INDEPENDENT VERTEX SETS IN SOME COMPOUND GRAPHS

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Abstract. Let $G$ be an $n$-vertex graph and $R_1, R_2, \ldots, R_n$ distinct rooted graphs. The compound graph $G[R_1, R_2, \ldots, R_n]$ is obtained by identifying the root of $R_i$ with the $i$-th vertex of $G$, $i = 1, 2, \ldots, n$. We determine the number of independent vertex sets and the independence polynomial of $G[R_1, R_2, \ldots, R_n]$. Several special cases of these results are pointed out.

1. Introduction

Consider finite graphs without loops. If $G$ is such a graph, then $V(G)$ will denote its vertex set. Any subset of $V(G)$, such that no two elements of it are mutually adjacent, is called an independent vertex set of the graph $G$. Let $\text{Ind}(G)$ be the set of all independent vertex sets of $G$.

The number $\sigma(G)$ of independent vertex sets of the graph $G$, i.e. the cardinality of $\text{Ind}(G)$, has been examined in a number of recent papers [1–11]. In particular, Prodinger and Tichy [7, 11] called the quantity $\sigma(G)$ “the Fibonacci number of the graph $G$”. The motivation for this was the fact that if $P_n$ is the path-graph with $n$ vertices, then $\sigma(P_n)$ is equal to the $(n+1)$-th Fibonacci number.

In the present paper we determine the number of independent vertex sets of the compound graph $G[R_1, R_2, \ldots, R_n]$ constructed in the following manner.

Let $G$ be a graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let further $R_1, R_2, \ldots, R_n$ be distinct rooted graphs; the root of $R_i$ is denoted by $r_i$, $i = 1, 2, \ldots, n$. Then $G[R_1, R_2, \ldots, R_n]$ is the graph obtained by identifying the vertex $v_i$ of $G$ with the root $r_i$ of $R_i$, simultaneously for $i = 1, 2, \ldots, n$ (see Fig. 1).

Denote by $R_i^*$ the graph obtained by deleting from $R_i$ the root-vertex $r_i$ and the edges incident to it. Denote by $R_i^e$ the graph obtained by deleting from $R_i$ the root-vertex $r_i$, the vertices adjacent to $r_i$ and all the incident edges. Then the main result of our work can be formulated as follows.

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**Theorem 1.** Let $I$ be an independent vertex set of the graph $G$. Define

$$
\sigma_i(I) = \begin{cases} 
\sigma(R_i^i) & \text{if } v_i \notin I, \\
\sigma(R_i^*) & \text{if } v_i \in I.
\end{cases}
$$

Then

$$
\sigma(G[R_1,R_2,\ldots,R_n]) = \sum_{I \in \text{Ind}(G)} \prod_{i=1}^{n} \sigma_i(I).
$$

Instead of Theorem 1 we prove a somewhat stronger result, namely Theorem 2. In order to do this we need some preparations.

2. The independence polynomial

Denote by $n(G,k)$ the number of distinct $k$-element independent vertex sets of the graph $G$. Then the polynomial

$$
\omega(G) = \omega(G,x) = \sum_{k \geq 0} n(G,k) x^k
$$

is called the independence polynomial of the graph $G$ [3, 5, 6]. Evidently, $\omega(G,1) = \sigma(G)$.

The basic properties of the independence polynomial have been determined by Gutman and Harary [5] and recently by Hoede and Li [6].

Two of these properties will be needed in the subsequent considerations:

(a) If $v$ is a vertex of the graph $G$ and $N_v$ is the set containing $v$ and its first neighbors, then

$$
\omega(G) = \omega(G - v) + x \omega(G - N_v).
$$

(b) If $G_1 \cup G_2$ is the graph composed of components $G_1$ and $G_2$, then

$$
\omega(G_1 \cup G_2) = \omega(G_1) \omega(G_2).
$$
THEOREM 2. Let $I$ be an independent vertex set of the graph $G$. Define

$$
\omega_i(I) = \begin{cases} 
\omega(R_i^q) & \text{if } v_i \notin I, \\
\omega(R_i^*) & \text{if } v_i \in I.
\end{cases}
$$

Then

$$
\omega(G[R_1, R_2, \ldots, R_n], x) = \sum_{I \in \text{Ind}(G)} x^{|I|} \prod_{i=1}^{n} \omega_i(I)
$$

where $|I|$ stands for the number of elements of $I$.

Evidently, Theorem 1 is a special case of Theorem 2, obtained by setting $x = 1$ in formula (5).

3. Proof of Theorem 2

We demonstrate the validity of Theorem 2 by induction on the number $n$ of vertices of the graph $G$. If $n = 1$, then $V(G) = \{v_1\}$ and therefore $G[R_1, R_2, \ldots, R_n]$ coincides with $R_1$. On the other hand, for $n = 1$ the set $\text{Ind}(G)$ consists of only two elements, namely $I_1 = \emptyset$ and $I_2 = \{v_1\}$. Bearing in mind (4) we have $\omega_1(I_1) = \omega(R_1^q)$ and $\omega_1(I_2) = \omega(R_1^*)$. Consequently, the right-hand side of (5) is equal to $\omega(R_1^q) + x \omega(R_1^*)$. Because of (2) this latter expression is equal to $\omega(G)$.

Thus the statement of Theorem 2 is true for $n = 1$. In a similar manner one can check that Theorem 2 is satisfied for $n = 2$ and $n = 3$.

Assume now that Theorem 2 holds for all graphs $G$ with less than $n$ vertices. In order to accomplish the inductive proof we have to show that this assumption implies the validity of Theorem 2 for the graphs $G$ having $n$ vertices.

Suppose that $n \geq 3$ and apply formula (2) to the vertex $v_n$ of the graph $G$. Without loss of generality we may label the vertices of $G$ so that $v_n$ is adjacent to $v_{n-1}, \ldots, v_{n-d}$. Then by using (3),

$$
\omega(G[R_1, R_2, \ldots, R_n]) = \omega(R_n^*) \omega((G - v_n)[R_1, R_2, \ldots, R_n]) + \omega(R_n^q) \omega(R_{n-1}^*) \cdots \omega(R_{n-d}^q) \omega((G - N_{v_n})[R_1, R_2, \ldots, R_n]).
$$

The subgraphs $G - v_n$ and $G - N_{v_n}$ have $n - 1$ and $n - 1 - d$ vertices, respectively. Therefore according to the induction hypothesis:

$$
\omega((G - v_n)[R_1, R_2, \ldots, R_n]) = \sum_{I \in \text{Ind}(G - v_n)} x^{|I|} \prod_{i=1}^{n-1} \omega_i(I)
$$

$$
\omega((G - N_{v_n})[R_1, R_2, \ldots, R_n]) = \sum_{I \in \text{Ind}(G - N_{v_n})} x^{|I|} \prod_{i=1}^{n-d-1} \omega_i(I).
$$

The set $\text{Ind}(G)$ can be partitioned into two disjoint subsets $\text{Ind}^\circ(G)$ and $\text{Ind}^\ast(G)$, such that $\text{Ind}^\circ(G)$ is the set of independent vertex sets of $G$ which do not contain
the vertex \( v_n \), whereas \( \text{Ind}^\circ(G) \) is the set of those independent vertex sets of \( G \) which do contain \( v_n \). It is easy to see that

\[
\text{Ind}^\circ(G) = \text{Ind}(G - v_n) \tag{9}
\]

\[
\text{Ind}^\bullet(G) = \{ I \cup \{ v_n \} | I \in \text{Ind}(G - N_{v_n}) \}. \tag{10}
\]

Bearing in mind (7)-(10), the relation (6) is transformed into

\[
\omega(G[R_1, R_2, \ldots, R_n]) = \omega(R^n_1) \sum_{I \in \text{Ind}^\bullet(G)} x^{|I|} \prod_{i=1}^{n-1} \omega_i(I) \\
\quad + x \omega(R^n_1) \omega(R^n_{n-1}) \cdots \omega(R^n_{n-d}) \sum_{I \in \text{Ind}^\bullet(G)} x^{|I|-1} \prod_{i=1}^{n-d-1} \omega_i(I). \tag{11}
\]

For all \( I \in \text{Ind}^\circ(G) \), \( v_n \notin I \) and therefore \( \omega_n(I) = \omega(R^n_n) \). For similar reasons, the relations \( \omega_n(I) = \omega(R^n_n) \) and \( \omega_j(I) = \omega(R^n_j) \), \( j = 1, \ldots, d \), are satisfied for all \( I \in \text{Ind}^\bullet(G) \). Consequently, equation (11) becomes

\[
\omega(G[R_1, R_2, \ldots, R_n]) = \sum_{I \in \text{Ind}^\circ(G)} x^{|I|} \prod_{i=1}^{n} \omega_i(I) + \sum_{I \in \text{Ind}^\bullet(G)} x^{|I|} \prod_{i=1}^{n} \omega_i(I)
\]

and formula (5) follows from the fact that \( \text{Ind}^\circ(G) \cup \text{Ind}^\bullet(G) = \text{Ind}(G) \). This completes the proof of Theorem 2.

4. Special cases

4.1. All \( R_i \) are isomorphic. The graph \( G[R_1, R_2, \ldots, R_n] \) in which all \( R_i \), \( i = 1, 2, \ldots, n \) are isomorphic to the rooted graph \( R \) is denoted by \( G[R] \). For the compound graphs \( G[R] \) formula (5) is much simplified by the fact that the product \( \prod_{i=1}^{n} \omega_i(I) \) depends only on the cardinality \( k \) of the independent vertex set \( I \) and is equal to \( \omega(R - r)^{n-k} \omega(R - N_r)^k \) where \( r \) stands for the root of \( R \). Since the number of \( k \)-element independent vertex sets of the graph \( G \) is equal to \( n(G,k) \) we further have

\[
\omega(G[R]) = \sum_{k \geq 0} x^k n(G,k) \omega(R-r)^{n-k} \omega(R-N_r)^k. \tag{12}
\]

This, bearing in mind the definition of \( \omega(G) \), immediately leads to Corollary 2.1.

**Corollary 2.1.** \( \omega(G[R],x) = \omega(R-r,x)^n \omega(G,\omega(R-N_r))/\omega(R-r)) \).

**Corollary 2.2.** If \( k^* \) is the maximum cardinality of an independent vertex set of the graph \( G \), then the polynomial \( \omega(R-r)^{n-k^*} \) divides the polynomial \( \omega(G[R]) \).

4.2. The corona. The corona \( G \circ Q \) of the graphs \( G \) and \( Q \) is obtained from \( G \) and \( n \) copies of \( Q \), so that each vertex of \( G \) is joined to all vertices of a copy of \( Q \). Whence, \( G \circ Q \) is a special case of \( G[R] \) when the root \( r \) of \( R \) is adjacent to all other vertices of \( R \). In this notation, \( Q = R - r \).
COROLLARY 2.3. \( \omega(G \circ Q, x) = \omega(Q, x)^n \omega(G, 1/\omega(Q)) \).

COROLLARY 2.4. If \( k^* \) is the maximum cardinality of an independent vertex set of the graph \( G \), then the polynomial \( \omega(Q)^{n-k^*} \) divides the polynomial \( \omega(G \circ Q) \).

4.3. Some more special cases. If \( G \) is the complete graph \( K_n \), then then \( \text{Ind}(G) \) consists of \( n+1 \) elements: the empty set and \( n \) one-element sets, each containing one vertex of \( G \). Formula (5) gives then

\[
\omega(K_n[R_1, R_2, \ldots, R_n]) = x^0 \prod_{i=1}^{n} \omega_i(\emptyset) + \sum_{j=1}^{n} \prod_{i \neq j} \omega_i(\{v_j\}).
\]

Bearing in mind (4) we arrive at

COROLLARY 2.5. \( \omega(K_n[R_1, R_2, \ldots, R_n]) = \left[1 + x \sum_{j=1}^{n} \frac{\omega(R_j^*)}{\omega(R_j^0)} \right] \prod_{i=1}^{n} \omega(R_i^0). \)

It is easy to deduce combinatorial formulas for the \( n(G, k) \)-numbers of the path \( P_n \) and the circuit \( C_n \) [2, 5]. Then equations (12) and (2) lead to

COROLLARY 2.6.

\[
\omega(P_n(R)) = \sum_{k \geq 0} \binom{n-1-k}{k} \omega(R-r)^{n-k} [\omega(R) - \omega(R-r)]^k,
\]

\[
\omega(C_n(R)) = \sum_{k \geq 0} \frac{n}{n-k} \binom{n-1-k}{k} \omega(R-r)^{n-k} [\omega(R) - \omega(R-r)]^k.
\]

REFERENCES

[1] K. Engel, On the Fibonacci number of an \( M \times N \) lattice, Fibonacci Quart. 28 (1990), 72-78.