EMBEDDING DERIVATIVES OF M-HARMONIC TENT SPACES INTO LEBESGUE SPACES

Miroljub Jevtić

Abstract. A characterization is given of those measures $\mu$ on $B$, the open unit ball in $\mathbb{C}^n$, such that differentiation of order $m$ maps the $M$-harmonic tent space $\mathcal{H}^p$ boundedly into $L^q(\mu)$, $0 < p < q < \infty$.

1. Introduction. Let $B$ be the open unit ball in $\mathbb{C}^n$ with (normalized) volume measure $\nu$ and let $S$ denote its boundary. For the most part we will follow the notation and terminology of Rudin [5]. If $\alpha > 1$ and $\xi \in S$ the Koranyi approach regions are defined by

$$D_\alpha(\xi) = \{z \in B : |1 - \langle z, \xi \rangle| < \frac{1}{2}\alpha (1 - |z|^2)\}.$$

For any function $f$ on $B$, we define a scale of maximal functions by

$$M_\alpha f(\xi) = \sup\{|f(z)| : z \in D_\alpha(\xi)\}.$$

For simplicity of notation, we write simply $D(\xi)$ for $D_2(\xi)$ and $Mf$ for $M_2 f$. For $0 < p < \infty$, the tent space $T^p = T^p(B)$ is defined to be the space of all continuous functions $f$ on $B$ such that $Mf \in L^p(\sigma)$. Here $\sigma$ denotes the rotation invariant probability measure on $S$. We note that the use of approach regions of “aperture” 2 in the definition of $T^p$ is merely a convenience: approach regions of any other aperture would yield the same class of functions with an equivalent norm.

Let $\Delta$ be the invariant Laplacian on $B$. That is, $(\Delta f)(z) = \Delta(f \circ \varphi_z)(0)$, $f \in C^2(B)$, where $\Delta$ is the ordinary Laplacian and $\varphi_z$ the standard automorphism of $B$ ($\varphi_z \in \text{Aut}(B)$) taking 0 to $z$ (see [5]). A function $f$ defined on $B$ is $M$-harmonic, $f \in \mathcal{M}$, if $\Delta f = 0$.

We shall call $\mathcal{H}^p = \mathcal{M} \cap T^p$ $M$-harmonic tent space.

For $f \in \mathcal{M}$ let

$$\partial f(z) = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}, \frac{\partial f}{\partial \bar{z}_1}, \ldots, \frac{\partial f}{\partial \bar{z}_n}\right)$$

AMS Subject Classification (1991): Primary 32A35
and for any positive integer $m$ we write
\[ \vartheta^m f(z) = (\vartheta^\alpha \vartheta^\beta f(z))_{|\alpha|+|\beta|=m} \quad \text{and} \quad |\vartheta^m f(z)|^2 = \sum_{|\alpha|+|\beta|=m} |\vartheta^\alpha \vartheta^\beta f(z)|^2, \]
where $\vartheta^\alpha \vartheta^\beta f(z) = \frac{\partial^{|\alpha|+|\beta|} f(z)}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n} \partial \bar{z}_1^{\beta_1} \cdots \partial \bar{z}_n^{\beta_n}}$, $\alpha$ and $\beta$ are multi-indices.

Let $\mu$ be a positive measure on $B$ and consider the problem of determining what conditions on $\mu$ imply $|\vartheta^\beta f| \in L^p(\mu)$, whenever $f \in H^p$. A standard application of the closed graph theorem leads to the following equivalent problem: Characterize such $\mu$ for which there exists a constant $C$ satisfying
\[ \left( \int_B |\vartheta^\beta f|^q \, d\mu \right)^{1/q} \leq C \left( \int_S |Mf|^p \, d\sigma \right)^{1/p} = C \|f\|_{H^p}. \]

The purpose of this paper is to present a solution of this problem in the case $0 < p < q < \infty$. To state it we need some more notations.

For $\xi \in S$ and $\delta > 0$ the following nonisotropic balls are defined
\[ B(\xi, \delta) = \{z \in B : |1 - \langle z, \xi \rangle| < \delta\}, \quad Q(\xi, \delta) = \{\eta \in S : |1 - \langle \eta, \xi \rangle| < \delta\}. \]
For each $E \subset S$ we define the $\alpha$-tent over $E$ by $T_\alpha(E) = \left( \bigcup \left\{ D_\alpha(\xi) : \xi \in E \right\} \right)^c$, the complement being taken in $B$. We write $T(E)$ for $T_2(E)$. For $z \in B$ and $r$, $0 < r < 1$, $E_r(z) = \{w \in B : |\varphi_z(w)| < r\}$, we will let $|E_r(z)| = \nu(E_r(z))$.

Throughout the paper $r$ will be fixed and we will occasionally write $E(z)$ instead of $E_r(z)$. Constants will be denoted by $C$ which may indicate a different constant from one occurrence to the next.

**Theorem.** Let $0 < p < q < \infty$. For a positive measure $\mu$ on $B$ and a positive integer $m$, necessary and sufficient condition for
\[ \left( \int_B |\vartheta^m f|^q \, d\mu \right)^{1/q} \leq C \|f\|_{H^p} \]
is that there exists a constant $C$ for which
\[ \mu(E(z)) \leq C(1 - |z|)^{mq/p+mq}, \quad z \in B. \]

For holomorphic functions Theorem was proved by Shirokov and Luecking (see [3, 4, 6, 7]).

**2. Proof of Theorem.** The following two preliminary lemmas will be needed in the proof of Theorem.
LEMMA 2.1 [2]. Let $k \geq m$ be non-negative integers, $0 < p < \infty$ and $0 < r < 1$. There exists a constant $C = C(k,m,p,r,n)$ such that if $f \in \mathcal{M}$ then

$$|\partial^k f(w)|^p \leq C(1 - |w|)^{(m-k)p} \int_{E_f(w)} |\partial^m f(z)|^p (1 - |z|)^{-n-1} \, dv(z), \quad \text{for all } w \in B.$$

LEMMA 2.2. Let $1 < \alpha < \infty$, $0 < p < \infty$ and let $\mu$ be a finite Borel measure in $B$. In order that there exist a constant $C$ such that

$$\int_B |f(z)|^p \, d\mu(z) \leq C \|f\|_{\mathcal{M}_p}^p, \quad \text{for all } f \in \mathcal{M}_p,$$

it is necessary and sufficient that there exists a constant $C$ for which

$$\mu(B(\xi, \delta)) \leq C \delta^{n\alpha}, \quad \xi \in C, \ \delta > 0.$$

Proof. The necessity of the condition (2.2) follows upon applying the inequality (2.1) to appropriate $f$ (see [1]).

It is easy to see that the condition (2.2) is equivalent to

$$\mu(T(Q(\xi, \delta))) \leq C \delta^{n\alpha},$$

for some constant $C$ and for all $\xi \in \mathcal{S}$, $\delta > 0$. The sufficiency can be gotten by the following argument. For $\lambda > 0$, let $E_\lambda = \{\xi \in \mathcal{S} : Mf(\xi) > \lambda\}$. By Whitney decomposition theorem there is a family $\mathcal{B}$ of sets $Q = Q(\xi, \delta)$ such that $E_\lambda = \bigcup Q \in \mathcal{B}$ and a countable disjoint subfamily \{\{Q_n = Q_n(\xi_n, \delta_n)\} of $\mathcal{B}$ such that each $Q$ in $\mathcal{B}$ is in some $\alpha Q_n = Q_n(\xi_n, \alpha \delta_n)$. Then \{z \in B : |f(z)| > \lambda\} $\subset \bigcup T(cQ_n)$, so

$$\mu(\{|f| > \lambda\}) \leq \sum_n \mu(T(cQ_n)) \leq C \sum_n [\sigma(cQ_n)]^\alpha \leq C \left( \sum_n [\sigma(Q_n)]^\alpha \right) \leq C(\sigma(E_\lambda))^\alpha.$$

Integrating this inequality with respect to $\lambda^{p-1} \, d\lambda$ we get

$$\int_B |f|^{p\alpha} \, d\mu = p \int_0^\infty \lambda^{p-1} \mu(\{|f| > \lambda\}) \, d\lambda \leq C \sum_{n=-\infty}^\infty 2^{kp} \mu(\{|f| > \lambda\}) \leq C \left( \sum_{k=-\infty}^\infty 2^{kp} \mu(\{|f| > \lambda\}) \right)^{1/\alpha} \leq C \left( \sum_{k=-\infty}^\infty 2^{kp} \sigma(\{|Mf| > \lambda\}) \right)^{\alpha} \leq C \|Mf\|_{\mathcal{M}_p}^{p\alpha}.$$

Proof of Theorem. The necessity of the condition (1.2) follows from the Shirokov-Luecking theorem mentioned above.
The sufficiency is obtained by the same arguments as in [4]:

\[(1 - |z|)^{mq}|\partial^m f(z)|^q \leq C \int_{E(z)} |f(w)|^q (1 - |w|)^{-n-1} d\nu(w),\]

by Lemma 2.1. Integrating both sides with respect to \((1 - |z|)^{-mq} d\mu(z)\) and using Fubini's theorem on the right, we obtain

\[\int_B |\partial^m f(z)|^q d\mu(z) \leq C \int_B |f(w)|^q \mu(E(w))(1 - |w|)^{-m_q - n - 1} d\nu(w).\]

By Lemma 2.2 and Theorem 2 of [1], we conclude that

\[\int_B |\partial^m f(z)|^q d\mu(z) \leq C\|f\|_{H^p}^q.\]

REFERENCES