CONVOLUTION IN COLOMBEAU'S SPACES OF GENERALIZED FUNCTIONS
PART II. THE CONVOLUTION IN $\mathcal{G}_a$

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Abstract. We investigate various definitions of convolution and the Fourier transform in spaces $\mathcal{G}_a$ which are studied in the first part of the paper.

0. Introduction

Colombeau's theory of generalized functions [3] made on the problem of multiplication of distributions has a lot of applications in the theory of linear and nonlinear partial differential equations; see the recent monograph [2] and the references there. In this paper we are concerned with the convolution in spaces $\mathcal{G}_a$ of Colombeau's generalized functions and the relations between the convolution of Schwartz distributions and the generalized convolution of corresponding generalized functions. For this reason some problems on the convolution of Schwartz distributions are examined.

For the notion and the properties of the spaces $\mathcal{G}_a$ and the $a$-integral we refer to Part I. In Section 1, we give several new definitions of convolution in the space $\mathcal{G}_a$. In Section 2, the relations between different definitions of convolution and the convolution of generalized functions which are determined by convolvable Schwartz distributions are treated. In Section 3, we introduce the $a, \mu$-Fourier transform of elements from $\mathcal{G}_a$ and give the well known exchange formulae for $a = t$.

1. Definitions of convolution

Colombeau has given two definitions of convolution [3]: the convolution when one generalized function has a compact support, and the tempered convolution. Let $F, G$ be in $\mathcal{G}$, and let one of them, suppose $G$, have compact support. Then the $c$-convolution is defined by

$$F \circledast G(x) = \int_K F(x - y) G(y) \, dy, \quad x \in \mathbb{R}^n,$$

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where $K$ is a compact set which contains $\text{supp}(G)$ in his interior. It is proved in [3] that this convolution exists and is a generalized function.

Let $F, G$ be in $\mathcal{G}_r$. The $\tau$-convolution is defined by

$$F \ast G(x) = \frac{1}{\tau} \int F(x - y) G(y) \, dy, \quad x \in \mathbb{R}^n.$$ 

It is proved in [3] that this convolution exists and is a generalized tempered function.

Now, we shall introduce several new definitions of convolution. Let $G_1, G_2$ be in $\mathcal{G}$ and let $K_1, K_2$ be their supports. We say that they have compatible supports if for every bounded set $I$ there exists an open bounded set $J$ such that $y \in I$ implies $(y - K_1) \cap K_2 \subset J$. For such $G_1$ and $G_2$ we define $G_1 \ast G_2 = \int_I G_1(x - y) G_2(y) \, dy, \quad x \in I$. One can prove that the definition is correct in the same way as Colombeau has proved the correctness of the integration on a compact set [3]. In that case we have $G_1 \ast G_2 = G_2 \ast G_1$ and $P(D)(G_1 \ast G_2) = (P(D)G_1) \ast G_2$. If $g_1, g_2 \in D'$ have compatible supports, then $G_1 = Cd(g_1), G_2 = Cd(g_2)$ also have compatible supports. By using [1] one can prove that for such $g_1$ and $g_2$, $G_1 \ast G_2 \approx g_1 \ast g_2$.

Let $G_1, G_2$ be in $\mathcal{G}_a$. We define

$$G_1^{\alpha, \mu} G_2(x) = \int^{\alpha, \mu} G_1(x - y) G_2(y) \, dy, \quad x \in \mathbb{R}^n,$$

where $\mu, \varepsilon > 0$, is a unit net which corresponds to $a$ (see Part I).

**Proposition 1.** Assume $G_1, G_2 \in \mathcal{G}_a$. Then:

a) $G_1^{\alpha, \mu} G_2 \in \mathcal{G}$;

b) $\partial \gamma (G_1^{\alpha, \mu} G_2) = (\partial \alpha G_1^{\alpha, \mu} G_2)$, where $\alpha \in \mathbb{N}^n$;

c) $\mathcal{T}_h^{\alpha, \mu} (G_1^{\alpha, \mu} G_2) = (\mathcal{T}_h \gamma G_1^{\alpha, \mu} G_2)$, where $h \in \mathcal{H}, j \in \{1, \ldots, n\}$ (see Part I);

d) Let $G_1$ and $G_2$ be in $\mathcal{G}_a$. Then $G_1^{\alpha, \mu} G_2 \in \mathcal{G}_a$.

The assertion in a) means the following: for $G_1, G_2 \in \mathcal{E}_a, N_1, N_2 \in \mathcal{N}$

$$(G_1 + N_1) \ast^{\alpha, \mu} (G_2 + N_2) = G_1 \ast^{\alpha, \mu} G_2 + (G_1 \ast^{\alpha, \mu} N_2 + N_1 \ast^{\alpha, \mu} G_2 + N_1 \ast^{\alpha, \mu} N_2) \in \mathcal{G},$$

because $G_1 \ast^{\alpha, \mu} G_2 \in \mathcal{E}_M$ and $(G_1 \ast^{\alpha, \mu} N_2 + N_1 \ast^{\alpha, \mu} G_2 + N_1 \ast^{\alpha, \mu} N_2) \in \mathcal{N}$. Assertion in d) is similar.

**Proof.** We shall prove only a). Let $G_1$ and $G_2$ be in $\mathcal{E}_a$. We adopt the notation for $G_1$ and $\partial^\beta$ in (12) from part I by using symbols with subindex 1, and for $G_2$ with subindex 2. Then, for given compact set $K$ and every $\beta \in \mathbb{N}_0^n$

let $N = [\gamma_1 + \gamma_2 + N_1 + N_2 + n] + 1$. Because of Lebesgue’s theorem for the differentiation under the integral sign, we have

$$\left| \partial^\beta \int G_1(\phi, x - y) G_2(\phi, y) \mu_c(y) \, dy \right| \leq \sup_{|y| \leq a(\phi/\varepsilon)^{1 + \tau}} \left| c_1 \theta_1(|x - y|) \right| \varepsilon^{-N_1} \cdot \sup_{|y| \leq a(\phi/\varepsilon)^{1 + \tau}} \left| c_2 \theta_2(|y|) \right| \varepsilon^{-N_2} c_0 \varepsilon^{-n} b^n$$

$$\leq c \varepsilon^{-N}, \quad \phi_c \in \mathcal{A}_N, \quad x \in K, \quad \varepsilon \in (0, \infty),$$
where $c = c_1 c_2 c_3 c_4 b^n \eta_1 + \gamma_1 + \gamma_2$, $\eta = \min \{\eta_1, \eta_2\}$; $c_3$, $c_4$, $\gamma_1$, $\gamma_2$ are given by $\theta(\sigma + a(x)) \leq c_3 x^{\gamma_1}, \theta(a(x)) \leq c_4 x^{\gamma_2}, c = \sup \{|x| : x \in K\}$; $c_6 = \pi^{n/2} ((n + 2)/2).

Thus we have proved $G_1 \ast_{\mu} G_2 \in E_M$. Similarly, one can prove that if $G_1$ or $G_2$ belongs to $N_{\alpha}$, then $G_1 \ast_{\mu} G_2 \in N$. □

**Corollary.** If $\tilde{\Theta}_{\alpha} = \Theta_{\alpha}$ then $G_1 \ast_{\mu} G_2 \in \tilde{G}_{\alpha}$. Particularly, let $G_1$ and $G_2$ be in $\tilde{G}_t$. Then $G_1 \ast_{\mu} G_2 \in \tilde{G}_t$.

If for every pair of unit nets $\mu_{1 \varepsilon}, \mu_{2 \varepsilon}, G_1 \ast_{\mu \varepsilon} G_2 \approx G_1 \ast_{\mu \varepsilon} G_2$, then we say that there exists the associated $\alpha$-convolution $G_1 \ast G_2 = G_1 \ast_{\mu \varepsilon} G_2$. If for every pair of unit nets $\mu_{1 \varepsilon}, \mu_{2 \varepsilon}, G_1 \ast_{\mu \varepsilon} G_2 - G_1 \ast_{\mu \varepsilon} G_2 \in N (\in N_{\alpha})$, then there exist the $\alpha$-convolution in $\tilde{G}$ (in $G_{\alpha}$) $G_1 \ast G_2 = G_1 \ast_{\mu \varepsilon} G_2 = G_1 \ast_{\mu \varepsilon} G_2$.

If the equality holds in g.d. (g.t.d.) sense, then there exist g.d. (g.t.d.) $\alpha$-convolution $G_1 \ast G_2$.

**2. Relations between different convolutions**

Colombeau has proved [3] that if $G_1, G_2$ are from $G_{\tau}$ and one of them has a compact support, then $G_1 \ast_{\mu} G_2 = G_1 \ast_{\tau} G_2$. Let $G_1, G_2 \in G_{\alpha}$ and one of them has a compact support. Then there exists $G_1 \ast_{\mu} G_2$ and $G_1 \ast_{\mu} G_2 = G_1 \ast_{\mu} G_2$ for every $\mu_{\varepsilon}, \varepsilon > 0$; thus $G_1 \ast_{\mu} G_2 = G_1 \ast G_2$.

**Example.** Let $G(\phi_{\varepsilon}, x) = 1, x \in \mathbb{R}^n, \varepsilon > 0$. Clearly, $G \in G_{r}(\mathbb{R})$. Then

$$G \ast_{t} G(\phi_{\varepsilon}, x) = \int F(\phi)(\varepsilon x) \, dx = 2\pi \phi(0)/\varepsilon, \quad \varepsilon > 0,$$

$$G \ast_{\mu} G(\phi_{\varepsilon}, x) = \int_{-1/\varepsilon a}^{1/\varepsilon a} dx = 2/\varepsilon a, \quad \varepsilon > 0,$$

where $\mu_{\varepsilon}, \varepsilon > 0$, is a unit net. So $G \ast_{t} G$ is not associated with $G \ast_{\mu} G$. □

Since we shall compare the Schwartz’s convolution of distribution and the $\alpha, \mu$-convolutions of corresponding generalized functions, we need several assertions concerning Schwartz’s distributions.

If $\phi \in A_1$, then we put $\delta_{\nu}(x) = \phi_{1/\nu}(x) = \nu^n \phi(\nu x), x \in \mathbb{R}^n, \nu \in \mathbb{N}$. This is a $\delta$-sequence (for the general definitions we refer to [1]). For a unit net $\mu_{\varepsilon}$ we put $\varepsilon = 1/\nu, \nu \in \mathbb{N}$ and the corresponding sequence will be called a unit sequence and denoted by $\mu_{\nu}$ (instead of $\mu_{1/\nu}$). Such sequences belong to the set of special approximate unit sequences introduced in [9] (see [4]): a sequence from $\mathcal{D}', \nu \in \mathbb{N}$, is a special approximate unit if

\begin{align}
(i) \text{ For every compact set } K \subset \mathbb{R}^n \text{ there is } \nu_K > 0 \text{ such that } \mu_{\nu}(x) = 1, \\
\quad x \in K, \nu > \nu_K; \\
(ii) \text{ For every } m \in \mathbb{N}_0, p_m(\mu_{\nu}) \leq c_m, \nu \in \mathbb{N}, \text{ where } p_m(\varphi) = \sup \{|\partial^a \varphi(x)| : |a| \leq m, x \in \mathbb{R}^n\}, \varphi \in \mathcal{D}. 
\end{align}
Proposition 2. Let $h_k, k \in \mathbb{N}$, be a sequence of distributions from $\mathcal{D}'$. If

\begin{align*}
&\text{(a)} \quad \text{there exists } m \in \mathbb{N}_0 \text{ such that for every } \varepsilon > 0 \text{ there exist a compact set } \ K \subset \mathbb{R}^n \text{ and } k_0 \in \mathbb{N} \text{ with the property: } \varphi \in \mathcal{D}, \text{ supp}(\varphi) \cap K = \emptyset \Rightarrow \langle h_k, \varphi \rangle \leq \varepsilon p_m(\varphi) \text{ if } k > k_0, \\
&\text{(b)} \quad \text{for every special approximate unit } \mu_\nu \text{ the sequence } \langle h_k, \mu_\nu \rangle, \text{ converges} \\
&\quad \quad \quad \text{(when } \nu \to \infty \text{) uniformly for } k \in \mathbb{N}.
\end{align*}

Proof. The proof is similar to the proof of “(b) $\Rightarrow$ (c)” in [4, (1.1) Proposition] with the remark that for $\theta_\nu$, as in this proof and a special approximate unit $\mu_\nu$ we have $\langle h_\nu, \theta_\nu, \mu_p - \mu_q \rangle = 0$ for $p, q > \nu_0$. \□

It is proved in [4] that the definitions of convolutions of Schwartz, Vladimirov, Schiraishi, Chevalley and Mikusinski are equivalent (see also [5]). Recall, [4, (1.1) Proposition and (1.3) Theorem], $f, g \in \mathcal{D}'$ are convolvable iff one of the following equivalent conditions is satisfied:

\begin{enumerate}
\item [(I)] For every $\varphi \in \mathcal{D}$, $f(x)g(y)\varphi(x+y) \in \mathcal{D}'_{L^1}, \ x, y \in \mathbb{R}^n$;
\item [(II)] For every $\varphi \in \mathcal{D}$ and every special approximate unit $\mu_\nu \in \mathcal{D}(\mathbb{R}^{2n})$,

\[ \langle f(x)g(y), \varphi(x+y)\mu_\nu(x,y) \rangle \text{ converges when } \nu \to \infty; \]
\item [(III)] For every $\varphi \in \mathcal{D}$ there is an $m \in \mathbb{N}_0$ such that for every $\varepsilon > 0$ there is a compact set $K \subset \mathbb{R}^{2n}$ such that if $\psi \in \mathcal{D}(\mathbb{R}^{2n})$, and $\text{supp}(\psi) \cap K = \emptyset$, then $\langle (f(x)g(y), \varphi(x+y)\psi(x,y)) \rangle \leq \varepsilon p_m(\psi)$.
\end{enumerate}

Note that there are several other equivalent conditions.

Proposition 3. a) Let $f, g \in \mathcal{D}'$ be convolvable, and let $\delta_k$ be a delta sequence. Then for $h_k(x,y) = (f \ast \delta_k)(x)(g \ast \delta_k)(y), k \in \mathbb{N}, \ x, y \in \mathbb{R}^n$, the condition (a) from Proposition 2 holds (with the same $m, K$ as in (III)). Particularly, for any strong approximate unit $\mu_\nu, \nu \in \mathbb{N}, \text{ from } \mathcal{D}(\mathbb{R}^{2n}) \text{ we have that } \langle (f \ast \delta_k)(x)(g \ast \delta_k)(y), \mu_\nu(x,y), \nu \to \infty, \text{ converges uniformly for } k \in \mathbb{N}.

b) $\langle f \ast g, \varphi \rangle = \lim_{\nu \to \infty} \langle (f \ast \delta_\nu)(x)(g \ast \delta_\nu)(y) \varphi(x+y), \mu_\nu(x,y) \rangle, \varphi \in \mathcal{D}(\mathbb{R}^n)$.

Proof. a) By using the condition (III) and the notation from there, we have

\[ \langle f(x)g(y)\varphi(x+y), \psi(x,y) \rangle = \langle f(x)g(y), \varphi(x+y)\psi(x,y) \rangle = \langle \partial^i F(x)\partial^s G(y), \theta(x,y) \rangle, \]

where $\theta(x,y) = \varphi(x+y)\psi(x,y) \in \mathcal{D}(\mathbb{R}^{2n}), \text{ supp}(\theta) \subset I_x \times I_y$; $I_x$ and $I_y$ are bounded intervals in $\mathbb{R}^n$ and $f = \partial^i F \in I_x$ and $g = \partial^s G \in I_y$ for some $l, s \in \mathbb{N}_0$ and some continuous functions $F$ and $G$. Thus we obtain

\[ |\langle f(x)g(y), \varphi(x+y)\psi(x,y) \rangle| = \left| \iint_{I_x \times I_y} F(x)G(y)(-\partial_x^i)(-\partial_y^s)\theta(x,y) \ dx \ dy \right| \leq \varepsilon p_m(\psi). \]
Assume now that (a) from Proposition 2, with \( \varpi \) replaced with \( 2\varpi \), does not hold. Since on \( I_x \), respectively on \( I_y \), we have \( f \ast \delta_k = \partial (F \ast \delta_k), g \ast \delta_k = \partial (G \ast \delta_k) \), \( k \in \mathbb{N} \), after the same procedure we get that there is a subsequence \( r_k, k \in \mathbb{N} \), of natural numbers such that
\[
\left| \iint_{I_x \times I_y} (F \ast \delta_{r_k})(x) (G \ast \delta_{r_k})(y) (-\partial_x^k)(-\partial_y^k) \theta(x,y) \, dx \, dy \right| > 2\varepsilon p_m(\psi), \quad k \in \mathbb{N}.
\]
This is in a contradiction with the fact that \( F \ast \delta_{r_k} \rightarrow F \) uniformly on \( I_x \) and \( G \ast \delta_{r_k} \rightarrow G \) uniformly on \( I_y \), when \( k \rightarrow \infty \). The assertion a) is proved.

b) The previous part implies that for every \( k \in \mathbb{N} \), \( f \ast \delta_k \) and \( g \ast \delta_k \) are convolvable. Put
\[
ak_{k,\nu} = \langle (f \ast \delta_k)(x)(g \ast \delta_k)(y) \varphi(x+y), \mu_{\nu}(x,y) \rangle, \quad k, \nu \in \mathbb{N}.
\]
We have
\[
ak_{k,\nu} \overset{\nu \rightarrow \infty}{\longrightarrow} a_k = \langle (f \ast \delta_k) \ast (g \ast \delta_k), \varphi \rangle \overset{k \rightarrow \infty}{\longrightarrow} \langle f \ast g, \varphi \rangle,
\]
\[
ak_{k,\nu} \overset{\nu \rightarrow \infty}{\longrightarrow} a_\nu = \langle f(x)g(y)\varphi(x+y), \mu_\nu(x,y) \rangle \overset{\nu \rightarrow \infty}{\longrightarrow} \langle f \ast g, \varphi \rangle.
\]
Since Proposition 3 implies \( a_{k,\nu} \overset{\nu \rightarrow \infty}{\longrightarrow} a_k \) uniformly for \( k \in \mathbb{N} \), from the well known properties of a double sequence we have
\[
\lim_{k \rightarrow \infty} \lim_{\nu \rightarrow \infty} a_{k,\nu} = \lim_{\nu \rightarrow \infty} \lim_{k \rightarrow \infty} a_{k,\nu} = \lim_{\nu \rightarrow \infty} a_{\nu,\nu} = \lim_{k \rightarrow \infty} a_{k,\nu}.
\]
All above implies the assertion b). \( \square \)

By [8] we have (for \( \varphi \in \mathcal{D} (\mathbb{R}^n) \))
\[
\langle f \ast g, \varphi \rangle = \lim_{k \rightarrow \infty} \langle f(x)(g \ast \varphi)(x), \mu_k(x) \rangle = \lim_{k \rightarrow \infty} \langle f(x)g(x)\varphi(x+y), \mu_k(x) \rangle,
\]
where \( \mu_k, k \in \mathbb{N} \), is a strong unit sequence from \( \mathcal{D} (\mathbb{R}^n) \).

In the same way as Proposition 3 one can prove

**Proposition 4.** \( \lim_{m \rightarrow \infty} \langle f_m(x)g_m(y), \mu_m(x+y) \rangle = \langle f \ast g, \varphi \rangle, \varphi \in \mathcal{D} (\mathbb{R}^n) \),

where \( f_m = f \ast \delta_m, g_m = g \ast \delta_m, m \in \mathbb{N} \) and \( \mu_m, m \in \mathbb{N} \), is a strong unit sequence from \( \mathcal{D} (\mathbb{R}^n) \).

**Proposition 5.** Let \( f, g \) be in \( \mathcal{D}' \), \( G_1 = \text{C}(f), G_2 = \text{C}(g) \). If there exists \( f \ast g \) in the distributional sense and for some \( a \in A \), \( G_1 \) and \( G_2 \) are in \( G_a \), then there exists \( G_1 \overset{a_{\mu}}{\ast} G_2 \), and \( G_1 \overset{a_{\mu}}{\ast} G_2 \approx f \ast g \), for all unit nets \( \mu \), \( \varepsilon > 0 \), and thus \( G_1 \overset{a_{\mu}}{\ast} G_2 \approx f \ast g \).

**Proof.** Proposition 1 a) implies \( G_1 \overset{a_{\mu}}{\ast} G_2 \in \mathcal{G} \) for every unit net \( \mu, \varepsilon > 0 \), and Proposition 4 implies that
\[
\lim_{\varepsilon \rightarrow 0} \iint G_1(\phi, x-y)G_2(\phi, x+y) \, dx \, dy = \langle f \ast g, \varphi \rangle.
\]
This implies the assertion. \( \square \)
COROLLARY. With the same notation as in Proposition 5 we have \( G_1 ^* G_2 = G_2 ^* G_1 \).

Proof. It follows from \( f * g = g * f \) in \( D' \).

COROLLARY. If \( g_1, g_2 \) are in \( D'_F \) and \( g_1 * g_2 \) exists in \( D \); then there exists an \( a \in A \) such that for any \( \mu, G_1 ^a * G_2 \) exists. Thus \( G_1 ^a G_2 \approx g_1 * g_2 \), where \( G_1 = Cd(g_1), G_2 = Cd(g_2) \).

3. Fourier transform

We shall define the \( a, \mu \)-Fourier transform of elements from \( G_a \). Let \( G \in G_a \). We define \( F_{a, \mu} : G_a \rightarrow G_t \) by \( F_{a, \mu}(G)(x) = \int G(y)e^{-ixy} dy, x \in \mathbb{R}^n \).

PROPOSITION 6. \( F_{a, \mu} : G_a \rightarrow G_t \).

Proof. Let \( G \in N_a \). For \( c > 0 \) we have
\[
\left| \int G(\phi_z, y)e^{-ixy}\mu_z(y) dy \right| \leq \int |G(\phi_z, y)| dy \leq c|G(\phi_z, y)|dy \\
\leq c^c|\mu_z(y)|^c \leq c(1 + |x|)^c|\mu_z(y)|^c - N.
\]

This means that \( F_{a, \mu}(N_a) \subset N_t \). Similarly, we have \( F_{a, \mu}(E_a) \subset E_t \).

If for every two unit nets \( \mu_1, \mu_2, \varepsilon > 0 \), which correspond to \( a \), \( F_{a, \mu_1}(G) = F_{a, \mu_2}(G) \) (g.t.d.), then we say that there exists the \( a \)-Fourier transform in g.t.d. sense \( F_{a}(G) = F_{a, \mu_1}(G) \). In the sequel, we shall consider \( t \)-Fourier transform.

PROPOSITION 7. Let \( G_1, G_2 \) be in \( G_t \) and \( \mu_z, \varepsilon > 0 \), be a unit net. Then:

a) \( \langle F_{t, \mu}(G), \varphi \rangle = \langle G, F(\varphi) \rangle \). Particularly, for every \( G \in G_t \) there exists \( F_t(\varphi) \) (in g.t.d., sense);

b) If \( F_t(G_1) = F_t(G_2) \) (g.t.d.), then \( G_1 = G_2 \) (g.t.d.);

c) \( F_t(G_1 ^{\alpha} G_2) = F_t(G_1)F_t(G_2) \) (g.t.d.);

d) \( F_t(\varphi^a G) = (iz)^a F_t(G) \) (g.t.d.), \( \alpha \in \mathbb{N}_0 \).

Proof. a) We have
\[
\langle F_{t, \mu}(G), \varphi \rangle = \int \left( \int G(y)e^{-ixy} dy \right) \varphi(x) dx = \int G(y)F(\varphi)(y) dy \equiv \int G(y)F(\varphi)(y) dy = \langle G, F(\varphi) \rangle.
\]

b) This assertion follows from Proposition 7a) and the fact that the Fourier transform is a bijection of \( S \) onto \( S \).
c) The corollary of Proposition 1 implies \( G_1^{t_{\mu}} \ast G_2 \in \mathcal{G}_t \). For any \( \varphi \in \mathcal{S} \) we have
\[
\langle F_t(G_1^{t_{\mu}} \ast G_2), \varphi \rangle = \langle G_1^{t_{\mu}} \ast G_2, F(\varphi) \rangle \\
= \iiint \left( \int G_1(x_1 - y) G_2(y) \varphi(z) \, dy \right) e^{-ixz} \, dz \, dx \\
= \iiint \left( \int G_1(x_1 - y) G_2(y) \varphi(z) e^{-i((x_1 - y)z) - iyx} \, dy \right) \, dz \, dx_1.
\]
If we put \( x = x_1 - y, y = y, z = z \), we obtain
\[
\langle F_t(G_1^{t_{\mu}} \ast G_2), \varphi \rangle = \iiint \left( \int G_1(x) G_2(y) \varphi(z) e^{-ixz} e^{-iyx} \, dy \right) \, dz \, dx \\
= \int \left( \int F_t(G_2)(z) G_1(x) \varphi(z) e^{-ixz} \, dz \right) \, dx = \int F_t(F_t(G_2 \varphi))(x) G_1(x) \, dx \\
= \int F_t(G_1)(z) F_t(G_2)(z) \varphi(z) \, dz.
\]
This follows from Proposition 7a), since \( F_t(G) \varphi \) is a rapidly decreasing function for fixed \( \varepsilon > 0 \).

d) We have
\[
\langle F_t(\partial^\alpha G), \varphi \rangle = \langle \partial^\alpha G, F(\varphi) \rangle = \langle G, \partial^\alpha F(\varphi) \rangle (-1)^{\lvert \alpha \rvert} \\
= \langle G, F((ix)^{\alpha} \varphi) \rangle (-1)^{\lvert \alpha \rvert} = (-1)^{\lvert \alpha \rvert} \langle (ix)^{\alpha} F_t(G), \varphi \rangle. \quad \Box
\]

Proposition 7 implies the following one.

**Proposition 8.** If \( G_1, G_2, G_3 \in \mathcal{G}_t \) and \( \mu, \varepsilon > 0 \), is a unit net, then:

(i) \( G_1^{t_{\mu}} \ast G_2 = G_2^{t_{\mu}} \ast G_1 \) (g.t.d.);

(ii) \( (G_1^{t_{\mu}} \ast G_2)^{t_{\mu}} \ast G_3 = G_1^{t_{\mu}} \ast (G_2^{t_{\mu}} \ast G_3) \) (g.t.d.);

(iii) \( \partial^\alpha G_1^{t_{\mu}} \ast G_2 = \partial^\alpha G_1^{t_{\mu}} \ast G_2 \) (g.t.d.), \( \alpha \in \mathbb{N}_0^d \).

Let us define the inverse \( a, \mu \)-Fourier transform of elements from \( \mathcal{G}_a \). Let \( G \in \mathcal{G}_a \). We define \( F^{-1}_{a,\mu} : \mathcal{G}_a \to \mathcal{G}_t \) by
\[
F^{-1}_{a,\mu}(G)(x) = (2\pi)^{-n} \int \mathcal{G}(y) e^{ixy} \, dy, \quad x \in \mathbb{R}^n.
\]
All the facts which hold for \( F_{a,\mu} \), hold also for \( F^{-1}_{a,\mu} \). Furthermore, we have
\[
\langle F_t(F^{-1}_{a,\mu}(G)), \varphi \rangle = \langle F^{-1}_{a,\mu}(G), F(\varphi) \rangle = \langle G, \varphi \rangle, \quad G \in \mathcal{G}_t, \varphi \in \mathcal{D},
\]
i.e. \( F^{-1}_{a,\mu} \) is the inverse of \( F_t \) in the g.t.d. sense. For unit nets \( \mu_1, \varepsilon, \mu_2, \varepsilon > 0 \), we have
\[
\langle G_1^{t_{\mu_1}} \ast G_2, \varphi \rangle = \langle F_t(F^{-1}_{a,\mu}(G_1^{t_{\mu_1}} \ast G_2)), \varphi \rangle = \langle F_t(F^{-1}_{a,\mu}(G_1^{t_{\mu_1}} \ast G_2)), \varphi \rangle
\]
\[
= \langle F_t(F^{-1}_{a,\mu}(G_1^{t_{\mu_1}} \ast G_2)), \varphi \rangle = \langle G_1^{t_{\mu}} \ast G_2, \varphi \rangle.
\]
This implies that there exists the g.t.d. t-convolution for every $G_1$, $G_2$ from $\mathcal{G}_k$:

$$G_1 *_t G_2 = G_1 *_{\mathcal{G}_k} G_2.$$  

**Proposition 9.** Let $G$ be in $\mathcal{G}_k$ such that $G \approx g$, $g \in S'$. Then $x_j G \approx x_j g$; $F_t(G) \approx F(g)$, and $F_t(\overline{\mathcal{F}}_j G)(x) \approx i x_j F_t(G)(x)$, for $h \in \mathcal{H}, j \in \{1, \ldots, n\}$.

**Proof.** One can easily prove the first two assertions by using Proposition 8 of Part I. So we shall prove only the last one. Let $\varphi \in D$. Then, by Proposition 7 d),

$$\langle F_t(\overline{\mathcal{F}}_j G), \varphi \rangle(\phi) = \int F_t(\partial_j G(\phi, \cdot) * \phi_h(\varepsilon))(x) \varphi(x) \, dx$$

$$= \int_{\text{supp}(\phi)} i x_j F_t(G)(\phi, x) F_t(\phi_h(\varepsilon))(x) \varphi(x) \, dx.$$  

We shall use the fact that for any compact set $K$, $|1 - F(\phi_h(\varepsilon))(x)| \leq c h(\varepsilon)^n$, $x \in K$. 

Since for $g \in S'$, $F(\partial_j g)(x) = i x_j F(g)(x)$, $x \in \mathbb{R}^n$ and (i) and (ii) hold, we have that there exists a function $B \in L^1$, which depends on $\varphi$, and there exist an $N \in \mathbb{N}$, and an $\eta > 0$ such that

$$|ix_j F_t(G)(\phi, x) \varphi(x)| \leq B(x), \quad x \in \mathbb{R}^n, 0 < \varepsilon < \eta, \phi \in A_N.$$  

Let $A(\phi) = \langle (F_t(\overline{\mathcal{F}}_j G), \varphi) - (i x_j F_t(G), \varphi) \rangle(\phi)$. Then

$$|A(\phi_\varepsilon)| = \int_{\text{supp}(\phi)} B(x) |1 - F(\phi_h(\varepsilon))(x)| \, dx,$$

and $\lim_{\varepsilon \to 0, \phi \in A_N} |A(\phi_\varepsilon)| = 0$, because $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. \(\square\)

REFERENCES


