}
\]
\[+ \frac{eN^r A_r}{n-2}((u_a A_d + u_d A_a) g_{bc} + (u_c A_b + u_b A_c) g_{ad})
\]
\[- (u_a A_c + u_c A_a) g_{hd} - (u_b A_d + u_d A_b) g_{ac})
\]
\[+ \frac{2 e \tau}{n-2}(A_b A_c g_{hd} + A_a A_d g_{bc} - A_a A_c g_{hd} - A_b A_d g_{ac}).
\] (4.34)

We have now two possibilities: (a) \( \tau(p) = 0 \) and (b) \( \tau(p) \neq 0 \). (a) In this case (3.3), by (4.34), becomes
\[
\nabla_e C_{a b c d} = \varepsilon \beta u_e \left( g_{a d} L_{b c} + g_{b c} L_{a d} - g_{a c} L_{b d} - g_{b d} L_{a c}
\right.
\]
\[+ \left. (k(g_{a d} g_{bc} - g_{ac} g_{bd})) / ((n-2)(n-3)) \right),
\] (4.35)
where $L_{bc} = (eN^r A_r)(u_b A_c + u_c A_b)/(n - 2) - K_{bc}/(n - 3)$. From (4.35) it follows that $\nabla C = 0$ at $p$. (b) Using (4.34), we can present (3.3) in the form

$$\nabla_C A_{abcd} = \varepsilon u_e \left( \frac{k}{(n - 2)(n - 3)} (g_{ad} g_{bc} - g_{ac} g_{bd}) + 2r R_{abcd} \\
+ \frac{eN^r A_r}{n - 2} (g_{ad} (u_c A_b + u_b A_c) + g_{bc} (u_a A_d + u_d A_a) - g_{ac} (u_b A_d + u_d A_b) - \frac{1}{n - 3} (g_{ad} K_{bc} + g_{bc} K_{ad} - g_{ac} K_{bd} - g_{bd} K_{ac}) \\
+ \frac{2\varepsilon}{n - 2} (g_{ad} A_b A_c + g_{ac} A_b A_d - g_{ac} A_b A_d - g_{bd} A_a A_c) \right),$$

which can be rewritten in the following form

$$\nabla_C A_{abcd} = \varepsilon u_e \left( \frac{k - 2r K}{(n - 2)(n - 3)} (g_{ad} g_{bc} - g_{ac} g_{bd}) + 2r C_{abcd} + g_{ad} L_{bc} + g_{bc} L_{ad} - g_{ac} L_{bd} - g_{bd} L_{ac} \right) \quad (4.36)$$

where

$$L_{bc} = \frac{2r}{n - 3} S_{bc} - \frac{1}{n - 3} K_{bc} + \frac{eN^r A_r}{n - 2} (u_c A_b + u_b A_c) + \frac{2\varepsilon}{n - 2} A_b A_c.$$

But from (4.36) we obtain $\nabla_C A_{abcd} = 2\varepsilon u_e C_{abcd}$, which states that $C$ is recurrent. The last remark completes the proof.

Finally, combining Lemmas 5.6.5(ii) with Proposition 2 we immediately get the following theorem.

**Theorem 1.** Let $(M, g)$ be a hypercylinder in a parabolic essentially conformally symmetric manifold $(N, \tilde{g})$, $n \geq 5$ and let $\overline{U}$ be the subset of $N$ consisting of all points of $N$ at which the Ricci tensor $\tilde{S}$ of $(N, \tilde{g})$ is not recurrent. If $\overline{U} \cap M$ is a dense subset of $M$ then $(M, g)$ is a conformally recurrent manifold.

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