GENERALIZED HERMITE POLYNOMIALS

Gospava B. Đorđević

Abstract. We consider a new generalization of the classical Hermite polynomials and prove the basic characteristics of such polynomials $h_{n,m}^\lambda(x)$ (the generating function, an explicit representation, some recurrence relations, and the corresponding differential equation). For $m = 2$, the polynomial $h_{n,m}^\lambda(x)$ reduces to $H_n(x,\lambda)/n!$, where $H_n(x,\lambda)$ is the Hermite polynomial with a parameter. For $\lambda = 1$, $h_{n,2}^1(x) = H_n(x)/n!$, where $H_n(x)$ is the classical Hermite polynomial. Taking $\lambda = 1$ and $n = mN + q$, where $N = \lfloor n/m \rfloor$ and $0 \leq q \leq m - 1$, we introduce the polynomials $F_N^{(m,q)}(t)$ by $h_{n,m}^\lambda(x) = (2x)^q F_N^{(m,q)}((2x)^m)$, and prove that they satisfy an $(m+1)$-term linear recurrence relation.

1. Polynomials $h_{n,m}^\lambda(x)$. At the beginning, we define polynomials $h_{n,m}^\lambda(x)$ in the following manner.

Definition 1.1. The polynomials $h_{n,m}^\lambda(x)$, $\lambda \in \mathbb{R}^+$, $n,m \in \mathbb{N}$, are defined by the generating function

$$F(x,t) = e^{\lambda(2xt-t^m)} = \sum_{n=0}^{\infty} h_{n,m}^\lambda(x)t^n.$$  (1.1)

From above we get

$$F(x,t) = e^{\lambda(2xt-t^m)} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^k \lambda^n (2x)^{n-mk}}{\lambda^{m-1}k!(n-mk)!} t^n. \right.$$  

Thus, we obtain the following explicit representation

$$h_{n,m}^\lambda(x) = \lambda^n \sum_{k=0}^{\lfloor n/m \rfloor} (-1)^k \frac{2x}{}(n-mk)!.$$  (1.2)

Starting from (1.1) we can prove the following theorem.

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THEOREM 1.1. The polynomials $h_{n,m}^\lambda(x)$ satisfy the three-term recurrence relation

$$ nh_{n,m}^\lambda(x) = \lambda(2xh_{n-1,m}^\lambda(x) - mh_{n,m,n}^\lambda(x)), \quad n \geq m $$

(1.3)

with initial values: $h_{n,m}^\lambda(x) = (2\lambda x)^n/n!$, $0 \leq n \leq m - 1$.

Now, we prove the following theorem:

**THEOREM 1.2.** The polynomials $h_{n,m}^\lambda(x)$ satisfy the following relations:

$$ 2n h_{n,m}^\lambda(x) = (2x) D h_{n,m}^\lambda(x) - m D h_{n+1,m}^\lambda(x); $$

(1.4)

$$ D^k h_{n,m}^\lambda(x) = (2\lambda)^k h_{n-k,m}^\lambda(x); $$

(1.5)

$$ \frac{(2x)^n}{n!} = \sum_{k=0}^{[n/m]} \frac{1}{k!} h_{n-mk,m}^l(x) \quad (m \geq 2); $$

(1.6)

$$ u^n h_{n,m}^l \left( \frac{z}{u} \right) = \sum_{k=0}^{[n/m]} \frac{1 - u^m}{k!} h_{n-mk,m}^l(x); $$

(1.7)

$$ h_{n,m}^l(x + y) = \sum_{k=0}^{n} \frac{(2y)^k}{k!} h_{n-k,m}^l(x), $$

(1.8)

where $D = d/dx$ is the differentiation operator.

Proof. Differentiating (1.1) with respect to $x$ and $t$ we find the next equalities:

(i) $\partial F(x,t)/\partial x = 2\lambda e^{2xt - t^m}$.

(ii) $\partial F(x,t)/\partial t = \lambda(2x - mt^{m-1})e^{\lambda(2xt - t^m)}$.

Combining these equalities we obtain (1.4).

Differentiating the polynomials $h_{n,m}^\lambda(x)$ given by (1.2) $k$-times we get (1.5).

The generating function (1.1) for $\lambda = 1$ reduces to

$$ e^{2xt - t^m} = \sum_{n=0}^{\infty} h_{n,m}^1(x)t^n, \quad \text{i.e.} \quad e^{2xt} = e^{tm} \sum_{n=0}^{\infty} h_{n,m}^1(x)t^n. $$

Developing both sides of the last equality in $t$, we obtain

$$ \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}t^n = \left( \sum_{n=0}^{\infty} \frac{t^m}{n!} \right) \left( \sum_{n=0}^{\infty} h_{n,m}^1(x)t^n \right) $$

$$ = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} \frac{1}{k!} h_{n-mk,m}^l(x) t^n. $$

Now, comparing coefficients of $t^n$ in the last equality we get (1.6).

Starting from $e^{2xt - t^m} = e^{2xt - t^m} \circ e^{m - u^m t^m}$, we get (1.7).

Finally, from the equality $e^{2(x+y)t - t^m} = e^{2x t^m} \circ e^{2y t}$, we get

$$ \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} (-1)^k \frac{(2x + 2y)^{n-mk}}{k!(n-mk)!} = $$
\[
\left( \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} (-1)^k \frac{(2x)^{n-k} (m-1)^k}{k!(n-k)!} \right) \left( \sum_{n=0}^{\infty} \frac{t^n (2y)^n}{n!} \right),
\]
wherefrom, after some calculations, we obtain (1.8).

**Corollary 1.1.** For \( m = 2 \) and \( \lambda = 1 \) the equalities (1.4)–(1.8) reduce to the corresponding relations for the classical Hermite polynomials.

At the end of this section we prove that the polynomials \( h_{n,m}^\lambda(x) \) have an interesting property.

**Theorem 1.3.** The polynomial \( h_{n,m}^\lambda(x) \) is a particular solution of linear homogeneous equation of \( m \)-th order given by

\[
L_n(y) = y^{[m]} - 2^m m^{-1} \lambda^{m-1} (x y - n y) = 0.
\]

**Proof.** Using (1.5) and the recurrence relation (1.3) we get

\[
L_n[h_{n,m}^\lambda(x)] = (2\lambda)^m h_{n-m,m}^\lambda(x) - 2^m m^{-1} \lambda^{m-1} x (2\lambda) h_{n-1,m}^\lambda(x)
+ 2^m m^{-1} \lambda^{m-1} n h_{n,m}^\lambda(x)
= 2^m m^{-1} \lambda^{m-1} (n h_{n,m}^\lambda(x) - 2\lambda h_{n-1,m}^\lambda(x) + m \lambda h_{n-1,m}^\lambda(x)) = 0.
\]

2. **Polynomials \( P_N^{m,q}(t) \).** In this section we introduce a class of polynomials \( \{P_N^{m,q}(t)\}_{N=0}^{\infty} \). Let us suppose that \( n = mN + q \), where \( N = \lfloor n/m \rfloor \) and \( 0 \leq q \leq m - 1 \). Starting from (1.2) and taking \( \lambda = 1 \), we have

\[
h_{n,m}^1(x) = (2x)^q \sum_{k=0}^{N} (-1)^k \frac{(2x)^m N^{-k}}{k!(mN + q + mk)!}
= (2x)^q \sum_{k=0}^{N} (-1)^k \frac{((2x)^m)^{N-k}}{k!(q + m(N + k))!}
= (2x)^q P_N^{[m,q]}(t), \quad \text{where} \ t = (2x)^m.
\]

In this way we come to

\[
P_N^{m,q}(t) = \sum_{k=0}^{N} (-1)^k \frac{t^{N-k}}{k!(q + m(k + 1))!}
\]

In fact, the polynomials \( P_N^{m,q}(t) \) depend on two parameters: \( m \in N \) and \( q \in \{0, 1, \ldots, m - 1\} \).

Using (1.3) for \( \lambda = 1 \), i.e., \( nh_{n,m}^1(x) = 2x h_{n-1,m}^1(x) - m h_{n-1,m}^1(x) \), where \( n \geq m \geq 1 \), we can prove the following theorem:

**Theorem 2.1** The polynomials \( P_N^{m,q}(t) \) satisfy the next recurrence relations:

\[
(mN + q) P_N^{[m,q]}(t) = P_N^{[m,q-1]}(t) - m P_N^{[m,q]}(t), \quad \text{for} \ 1 \leq q \leq m - 1,
\]

\[
m N P_N^{[m,0]}(t) = t P_N^{[m,m-1]}(t) - m P_N^{[m,0]}(t), \quad \text{for} \ q = 0.
\]
It is interesting to find a recurrence relation for the polynomials \( P_N^{(m,q)}(t) \) where the parameters \( m \) and \( q \) are fixed.

Using the same method as in [3] we can prove the following result:

**Theorem 2.2.** The polynomials \( P_N^{(m,q)}(t) \) satisfy an \((m+1)\)-term recurrence relation of the form

\[
\sum_{i=0}^{m} A_{i,N,q} P_{N+1-i}^{(m,q)}(t) = B_{N,q} t^i P_N^{(m,q)}(t),
\]

where \( B_{N,q} \) and \( A_{i,N,q} \) \((i = 0, 1, \ldots, m)\) are constants depending only on \( N \), \( m \) and \( q \).

According to the explicit representation of polynomials \( P_N^{(m,q)}(t) \) given by (2.1), we get:

**Proposition 2.3** The polynomials \( P_N^{(m,q)}(t) \) have no negative real zeros.

**References**


