THEOREMS CONCERNING CERTAIN SPECIAL TENSOR FIELDS ON RIEMANNIAN MANIFOLDS AND THEIR APPLICATIONS

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Abstract. Let $M$ be an $n$-dimensional Riemannian manifold and $F$ a symmetric $(0,2)$-tensor field on $M$, which satisfies the condition $R \cdot F = 0$. Let, additionally, $H$, $A$ and $B$ be symmetric $(0,2)$-tensor fields on $M$. If the tensor $B$ commutes with $F$ (cf. (1.3)) and $H$ satisfies the condition $R \cdot H = Q(A, B)$, then

$$(A_{jk} = \frac{\text{tr}(A)}{\text{tr}(B)} B_{jk}) (B_{ij} F_{jm} - \frac{\text{tr}(B, F)}{\text{tr}(B)} B_{im}) = 0$$

on the open subset of $M$ on which $\text{tr}(B) \neq 0$. It is also proved that, in certain separately Einstein manifolds, null geodesic collineation and projective collineations reduce to motions.

1. Preliminary results. Let $M$ be an $n$-dimensional Riemannian manifold with not necessarily definite metric $g$. We denote by $g_{ij}$, $g_{ij}^h$, $R_{ijk}^h$ and $S_{ij}$ the local components of the metric $g$, the Levi Civita connection $\nabla$, the Riemann-Christoffel curvature tensor $R$ and the Ricci tensor $S$ of $M$, respectively.

For $(0,p)$-tensor $T$ with local components $T_{i_1 \ldots i_p}$, we define $(0,p+2)$-tensor $R \cdot T$ by

$$(R \cdot T)_{i_1 \ldots i_p, mk} = (\nabla_m \nabla_k - \nabla_k \nabla_m) T_{i_1 \ldots i_p} = - T_{i_1 \ldots i_p, mn} R^m_{i_1 \ldots i_p, mk}$$

Moreover, for $(0,2)$-tensors $A$ and $B$ with local components $A_{ij}$ and $B_{ij}$ respectively, define $(0,4)$-tensor $Q(A, B)$ by

$$Q(A, B)_{ijkh} = A_{ih} B_{jk} + A_{jh} B_{ih} - A_{ik} B_{jh} - A_{jh} B_{ik}.$$ 

Lemma 1.1. Let $F$ be a symmetric $(0,2)$-tensor field on $M$ satisfying the condition

$$R \cdot F = 0.$$ 

(1.1)

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If $H$, $A$ and $B$ are symmetric $(0,2)$-tensor fields on $M$ satisfying the relations

$$R\cdot H = Q(A,B),$$

(1.2)

then

$$\tilde{a}B_{jk} - \tilde{b} A_{jk} = a B_{jr} F^r_{k} - b A_{jr} F^r_{k},$$

(1.4)

where $a = \text{tr}(A) = A_{ij} g^{ij}$, $b = \text{tr}(B) = B_{ij} g^{ij}$, $\tilde{a} = \text{tr}(A, F) = A_{ij} F^{ij}$, $\tilde{b} = \text{tr}(B, F) = B_{ij} F^{ij}$, $F^{r}_{j} = F_{sj} g^{sr}$ and $F^{ij} = F_{r} g^{ri} g^{sj}$, etc.

Proof. At first we note that by virtue of Ricci identity the relations (1.1) and (1.2) can be written in the following forms

$$F_{tr} R^r_{jmk} + F_{jr} R^r_{rkm} = 0,$$

(1.6)

$$H_{ir} R^r_{jkm} + H_{jr} R^r_{rkm} = A_{ik} B_{jm} + A_{jk} B_{im} - A_{im} B_{jk} - A_{jm} B_{ik}.$$  

(1.7)

We remark also that from (1.7) it follows

$$A_{rn} B^r_{k} = A_{rk} B^r_{m}.$$  

(1.8)

Transvecting (1.6) with $H^{ij}$, we have $F^{ir} H^{is} R^s_{rkm} = 0$. Transvecting again (1.7) with $F^{ij}$ and applying the above equality,

$$A_{rn} F^{rs} B_{rk} = A_{rk} F^{rs} B_{mn}.$$  

(1.9)

Moreover, we see from (1.1) that $R^{0}_{jmk}$ is antisymmetric with respect to indices $j, k$. Therefore the following equality holds good

$$(H^{ir} R^{r}_{jks} + H^{jr} R^{r}_{rks}) F^{m}_{s} = (H^{ir} R^{r}_{jsm} + H^{jr} R^{r}_{rsm}) F^{m}_{s}.$$  

Hence by (1.7)

$$A_{ir} F^{r}_{m} B_{jk} + A_{jr} F^{r}_{m} B_{ik} - A_{ik} B_{jr} F^{r}_{m} - A_{jk} B_{ir} F^{r}_{m}$$

$$= A_{im} B_{jr} F^{r}_{k} + A_{jm} B_{ir} F^{r}_{k} - A_{ik} F^{r}_{m} B_{jm} - A_{jk} F^{r}_{m} B_{im}.$$  

(1.10)

Transvecting this with $g^{im}$ and using (1.3), (1.8) and (1.9), we obtain (1.4). Hence (1.5) follows, completing the proof.

Theorem 1.2. Let $F$ be a symmetric $(0,2)$-tensor field on $M$ satisfying the condition (1.1). If $H$, $A$ and $B$ are symmetric $(0,2)$-tensor fields on $M$ satisfying the conditions (1.2) and (1.3), then

$$(A_{jk} - \frac{a}{b} B_{jk}) (B_{ir} F^{r}_{m} - \frac{\tilde{b}}{b} B_{im}) = 0$$

on the open subset of $M$ on which $b \neq 0.$
Proof. From (1.4) it follows that tensor $A$ commutes with $F$, that is, $A_{ik}F^r_i = A_{jk}F^r_j$. Moreover, from (1.4) we derive $A_{ik}F^r_k = \frac{1}{2}(aB_{ir}F^r_k + \tilde{a}A_{rk} - \tilde{a}B_{rk})$, which together with (1.5) substituted in (1.10) gives the equality

$$W_{jm}P_{mk} + W_{im}P_{jk} + W_{jk}P_{lm} + W_{nk}P_{jm} = 0,$$

where $W_{jm} = B_{jr}F^r_m - \frac{1}{2}B_{jm}$ and $P_{ik} = A_{ik} - \frac{a}{2}B_{ik}$. Now the antisymmetrization of (1.11) with respect to the indices $m, j$ yields an equation which compared with (1.11) leads to

$$W_{im}P_{jk} + W_{jk}P_{lm} = 0.$$ 

Hence, it follows that $W_{im}P_{jk} = 0$, completing the proof.

**Theorem 1.3.** Let $F$ be a symmetric $(0, 2)$-tensor field on $M$ satisfying the condition $R \cdot F = 0$. Let $H$, $\tilde{H}$ and $A$ be symmetric $(0, 2)$-tensor fields on $M$ satisfying the relations

(a) $R \cdot H = Q(A, g)$
(b) $R \cdot \tilde{H} = Q(A, F)$. Then $A = 0$ at every point of $M$ at which $F$ is nonsingular and nonproportional to the metric $g$.

Proof. We restrict our consideration to a point of $M$ at which $F$ is nonsingular and nonproportional to $g$. As an immediate consequence of Theorem 1.2, we have by (a)

$$A_{ij} = \frac{a}{n}g_{ij},$$

Moreover, from Lemma 1.1 by (a) it follows that

$$\tilde{a}n - af = 0,$$

and by (b) it follows that

$$\tilde{a}F_{jk} - fA_{jk} = aF_{jr}F^r_k - fA_{jr}F^r_k,$$

where $f = \text{tr}(F)$, $\tilde{f} = \text{tr}(F, F)$, $a = \text{tr}(A)$ and $\tilde{a} = \text{tr}(A, F)$. Because of (1.12) to prove the theorem it is sufficient to show that $a = 0$.

Consider the case $f = 0$. By (1.13), we have $\tilde{a} = 0$ and by (1.15) $a = 0$ or $\tilde{f} = 0$. If $\tilde{f} = 0$, then from (1.14) and nonsingularity of $F$, we find $a = 0$.

Let now $f \neq 0$. Comparing the relations (1.13) and (1.15), we have

$$a(f^2 - n\tilde{f}) = 0.$$ 

In the sequel we assume that $a \neq 0$. Then

$$f^2 - n\tilde{f} = 0.$$ 

Next, in virtue of Theorem 1.2 by (b), we get $(A_{ij} - \frac{a}{2}F_{ij}(F_{mn}F^r_n - \frac{\tilde{f}}{f}F_{mn}) = 0$. This, because of (1.12) and because $F$ is not proportional to $g$, can be written as
\( a(F_{nm}^r F^r_n - \frac{1}{2} F_{mn}^r) = 0. \) Hence \( F_{nm}^r F^r_n - \frac{1}{2} F_{mn}^r = 0. \) So, by virtue of (1.14) and (1.15), we get \( A_{jk} F^r_j = \frac{1}{2} A_{jk}. \) The last equation, together with (1.12) and (1.16) implies \( a = 0. \) This is a contradiction. Therefore \( a = 0. \) This completes the proof.

**Remark.** In the above, we have indeed proved that, under our assumptions, \( A = 0 \) at every point of \( M \) at which \( F_{\mu}^r F^r_\mu \neq 0 \) and \( F \) is nonproportional to \( g. \)

**Theorem 1.5.** Let \( F \) be a symmetric \((0, 2)\)-tensor field on \( M \) satisfying the condition \( R \cdot F = 0. \) Let \( H, \tilde{H} \text{ and } A \) be symmetric \((0, 2)\)-tensor field on \( M \) satisfying the relations (a) \( R \cdot H = Q(A, g) \) and (b) \( R \cdot \tilde{H} = Q(\tilde{A}, g), \) where \( \tilde{A}_{ij} = A_{\nu}^\rho F^\rho_{ij}. \) Then \( A = 0 \) at every point of \( M \) at which \( F \) is nonproportional to the metric \( g. \)

**Proof.** We restrict our considerations to a point of \( M \) at which \( F \) is nonproportional to \( g. \) At first we note that by (a) and Theorem 1.2 it follows that 

\[
A_{ij} = \frac{a}{n} g_{ij}, \quad \text{where } a = \text{tr}(A). \tag{1.17}
\]

Transvectoring now (1.17) with \( F^\rho_i \) we see that \( \tilde{A} \) is symmetric. Next, from Theorem 1.2 by (b), we get 

\[
A_{\nu}^\rho F^\rho_{ij} = \frac{a}{n} g_{ij}. \tag{1.18}
\]

Substituting now (1.17) into (1.18), we get \( a = 0. \) This, with the help of (1.17), gives our assertion.

**2. Applications.** In this section we apply the results obtained in the previous section. Let \( M \) be a Riemannian manifold with not necessarily definite metric \( g \) and of dimension \( n > 2. \) For vector field \( v \) on \( M \), denote by \( L_v \) the Lie derivative with respect to \( v. \)

A vector field \( v \) on \( M \) is said to be a motion if \( L_v g = 0, \) and affine collineation if \( L_v \nabla = 0. \) A curvature collineation on \( M \) is a vector field \( v \) which satisfies the condition \( L_v R = 0. \) An investigation of this transformation was strongly motivated by the important role of the Riemannian curvature tensor in the theory of general relativity [3, 4].

The assertion of the theorem below is quite obvious.

**Theorem A.** In a non-Ricci-flat Einstein manifold a curvature collineation is a motion.

Let \( M \) be a locally product Riemannian manifold in the sense of Tachibana [8]. Then, there exists an atlas of separating coordinate neighborhoods \( \{(U_i(x^i))\} \) such that in each \( (U_i(x^i)) \) the metric \( g \) can be written as 

\[
g = \sum_{a,b=1}^{p} g_{ab}(x^c)dx^a \otimes dx^b + \sum_{a,b=1}^{q} g_{ab}(x^\alpha)dx^a \otimes dx^\beta, \quad p + q = n, \quad 1 \leq p \leq n - 1.
\]

Define an \((0, 2)\)-tensor field on \( M \) by 

\[
[F_{ij}] = \begin{bmatrix} g_{ij} & 0 \\ 0 & -g_{ij} \end{bmatrix}
\]
in each $(U, (x^i))$. The tensor field $F$ is nonsingular, nonproportional to $g$, symmetric and parallel.

A locally product Riemannian manifold $M$ is called to be a separately Einstein manifold if its Ricci tensor has the following form

$$ S = cg + dF, $$

(2.1)

where

$$ c = \frac{(n_s - f\tilde{s})}{(n^2 - f^2)}, \quad d = \frac{(n\tilde{s} - f s)}{(n^2 - f^2)}, \quad f = \text{tr}(F) = p - q, \quad s = \text{tr}(S) \text{ and } \tilde{s} = \text{tr}(S, F).$$

In a separately Einstein manifold $M$, $c = \text{const}$ and $d = \text{const}$ if $p > 2$ and $q > 2$ (see [8]). Note that a separately Einstein manifold is Ricci-flat if and only if $c = d = 0$. In the case $d = 0$, it reduces to an Einstein one.

It has been proved (cf. [8]) that

**Theorem B.** In a separately Einstein manifold with $c = \text{const}$, $d = \text{const} \neq 0$ and $c^2 \neq d^2$, a curvature collineation is necessarily a motion.

According to Katzin and Levine [4], a vector field $v$ on $M$ is said to be a null geodesic collineation (NGC) if

$$ L_v \Gamma^h_{ij} = g^{hr} A_r g_{ij}, $$

(2.2)

where $A_r = \nabla_r p$ and $p$ is a function. For such a transformation, we have

$$ L_v R^h_{ijk} = A^h_k g_{ij} - A^h_i g_{jk}, $$

(2.3)

where $A^h_k = \nabla^h \nabla_k p$ and $A^h_i = A_{ik} g^{hr}$. If additionally $A_{ik} = 0$, then the NGC is said to be special. Note also that a special null geodesic collineation is a curvature collineation.

**Theorem 2.1.** Let $F$ be a symmetric $(0,2)$-tensor field on a Riemannian manifold $M$. Assume additionally that $F$ is nonproportional to the metric tensor $g$ at every point of $M$ and satisfies the condition $R \cdot F = 0$. Then any NGC on $M$ is special.

**Proof.** Applying the Lie derivative to the equation $R \cdot F = 0$ and making use of (2.3), we have $R \cdot \tilde{H} = Q(\tilde{A}, g)$, where $\tilde{H} = L_v F$ and tensor $\tilde{A}$ have the local components $A_r F^r_j$. Similarly, applying the Lie derivative to the equation $R \cdot g = 0$ and using (2.3) we find $R \cdot H = Q(\tilde{A}, g)$, where $H = L_v g$. In our situation, Theorem 1.5 yields $\tilde{A}_{ij} = 0$, which completes the proof.

From Theorem 2.1, we get

**Theorem 2.2.** In a locally product Riemannian manifold $M$ any NGC is special.

Combining Theorems A, B and 2.2, we derive

**Theorem 2.3.** In a separately Einstein manifold $M$ with $c = \text{const}$, $d = \text{const}$ and $c^2 \neq d^2$, an NGC is necessarily a motion.
Moreover, from Theorem 2.1, for \( F = S \) it follows

**Theorem 2.4.** If the Ricci tensor \( S \) of a Riemannian manifold \( M \) satisfies the relation \( R \cdot S = 0 \) and if \( S \) is nonproportional to \( g \) at every point of \( M \), then any NGC on \( M \) is special.

A Riemannian manifold is called semisymmetric [7] if the condition \( R \cdot R = 0 \) is satisfied on \( M \).

As an immediate consequence of Theorem 2.4, we get

**Theorem 2.5.** In a semisymmetric manifold \( M \) with the Ricci tensor \( S \) nonproportional to \( g \) at every point, any NGC is special.

A vector field \( v \) on a Riemannian manifold is said to be a projective collineation (PC) if

\[
L_v \Gamma^i_{jk} = \delta^i_j A_i + \delta^i_k A_j,
\]

where the 1-form \( A \) is defined by \( A_j = (n + 1)^{-1} \nabla_j (g^{rs} \nabla_r v_s) \). If \( A_j = 0 \), then the PC is an affine one. It is well-known that for any PC, we have

\[
L_v R^h_{ijk} = \delta^h_j A_{ik} - \delta^h_k A_{ij},
\]

where \( A_{ik} = \nabla_k A_i \). Projective collineation is said to be special, if \( A_{ij} = 0 \). Note also that a special projective collineation is a curvature collineation.

**Theorem 2.6.** Let \( F \) be a nonsingular, nonproportional to \( g \) at every point of \( M \) and symmetric \((0,2)\)-tensor field satisfying the condition \( R \cdot F = 0 \) on a Riemannian manifold \( M \). Then any PC on \( M \) is special.

**Proof.** Applying the Lie derivative to the relations \( R \cdot g = 0 \) and \( R \cdot F = 0 \) and using (2.5), we see that a PC satisfies \( R \cdot H = Q(A, g) \) and \( R \cdot \tilde{H} = Q(A, F) \), respectively, where \( H = L_v g \) and \( \tilde{H} = L_v F \). In view of Theorem 1.3, we obtain \( A_{ij} = 0 \), which gives our assertion.

From Theorem 2.6, we find

**Theorem 2.7.** In a locally product Riemannian manifold \( M \) any PC is special.

Combining Theorems A, B and 2.7, we derive

**Theorem 2.8.** In a separately Einstein manifold \( M \) with \( c = \text{const} \), \( d = \text{const} \) and \( c^2 \neq d^2 \), any PC is necessarily a motion.

Moreover, we prove

**Theorem 2.9.** Let the Ricci tensor \( S \) of a Riemannian manifold \( M \) be nonsingular, nonproportional to the metric \( g \) at each point and satisfy the relation \( R \cdot S = 0 \). Then, any PC on \( M \) is an affine collineation.

**Proof.** From Theorem 2.6, for \( F = S \), we have \( \nabla_j A_i = 0 \). This, by the Ricci identity, leads to \( A_i R^r_{ijk} = 0 \) and also \( A_i S^r_k = 0 \). Since \( S \) is nonsingular, \( A_i = 0 \). This completes the proof.

As an immediate consequence of Theorem 2.9, we get
Corollary 2.10. Let the Ricci tensor $S$ of a semisymmetric manifold $M$ be nonsingular and nonproportional to the metric $g$ at each point of $M$. Then, any PC on $M$ is an affine collineation.

For projective collineation in a locally symmetric or Ricci-symmetric manifolds ($\nabla R = 0$ or $\nabla S = 0$, respectively) see Sumitomo [6].

The next theorem can be deduced from Theorem 1.4.

Theorem 2.11. Let $M$ be a Riemannian manifold whose Ricci tensor $S$ satisfies the condition $R \cdot S = 0$. Assume additionally that, at each point of $M$, the scalar curvature $s \neq 0$ and $S$ is nonproportional to the metric $g$. Then, any PC on $M$ is special.

Corollary 2.12. Let the Ricci tensor $S$ of a semisymmetric manifold $M$ be nonproportional to the metric $g$ and the scalar curvature $s \neq 0$ at each point of $M$. Then, any PC on $M$ is special.

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Added in proof: Theorem 1.2 is a generalization of Grycak's theorem from [2]. Certain other generalization of his theorem can also be found in [1].

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