SOME COMMUTATIVITY THEOREMS FOR s-UNITAL RINGS WITH CONSTRAINTS ON COMMUTATORS

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Abstract. Continuing the investigation of [1], [2], [3] and [10], we prove here some commutativity theorems for s-unital rings \( R \) satisfying the polynomial identity \( x^ny^m y^t' = \pm x^t' [x, y^n]y^s' \), resp. \( x^t[x, y^n]y^s' = \pm y^s'[x, y^m]x^t' \), where \( m, n, s, s', t \) and \( t' \) are given non-negative integers such that \( m > 0 \) or \( n > 0 \) and \( t + n = s' + 1 \) or \( m + s = t' + 1 \) for \( m = n \). The additional assumption in these theorems concern some torsion freeness of commutators in \( R \).

1. Introduction. Throughout this paper \( R \) will be an associative ring (may be without identity 1), \( Z(R) \) will represent the center of \( R \), \( N(R) \) the set of all nilpotent elements of \( R \), and \( C(R) \) the commutator ideal of \( R \). By \( R' \) we denote the opposite ring of \( R \), i.e. the ring with the same elements and addition as \( R \), but with opposite multiplication \( \circ \) defined by \( x \circ y = yx \) for all \( x, y \) in \( R' \). We will omit the sign \( \circ \) of the multiplication in \( R' \), as it is usual for the sign \( \circ \) of the multiplication in \( R \).

A ring \( R \) is called left, resp. right s-unital if \( x \in Rx \), resp. \( x \in xR \) for all \( x \) in \( R \). If \( R \) is both left and right s-unital, then \( R \) is said to be s-unital. If \( R \) is s-unital (resp. left or right s-unital), then, for every finite subset \( F \) of \( R \), there exists an element \( e \) in \( R \) such that \( ex = xe = x \) (resp. \( ex = x \) or \( xe = x \) for all \( x \) in \( F \).

By \( [x, y] \) we denote the commutator \( xy - yx \) of two elements \( x, y \) in a ring \( R \). If \( n \) is a positive integer, then we say for \( R \) to have the property \( Q(n) \) if commutators in \( R \) are \( n \)-torsion free, i.e. if \( n[x, y] = 0 \) implies \( [x, y] = 0 \) for all \( x, y \) in \( R \). Obviously, any \( n \)-torsion free ring \( R \) has the property \( Q(n) \), and if a ring \( R \) has the property \( Q(n) \), then \( R \) has also the property \( Q(m) \) for all divisors \( m \) of \( n \). It is clear that \( R \) is left, resp. right s-unital if and only if \( R' \) is right, resp. left s-unital, and that, for any positive integer \( n \), \( R \) has the property \( Q(n) \) if and only if \( R' \) has this property.

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We investigate here the commutativity of a ring $R$ which satisfies the polynomial identity
\[ x^t[x^n, y]y' = \pm x^{s'}[x, y^m]y^s \quad \text{for all } x, y \text{ in } R, \quad (1) \]
resp.
\[ x^t[x^n, y]y' = \pm y^s[x, y^m]x^{s'} \quad \text{for all } x, y \text{ in } R. \quad (1') \]
For $t' = s' = 0$, the identity $(1)$, resp. $(1')$ becomes
\[ x^t[x^n, y] = \pm [x, y^m]y^s \quad \text{for all } x, y \text{ in } R, \quad (2) \]
resp.
\[ x^t[x^n, y] = \pm y^s[x, y^m] \quad \text{for all } x, y \text{ in } R. \quad (2') \]
If an identity with the sign $\pm$ occurring in it is denoted by $(k)$, then we denote by $(k_+)$, resp. $(k_-)$ this identity with the sign $+$, resp. $-$ instead of $\pm$.

In [10] Psomopoulos proved the following result.

**Theorem P** [10, Theorem 1 and Theorem 2]. Let $R$ be a ring with identity 1 satisfying the polynomial identity $(2_+)$ for some positive integers $m, n$ and some non-negative integers $s, t$. If $n > 1$ and $R$ is $n$-torsion free, then $R$ is commutative. Also, if $m, n$ are relatively prime, then $R$ is commutative.

For $s = t' = 0$, the identities $(1)$ and $(1')$ reduce to
\[ x^t[x^n, y] = \pm x^{s'}[x, y^m] \quad \text{for all } x, y \text{ in } R \quad (3) \]
and
\[ x^t[x^n, y] = \pm [x, y^m]x^{s'} \quad \text{for all } x, y \text{ in } R, \quad (3') \]
respectively. The commutativity of a left or right $s$-unital ring $R$ satisfying $(3)$ or $(3')$ has been investigated in [1]. Especially was proved

**Theorem AP** [1, Theorem 1]. Let $R$ be a left or right $s$-unital ring with polynomial identity $(3)$ or $(3')$. If $m > 1, n > 1$, and $R$ has the property $Q(m)$ for $n > 1$, then $R$ is commutative.

If $s = s' = 0$, the identities $(1)$ and $(1')$ reduce to the identity
\[ y^s[x^n, y]y' = \pm [x, y^m] \quad \text{for all } x, y \text{ in } R \quad (4) \]
considered in [2]. For $s = t = 0$, $(1)$ and $(1')$ become
\[ [x^n, y]y' = \pm x^{s'}[x, y^m] \quad \text{for all } x, y \text{ in } R \quad (5) \]
and
\[ [x^n, y]y' = \pm [x, y^m]x^{s'} \quad \text{for all } x, y \text{ in } R, \quad (5') \]
respectively. Passing to the opposite ring $R'$, the identities $(5)$ and $(5')$ can be rewritten in the form
\[ x^t[x^n, y] = \pm [x, y^m]x^{s'} \quad \text{for all } x, y \text{ in } R' \quad (6) \]
and
\[ y^{t'}[x^n, y] = \pm x^{s'}[x, y^m] \quad \text{for all } x, y \text{ in } R', \] (6')
respectively. For \( R \) instead of \( R' \), the last two identities were considered in [3].

For \( m = n = 0 \), any ring \( R \) satisfies both (1) and (1'). If
\[ [[x, y], x] = 0 \quad \text{for all } x, y \text{ in } R, \] (7)
especially, if all commutators in \( R \) are central, then the identities (1) and (1') can be rewritten in the form
\[ nx^{n+t-1}[x, y]y^{t'} = \pm mx^{s'}[x, y]y^{m+s-1} \quad \text{for all } x, y \text{ in } R. \] (8)

Thus, for \( m = n, m + s = t' + 1 \) and \( n + t = s' + 1 \), any ring \( R \) satisfying (7), especially any ring \( R \) with central commutators, satisfies both (1) and (1'). Therefore, for non-negative numbers in the identities (1) and (1') we all along assume that \( m > 0 \) or \( n > 0 \), and \( m \neq n \) if \( n + t - 1 = s' \) and \( m + s - 1 = t' \).

2. First we observe that under an additional assumption the integers \( m \) and \( n \) in Theorem P, can be interchanged. In fact, the theorem can be improved as follows:

\textbf{Theorem 1.} Let \( R \) be a ring satisfying (2) or (2') for \( m \geq 1, n \geq 1 \), and having the property \( Q(d) \) for \( d = (m, n) \). If, moreover, \( R \) is left or right \( s \)-unital for \( m + s > 1 \) and \( n + t > 1 \), then \( R \) is commutative.

\textit{Proof.} By an argument used in the proof of [1, Lemma 4], we can prove that, form \( m + s > 1 \) and \( n + t > 1 \), the ring \( R \) is \( s \)-unital. Hence, for this case, we can assume that \( R \) is a ring with identity 1 (see [7, Proposition 1]).

If \( n = 1 \) and \( t = 0 \), then \( R \) is commutative by a special version of [11, Hauptsatz 3] stated in [1] which will be cited here as Theorem S.

If \( n = 1 \) and \( t > 0 \), then we set in (2), resp. (2'), \( x + 1 \) for \( x \) and combine the identity obtained with (2), resp., (2') to get \( ((x + 1)^t - x^t)[x, y] = 0 \) for all \( x, y \) in \( R \). For \( t = 1 \) this means that \( [x, y] = 0 \) for all \( x, y \) in \( R \), and thus, \( R \) is commutative. If \( t > 1 \), then the last identity yields \([x, y] = f(x)[x, y]\) for all \( x, y \) in \( R \), where \( f(X) \in \mathbb{Z}[X] \) is a polynomial all monomials of which are of degree at least one. Hence, \( R \) is commutative by Theorem S.

Similarly, we can prove that \( R \) is commutative for \( m = 1 \).

Now, we suppose that \( m > 1 \) and \( n > 1 \). The proof, we give here for the sake of completeness, differs from the proof of Theorem P only in the final phase where we use Theorem S. To prove that \( C(R) \subseteq N(R) \), by [8, Theorem 1], it suffices to take
\[ x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{in } \mathbb{Z}^{2 \times 2} \]
for the case of the identity (2). In the case of the identity (2'), one should take \([1, 0] \) instead of \([0, 0] \).
Next we prove that $N(R) \subseteq Z(R)$. Let $a$ be an arbitrary element in $N(R)$. Then there exists a positive integer $p$ such that
\[ a^k \in Z(R) \quad \text{for all integers } k \geq p, \text{ } p \text{ minimal.} \quad (9) \]

If $p = 1$, then $a \in Z(R)$. Suppose that $p > 1$. We set $b = a^{p-1}$ to get a contradiction. Obviously,
\[ [b^k, x] = b^k [b, x] = [b, x] b^k = 0 \quad \text{for all } x \text{ in } R \text{ and all integers } k > 1. \quad (10) \]

In view of (10), the identity (2), resp. (2') yields
\[ x^t [x^n, b] = 0 \quad \text{for all } x \text{ in } R. \quad (11) \]

Therefore, setting $1 + b$ for $y$ in (2), resp. (2'), one gets, in account of (10) and (11),
\[ m[x, b](1 + sb) = 0 \quad \text{for all } x \text{ in } R, \quad \text{resp.} \quad m(1 + sb)[x, b] = 0 \quad \text{for all } x \text{ in } R. \]

Hence, by (10), $m[x, b]b = 0$, resp. $mb[x, b] = 0$ for all $x$ in $R$, and thus,
\[ m[x, b] = 0 \quad \text{for all } x \text{ in } R. \quad (12) \]

Similarly, from (2), resp. (2') one gets
\[ [b, y^m] y^n = 0 \quad \text{for all } y \text{ in } R, \quad (13) \]
resp.
\[ y^m [b, y^n] = 0 \quad \text{for all } y \text{ in } R. \quad (13') \]

By (13), resp. (13'), from (2), resp. (2'), we easily get
\[ n(1 + tb)[b, y] = 0 \quad \text{for all } y \text{ in } R, \quad \text{resp.} \quad n[b, y](1 + tb) = 0 \quad \text{for all } y \text{ in } R, \]

hence, by (10), $nb[b, y] = 0$ resp. $n[b, y]b = 0$ for all $y$ in $R$, and thus
\[ n[x, b] = 0 \quad \text{for all } x \text{ in } R. \quad (14) \]

Since $R$ has the property $Q(d)$ for $d = (m, n)$, then (12) and (14) imply
\[ [x, b] = 0 \quad \text{for all } x \text{ in } R, \quad \text{i.e.} \quad a^{p-1} \in Z(R), \quad (15) \]
which is a contradiction. Thus, we proved that
\[ C(R) \subseteq N(R) \subseteq Z(R). \quad (16) \]

In view of (16) and [9, Lemma 3], the identities (2) and (2') can be rewritten in the form
\[ n x^{n+t-1} [x, y] = \pm m[x, y] y^{m+s-1} \quad \text{for all } x, y \text{ in } R. \quad (17) \]
Now, setting $x + 1$ for $x$ in (17) and combining the identity obtained with (17), one gets
\[ n((x + 1)^{m+s-1} - x^{m+s-1})[x,y] = 0 \quad \text{for all} \ x, y \in R. \quad (18) \]

Similarly, from (17), interchanging $x$ and $y$, and taking in account (16), one derives
\[ m((x + 1)^{m+s-1} - x^{m+s-1})[x,y] = 0 \quad \text{for all} \ x, y \in R. \quad (19) \]

For $m = dm_1$, $n = dn_1$, the integers $m_1, n_1$ are relatively prime, and by $Q(d)$, (18) and (19) imply
\[ n_1[x,y] = f(x)[x,y] \quad \text{for all} \ x, y \in R \quad (20) \]
and
\[ m_1[x,y] = g(x)[x,y] \quad \text{for all} \ x, y \in R, \quad (21) \]
where $f(X), g(X)$ are polynomials in $Z[X]$ all monomials of which have degree at least one. Since $m_1, n_1$ are relatively prime, then from (20) and (21), for some integers $m_2, n_2$, it follows
\[ [x,y] = (n_2 f(x) + m_2 g(y))[x,y] \quad \text{for all} \ x, y \in R. \]

Hence, $R$ is commutative by Theorem 5.

In [6, Theorem 8] Harmanci showed that “If $n > 1$ and $R$ is a ring with 1 which satisfies the identities $[x^n, y] = [x, y^n]$ and $[x^{n+1}, y] = [x, y^{n+1}]$ for all $x, y \in R$, then $R$ must be commutative”. Bell [4, Theorem 6] extended this result to any pair of relatively prime integers $m$ and $n$ instead of $n$ and $n + 1$. The following result, generalizing Bell’s result, was proved in [1] as Theorem 8.

**Theorem 2.** Let $m > 1$ and $n > 1$ be fixed relatively prime integers, $m' \geq 1$, $n' \geq 1$, and $r, s$ and $t$ be given non-negative integers. If $R$ is an $s$-unital (resp. left or right $s$-unital) ring satisfying both identities
\[ x^t[x^{m'}, y] = \pm y^r[x, y^m]x^s \quad \text{and} \quad x^t[x^{m'}, y] = \pm y^r[x, y^m]x^a \quad \text{for all} \ x, y \in R, \]
or
\[ x^t[x^{m'}, y] = \pm x^r[x, y^m]y^s \quad \text{and} \quad x^t[x^{m'}, y] = \pm x^r[x, y^m]y^s \quad \text{for all} \ x, y \in R, \]
(if $r = 0$), then $R$ is commutative.

Now, we prove the following similar result generalizing also Bell’s result.

**Theorem 3.** Let $m, n, m', n'$ be fixed positive, and $s, s', t$ fixed non-negative integers. Further, let $R$ be a ring satisfying both identities
\[ x^t[x^{m'}, y] = \pm x^{s'}[x, y^m]x^s \quad \text{and} \quad x^t[x^{m'}, y] = \pm x^{s'}[x, y^m]x^a \quad \text{for all} \ x, y \in R \quad (22) \]
or
\[ x^t[x^{m'}, y] = \pm y^{s'}[x, y^m]x^s \quad \text{and} \quad x^t[x^{m'}, y] = \pm y^{s'}[x, y^m]x^s \quad \text{for all} \ x, y \in R. \quad (22') \]

If, moreover, $R$ is $s$-unital (resp. left or right $s$-unital for $s' = 0$), and has the property $Q(d)$, where $d = (m, n)$ (resp. $d = (m, n, m', n')$, for $s' = 0$), then $R$ is commutative.
Proof. Actually, $R$ is $s$-unital, and thus by [7, Proposition 1], we can assume that $R$ is a ring with identity 1.

For $m = 1$ or $n = 1$ (resp. $m' = 1$ or $n' = 1$ if $s' = 0$), we can see, as in the proof of Theorem 1 (using [5, Lemma] for $s' > 0$), that $R$ is commutative. For $m > 1$ and $n > 1$ (resp. $m' > 1$ and $n' > 1$ for $s' = 0$), instead of (12), (resp. (14)), we get now (12) and (14) (resp.
\[ m'[x, b] = 0 \quad \text{and} \quad n'[x, b] = 0 \quad \text{for all} \ x \in R. \]  
By the property $Q(d)$ this implies (15). Similarly (for $s' = 0$), instead of (21) (resp. (20)), we have (20) and (21) (and also
\[ m'_i[x, y] = f'(x)[x, y] \quad \text{for all} \ x, y \ in \ R, \]  
\[ n'_i[x, y] = g'(x)[x, y] \quad \text{for all} \ x, y \ in \ R, \]  
where $f'(X)$, $g'(x)$ are polynomials in $Z[X]$ all monomials of which are of degree at least equal to one, and $m' = dm'_1$, $n' = dn'_1$ for $d = (m, n, m', n')$).

Since $m_1$ and $n_1$ (resp. $m_1$, $n_1$, $m'_1$ and $n'_1$) are relatively prime, (12) and (14) (resp. (12), (14), (24) and (25) for $s' = 0$) imply commutativity of $R$ by Theorem S.

3. Now we prove a commutativity theorem for $s$-unital rings satisfying the polynomial identity (1), resp. $(1')$, where $m = n = 1$ and one of the exponents is equal to zero.

**Theorem 4.** Let $R$ be a ring satisfying the polynomial identity (1), resp. $(1')$ for $m = n = 1$ and $s' = 0$. Then $R$ is commutative in any of the following cases:

(a) $t \geq 1$, and for $s > 0$, $R$ is right, resp. left $s$-unital;

(b) $t = 0$, and $t' = 0$ or $s = 0$;

(c) $t = 0$, $t' > 0$, $s > 0$, $R$ is an $s$-unital (resp. left or right $s$-unital) ring which satisfies $(1_\pm)$ (resp. $(1'_{\pm})$), or, for $s - t'$ odd, $(1_+)$ (resp. $(1'_+)$), and has the property $Q(2)$;

(d) $t = 0$, $t' > 0$, $s > 0$, $s - t'$ even, and $R$ is an $s$-unital (resp. left or right $s$-unital) ring which satisfies $(1_+)$ and the property $Q((|s - t'| + 1)!)$ (resp. $(1'_+)$ and the property $Q((\max\{s, t'\})!)$).

**Proof.** Case (a): For $s = 0$, $R$ is commutative by Theorem S. If $s > 0$, then it is easy to see that $R$ is in fact $s$-unital, and thus, by [7, Proposition 1], we can assure that for $s > 0$, $R$ is a ring with identity 1.

Now, setting $x + 1$ for $x$ in (1), resp. $(1')$, and combining the identity obtained with (1), resp. $(1')$, one gets $(x + 1)^t - x^t)[x, y]y^{t'} = 0$ for all $x, y$ in $R$; hence, by [5, Lemma], we have $(x + 1)^t - x^t)[x, y] = 0$ for all $x, y$ in $R$.

For $t = 1$, the last identity means that $R$ is commutative, and for $t > 1$, this identity implies the commutativity of $R$ by Theorem S.

The cases (b), (c) and (d) follow from [1, Theorem 6].

Obviously, for $m = n = 1$ and any one zero exponent in (1), resp. $(1')$, we have an analogous result. All these results are corollaries of Theorem 4. We state here only the following one
COROLLARY 1. Let $R$ be a ring satisfying the polynomial identity (1), resp. (1') for $m = n = 1$ and $s = 0$. Then $R$ is commutative in any of the following cases:

(a) $t' \geq 1$, and for $s' > 0$, $R$ is left, resp. right $s$-unital;
(b) $t' = 0$, and $t = 0$ or $s' = 0$;
(c) $t' = 0$, $t > 0$, $s' > 0$, and $R$ is an $s$-unital (resp. left or right $s$-unital) ring which satisfies (1-) (resp. (1'_-)), or for $s' - t$ odd, (1+) (resp. (1'_+)) and has the property $Q(2)$;
(d) $t' = 0$, $t > 0$, $s' > 0$, $s' - t$ even, and $R$ is $s$-unital (resp. left or right $s$-unital) ring which satisfies (1+) and the property $Q(|s' - t| + 1)!$ (resp. (1'_+) and the property $Q((\max\{s', t\})!)$).

Proof. From (1), resp. (1') it follows

$$y^r[x, y]x^t = \mp y^r[x, y]x^{s'}$$

for all $x, y$ in $R'$, resp.

$$y^r[x, y]x^t = \pm x^{s'}[x, y]y^r$$

for all $x, y$ in $R'$, and thus, $R'$ is commutative by Theorem 4. Hence, $R$ is also commutative.

4. The assumption that in (1), resp. (1'), $s' = t' = 0$, makes Theorem 1 symmetrical with respect to $m$ and $n$. Here we assume that in (1), resp. (1'), $m, n, s$ and $t$ are given positive integers, and that one of the given non-negative integers $s'$ and $t'$ is equal to zero. The result we will prove is the following theorem.

THEOREM 5. Let $R$ be a ring with polynomial identity (1), resp. (1'), where $m, n, s$ and $t$ are given positive, and $s', t'$ are given non-negative integers one of them being equal to zero, and the other positive. Then $R$ is commutative in any of the following cases:

(a) $s' = 0$ and $R$ is right, resp. left $s$-unital and has the property $Q(n)$ for $n > 1$,
(b) $t' = 0$, and $R$ is left $s$-unital and has the property $Q(m)$ form $m > 1$;

Proof. Case (a): It is easy to see that $R$ is in fact $s$-unital, and thus we can assume that $R$ is a ring with identity 1. For $n = 1$, by the same argument used in the proof of Theorem 1, one can show that $R$ is commutative. If $n > 1$, using the property $Q(n)$, we can prove, as in the proof of Theorem 1, that $C(R) \subseteq N(R) \subseteq Z(R)$. Hence, the identities (1) and (1') can be rewritten in the form $nx^{n+t-1}y^{n+t-1} = \pm m[x, y]y^{n+s-1}$ for all $x, y$ in $R$. Now, setting in the last identity $x = 1$ for $x$ and combining the identity obtained with the one above, we get

$$n((x + 1)^{n+t-1} - x^{n+t-1})[x, y]y^r = 0$$

for all $x, y$ in $R$, hence, by [5, Lemma] and the property $Q(n)$,

$$(x + 1)^{n+t-1} - x^{n+t-1} = 0$$

for all $x, y$ in $R$. 

This yields commutativity of $R$ by Theorem S.

Case (b): Since, for $t' = 0$, (1), resp. (1') can be rewritten in the form

$$y^s[y^m, x]x^{s'} = \pm [y, x^n] x^t$$

for all $x, y$ in $R'$

resp.

$$y^s[y^m, x]x^{s'} = \pm x^t[y, x^n]$$

for all $x, y$ in $R$.

Hence, $R'$ resp. $R$ is commutative by the case (a), and thus, $R$ is commutative.

5. In this section the commutativity of an $s$-unital ring $R$ satisfying the polynomial identity (1) or (1') shall be shown for some other special values of non-negative integers $m, n, s, s'$, and $t'$. Since every of these results is similar to the corresponding result in [1], then they will be stated here without proof.

**Theorem 6.** Let $R$ be an $s$-unital ring satisfying the polynomial identity (1) or (1'). Then $R$ is commutative provided one of the following conditions is fulfilled:

(a) $m = 0$ and $R$ has the property $Q(n)$;

(b) $n = 0$ and $R$ has the property $Q(m)$.

**Theorem 7.** Let $R$ be an $s$-unital ring which satisfies the polynomial identity (1) or (1'). Suppose that at least one of the integers $n + t - s' - 1$ and $m + s - t' - 1$ is odd and that $R$ has the property $Q(2)$. If, moreover, $R$ has one of the properties $Q(m)$ and $Q(n)$, especially, if $(m, n) = 2^r$ for some non-negative integer $r$, then $R$ is commutative.

**Theorem 8.** Let $R$ be an $s$-unital ring with polynomial identity (1) or (1'). Suppose that $n + t \neq s' + 1$ or $m + s \neq t' + 1$, and that $R$ has the property $Q(k)$ for $k = [2^{n+t} - 2^{s'} + 1]$ or $k = [2^{m+s} - 2^{t'} + 1]$. If, moreover, $R$ has one of the properties $Q(m)$ and $Q(n)$, especially, if $(m, n) = 2^r r'$ for some non-negative integer $r$ and some odd divisor $r'$ of $k$, then $R$ is commutative.

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