SOME ESTIMATES OF THE INTEGRAL $\int_0^{2\pi} \log |P(e^{i\theta})|(2\pi)^{-1} d\theta$

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Abstract. We investigate some estimates of the integral $\int_0^{2\pi} \log |P(e^{i\theta})|\frac{d\theta}{2\pi}$, if the polynomial $P(z)$ has a concentration at low degrees measured by the $l_p$-norm, $1 \leq p \leq 2$. We also obtain better estimates for some concentrations than those obtained in [1].

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial with complex coefficients and let $d$ be a real number such that $0 < d \leq 1$. We say that $P(z)$ has a concentration $d$ of degrees of at most $k$, measured by the $l_p$-norm $(p \geq 1)$, if

$$\left(\sum_{j \leq k} |a_j|^p\right)^{1/p} \geq d \left(\sum_{j \geq 0} |a_j|^p\right)^{1/p}.$$  \hspace{1cm} (1)

Polynomials with concentrations of low degrees were introduced by B. Beauzamy and P. Enflo; this plays an important role in the construction of an operator on a Banach space with no non-trivial invariant subspace. We investigate here the estimates of the integral $\int_0^{2\pi} \log |P(e^{i\theta})|\frac{d\theta}{2\pi}$ of such polynomials. In the following, we shall normalize $P(z)$ under the $l_p$-norm and also assume that

$$\left(\sum_{j \geq 0} |a_j|^p\right)^{1/p} = 1.$$ \hspace{1cm} (2)

Otherwise, the concentration of polynomials is measured by some of the well-known norms: $|P|_p (p \geq 1)$, $|P|_2 = \|P\|_2$, $|P|_\infty$, $|P|_{l_\infty}$, $\ldots$. For details see [1].

Similarly, as in [1, Lemme 3] (case $p = 2$) and [2, Theorem 1] (case $p = 1$) we have the following results:

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Theorem 1. Let \( P(z) = \sum_{j \geq 0} a_j z^j \) be a polynomial which satisfies (1) and (2). Then:

\[
\int_0^{2\pi} \log |P(e^{i\theta})| \frac{d\theta}{2\pi} \geq \sup_{1 < t \leq 3} f_{a,k}(t), \quad \text{where}
\]

\[
f_{a,k}(t) = \begin{cases} t \log \left( \frac{t-1}{t+1} \right)^{k+1} - \frac{1}{2} t^2, & 1 < p \leq 2 \\ t \log \left( \frac{t-1}{t+1} \right)^{k+1}, & p = 1 \end{cases}
\]

(see also [3, Lemma 3.2; p. 28, 29]).

Theorem 2. Let \( P(z) \) be a polynomial as in Theorem 1. Then:

\[
\int_0^{2\pi} \log |P(e^{i\theta})| \frac{d\theta}{2\pi} \geq \sup_{1 < t < +\infty} f_{d,k,p}(t), \quad \text{where}
\]

\[
f_{d,k,p}(t) = \begin{cases} \frac{t}{p} \log dp \left( \frac{t+1}{t-1} \right)^{p(k+1) - 1} - \frac{1}{2} t^2, & 1 < p \leq 2 \\ t \log \left( \frac{t+1}{t-1} \right)^{k+1}, & p = 1 \end{cases}
\]

(for the case \( p = 1 \) see [2, Theorem 1]).

For proofs of the Theorems 1 and 2 we use (as in [1, Lemme 3] and [2, Theorem 1] (see also [3])) the following well known facts

1° \( a_j = \int_0^{2\pi} P(re^{i\theta}) \frac{d\theta}{2\pi} \) if \( 0 < r < 1 \).

2° \( |a_j| \leq |P(z_0)| \frac{1}{r^j} \), where \( |P(z_0)| = \max_{|z| = r} |P(z)| \).

3° The classical Jensen’s inequality and the known transformation:

\[
\log |P(z_0)| \leq \int_0^{2\pi} \log \left| P \left( \frac{z_0 + e^{i\theta}}{1 + z_0 e^{i\theta}} \right) \right| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log |P(e^{i\theta})| \frac{1 - r^2}{|1 - z_0 e^{i\theta}|^2} \frac{d\theta}{2\pi},
\]

where \( |z_0| = r \).

4° If \( 0 < r < 1 \) then \( \frac{1 - r}{1 + r} = \frac{1 - r^2}{|1 - z_0 e^{i\theta}|^2} \leq \frac{1 + r}{1 - r} \).

5° \( \int_0^{2\pi} \log |P(e^{i\theta})| \frac{d\theta}{2\pi} = \int_{\log |P| < 0} + \int_{\log |P| > 0}, \) and

\[
\int_{\log |P| > 0} = \frac{1}{2} \int_{\log |P| > 0} \log |P|^2 \frac{d\theta}{2\pi}
\]

\[
= \frac{1}{2} |P|^2 \frac{d\theta}{2\pi} = \frac{1}{2} |P|^2 \frac{d\theta}{2\pi} = \frac{1}{2} |P|^2 = \frac{1}{2} |P|^2.
\]
because the $l_p$ norm decreases with $p$.

Finally, we get the functions $f_{d,k}(t)$ and $f_{d,k,p}(t)$ after the change of variables $t = (1 + r)/(1 - r)$.

Taking $t = 2$ and $1 < p \leq 2$, we have the Beuzamy-Enflo’s estimate from [1]:

$$\int_0^{2\pi} \log |P(e^{i\theta})| \frac{d\theta}{2\pi} \geq 2 \log \frac{d}{e^{\cdot 3k+1}}.$$

From the following proposition and Corollaries 1 and 3, it follows that this is not the best possible estimate.

**Proposition 1.** Let $P(z)$ be a polynomial as in Theorem 1. Then there exists a $t_k \in [1,3]$ such that

$$\int_0^{2\pi} \log |P(e^{i\theta})| \frac{d\theta}{2\pi} \geq f_{d,k}(t_k) \geq \begin{cases} 2 \log \frac{d}{e^{\cdot 3k+1}} : & 1 < p \leq 2; \\ 2 \log \frac{d}{3k+1} : & p = 1. \end{cases}$$

**Proof.** First observe that $\lim_{t \to 1^+} f_{d,k}(t) = -\infty$ and the function $f_{d,k}(t)$ has the form

$$f_{d,k}(t) = t \log d + t(k+1) \log(t-1) - t(k+1) \log(t+1), \quad 1 < p \leq 2.$$

We find derivatives:

$$f'_{d,k} = \log d + (k+1)(\log(t-1) - 1 - \log(t+1))$$

$$f''_{d,k} = 2(k+1) + 2(k+1) - 1 + t(k+1) \left(\frac{1}{(t+1)^2} - \frac{1}{(t-1)^2}\right)$$

$$f'''_{d,k} = 3(k+1) - 3(k+1) + 2t(k+1) \left(\frac{1}{(t+1)^3} - \frac{1}{(t-1)^3}\right).$$

It is clear that $\lim_{t \to 1^+} f''_{d,k} = -\infty$ and $f'''_{d,k}(3) < 0$. Since $f'''_{d,k}(t) > 0$, $t \in [1,3]$, it follows that $f''_{d,k}(t) < 0$, hence $f'_{d,k}(t)$ decreases. We also observe that $\lim_{t \to 1^+} f'_{d,k}(t) = +\infty$. Hence, there exists exactly one $t_k \in [1,3]$ such that $f_{d,k}(t_k) = 0$ or $f'_{d,k}(t) > 0$ for each $t \in [1,3]$. This proves the proposition. The case $p = 1$ can be treated similarly.

**Corollary 1.** Let $P(z)$ be a polynomial as in Theorem 1. Then for every $d \in [0,1]$ and $k \in \{0,1,2,3,4,5,6,7\}$ there exists a $t_k \in [1,2]$, such that

$$\int_0^{2\pi} \log |P(e^{i\theta})| \frac{d\theta}{2\pi} \geq f_{d,k}(t_k) > 2 \log \frac{d}{e^{\cdot 3k+1}}, \quad 1 < p \leq 2.$$

For the case $p = 1$ a similar result does not hold.
Proof. Since
\[ f'_{d,k}(3) = \frac{4}{3} k - \frac{2}{3} - (k + 1) \log 3 + \log d = 0.235k - 1.773 + \log d, \quad \log 3 = 1.098 \]
it follows that \( f'_{d,k}(2) < 0 \), for each \( d \in [0,1] \) and \( k \in \{0,1,\ldots,7\} \). Hence,
\[
\max_{1 < t \leq 3} f_{d,k}(2) > f_{d,k}(2) = 2 \log \frac{d}{e \cdot 3^{k+1}}.
\]
If \( p = 1 \), we have
\[
f'_{d,k}(2) = (4/3 - \log 3) k + (4/3 - \log 3) + \log d = 0.235k + 0.235 + \log d \geq 0.
\]

**Corollary 2.** Let \( P(z) \) be a polynomial as in Theorem 1. Then for every \( d \in [0,1] \) and \( k > 7 \) for which \( \log(3k+1/d) \) is a rational number, there exists a \( t_k \in [1,3], t_k \neq 2 \), such that
\[
\int_0^{2\pi} \log |P(e^{i\theta})| \frac{d\theta}{2\pi} \geq f_{d,k}(t_k) > f_{d,k}(2), \quad 1 \leq p \leq 2.
\]

**Proof.** In both cases \((1 < p \leq 2, p = 1)\) we have that \( f'_{d,k}(2) = 0 \) iff \( 4/3 - 2/3 = \log 3^{k+1} / d \), that is \( 4/3 k + 4/3 = \log 3^{k+1} / d \).

**Corollary 3.** Let \( P(z) \) be a polynomial as in Theorem 1. Then for every \( d \in [0,1] \) there exists a \( k_1 \in \mathbb{N} \) such that for \( k > k_1 \):
\[
\int_0^{2\pi} \log |P(e^{i\theta})| \frac{d\theta}{2\pi} \geq f_{d,k}(3) = \begin{cases} 
\frac{3 \log d}{e^{3/2} (2k+1)}, & 1 < p \leq 2 \\
\frac{3 \log d}{2k+1}, & p = 1 \\
\frac{2 \log d}{3k+1}, & 1 < p \leq 2 \\
\frac{2 \log d}{3k+1}, & p = 1.
\end{cases}
\]

**Proof.** Since
\[
f'_{d,k}(3) = \frac{3}{4} k - \frac{9}{4} - (k + 1) \log 2 + \log d = 0.057k - 2.943 + \log d,
\]
we have that \( \max_{1 < t \leq 3} f_{d,k}(t) = f_{d,k}(3) \) \((1 < p \leq 2)\) iff \( f'_{d,k}(3) \geq 0 \). Hence, it follows that
\[
k_1 = \left\lfloor \frac{(9/4) + \log 2 - \log d}{(3/4) - \log 2} \right\rfloor = [51.634 - 17.543 \log d].
\]
Similarly, for \( p = 1 \) there exists the corresponding number \( k_1 \).
Corollary 4. Let $P(z)$ be a polynomial as in Theorem 1. Then, for every $d \in [0, 1]$ and $k \in \{0, 1, 2, \ldots, 51\}$, there exists a $t_k \in [1, 3]$, such that
\[
\int_0^{2\pi} \log |P(e^{i\theta})| \frac{d\theta}{2\pi} \geq f_{d,k}(t_k) > f_{d,k}(3) = 3 \log \frac{d}{e^{2/2k+1}}, \quad 1 < p \leq 2.
\]

Proof. This is clear from the equality
\[
f'_{d,k}(3) = \frac{3}{4} k - \frac{9}{4} = (k + 1) \log 2 = 0.057 k - 2.943 + \log d, \quad 1 < p \leq 2.
\]

Since for $p = 1$ we have that $f'_{d,k}(3) = 0.057 k + 0.057 + \log d$, it follows that the conclusion is not the same as in the case $1 < p \leq 2$.

We shall now analyse the estimate of the integral \(\int_0^{2\pi} \log |P(e^{i\theta})| \frac{d\theta}{2\pi}\) with the function $f_{d,k,p}(t)$ as in Theorem 2. The following results can be compared with [2, Th. 2, Lemmas 3 and 4]. Firstly, we represent $f_{d,k,p}(t)$ in the form:

\[
f_{d,k,p} = h_{d,p}(t) + g_k(t) - \frac{1}{p} \cdot t \cdot \log \left[1 - \left(\frac{t - 1}{t + 1}\right)^{\frac{1}{p(k+1)}}\right],
\]

where (see [2])

\[
h_{d,p} = t \log d - \frac{1}{2} t^2 + \frac{t}{p} \log [(t+1)^p - (t-1)^p]
\]

\[
g_k(t) = kt \log (t-1) - (k+1)t \log (t+1).
\]

It is clear that $f_{d,k,p}(t) > h_{d,p}(t) + g_k(t)$, $t > 1$. We shall now prove the following.

Proposition 2. The function $h_{d,p}(t) + g_k(t)$ takes its maximum value at a point (unique) $t_k$ such that $t_k \to +\infty$, when $k \to +\infty$.

Proof. We essentially use the same argument as in [2]. From [2] it follows that $g_k'(t) < 0$, $t > 1$. Now, we find derivatives for $h_{d,p}(t)$

\[
h'_{d,p}(t) = \log d - t - \frac{1}{p} \log [(t+1)^p - (t-1)^p] + t \cdot \frac{(t+1)^{p-1} - (t-1)^{p-1}}{(t+1)^p - (t-1)^p}.
\]

\[
h''_{d,p}(t) = -1 + 2 \cdot \frac{(t+1)^{p-1} - (t-1)^{p-1}}{(t+1)^p - (t-1)^p} - \frac{t(p-1)}{A^2(t)} \left([(t+1)^{p-2} - (t-1)^{p-2}] A(t) - \frac{t[(t+1)^{p-1} - (t-1)^{p-1}]}{A(t)} \right),
\]

where $A(t) = (t+1)^p - (t-1)^p$.

Since $p \in [1, 2]$, $t > 1$, it is clear that

\[
h''_{d,p}(t) < 0 \quad \text{iff} \quad -1 + 2 \cdot \frac{(t+1)^{p-1} - (t-1)^{p-1}}{A(t)} < 0.
\]
But, this is true iff \( \varphi_p(t) < 0 \), where
\[
\varphi_p(t) = 2(t + 1)^{p-1} - 2(t - 1)^{p-1} - (t + 1)^p + (t - 1)^p.
\]
Hence, we find that
\[
\varphi_p'(t) = 2(p - 1)(t + 1)^{p-2} - 2(p - 1)(t - 1)^{p-2} + p[(t - 1)^{p-1} - (t + 1)^{p-1}] < 0.
\]
This shows that \( h_{d,p}'(t) + g_k''(t) < 0 \). Since
\[
\lim_{t \to +1} \left( h_{d,p}'(t) + g_k'(t) \right) = +\infty \quad \text{and} \quad \lim_{t \to +1} \left( h_{d,p}'(t) + g_k'(t) \right) = -\infty,
\]
equation \( h_{d,p}'(t) + g_k'(t) = 0 \) has exactly one solution \( t_k \). From the equality \( h_{d,p}'(t) + g_k'(t) = 0 \) we get with \( t = t_k \),
\[
k = \frac{(t^2 - 1) \log(t + 1) + \log \left( \frac{t}{(t - 1)^{2k} - 1} \right) - \frac{1}{2} t^2}{2t + (t^2 - 1) \log(t - 1) - (t^2 - 1) \log(t + 1)},
\]
wherefrom we easily deduce that \( t_k \to +\infty \).

**Remark 1.** From the Proposition 1 it follows that the function \( f_{d,k,p}(t) \) \((1 < p \leq 2)\) has the same behaviour as the function \( f_{d,k}(t) \) from [2]. If \( p = 2 \) we get
\[
f_{d,k,2}(t) = t \log \frac{2d}{t - 1} \sqrt{\frac{t}{((t + 1)/(t - 1))^{2k+2} - 1} - \frac{1}{2} t^2},
\]
which is the answer to the remark from [2, p. 223].

For the function \( f_{d,k,2}(t) \) we have the following results

**Proposition 3.** Let \( f_{d,k,2}(t) \) be the function from Theorem 2 \((p = 2)\). Then, when \( k \to +\infty \)

1° \( \frac{4}{3} t_k \to 1 \);

2° \( t_k \log \left( 1 - \frac{t_k - 1}{t_k + 1} \right)^{2(k+1)} \to 0 \);

3° \( f_{d,k,2}(t_k) \) and \( h_{d,2}(t_k) + g_k(t_k) \) are asymptotically equivalent.

Namely, \( f_{d,k,2}(t) = t \log d - \frac{1}{2} t^2 + \frac{t}{2} \log 4t + g_k(t) \), where \( g_k(t) \) is same as in [2]. The proof is similar as in [2], i.e. it uses the Taylor expansion of \( \log(1 \pm x) \), \( x \to 0 \).

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**References**


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