LOGICS WITH TWO TYPES OF INTEGRAL OPERATORS

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Abstract. We prove completeness theorems for absolutely continuous and singular biprobability models of a logic with integrals. Also in both cases, we prove the finite compactness theorem for a set of sentences of the form \( r \in [r, \bar{r}] \).

We assume throughout the paper that \( A \) is a countable admissible set with \( \omega \in A \). In [2], Keisler introduced a logic \( L_{A/I} \) which has an integral operator which builds terms with bound variables. In our case two types of integral operators \( f_1 \ldots dx \) and \( f_2 \ldots dx \) are allowed.

A biprobability model for \( L_{A/I_1/I_2} \) logic is a model \( \mathfrak{A} = \langle A, R_t, c_t, \mu_1, \mu_2 \rangle \) \( t \in I \), \( j \in J \), where \( \langle A, R_t, c_j \rangle \) is a first-order model without operations and \( \mu_1, \mu_2 \) are probability measures on \( A \). We shall see a difference in semantics for \( L_{A_1/I_2}^a \) and \( L_{A_1/I_2}^s \) by means of the following definition.

Definition 1. (a) An absolutely continuous biprobability model for \( L_{A_1/I_2}^a \) is a biprobability model \( \mathfrak{A} \) such that \( \mu_1 \) is absolutely continuous with respect to \( \mu_2 \), i.e. \( \mu_1 \ll \mu_2 \).

(b) A singular biprobability model for \( L_{A_1/I_2}^s \) is a biprobability model \( \mathfrak{A} \) such that \( \mu_1 \) is singular with respect to \( \mu_2 \), i.e. \( \mu_1 \perp \mu_2 \). \( \Box \)

In both cases, quantifiers are interpreted by

\[
(\int \tau(x, \bar{a}) \, dx)^\mathfrak{A} = \int \tau(b, \bar{a})^\mathfrak{A} \, d\mu_k(b) \quad \text{for } k = 1, 2,
\]

where \( \tau(x, \bar{y}) \) is a term and \( \bar{a} \in A^n \).

Diagonal products \( \mu_k^{(n)} \), which are the corresponding restrictions of completions of \( \mu_k^\mathfrak{A} \)'s \( (k = 1, 2) \) to \( \sigma \)-algebras generated by the measurable rectangles and the diagonal sets \( \{ \bar{x} \in A^n : x_i = x_j \} \), can be replaced by sequences of probability measures on \( A^n \)'s which satisfy the Fubini theorem. That generalization of a probability structure is relevant for us.

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Definition 2. A graded biprobability model for $L_{A I_1I_2}$ is a model
$\mathfrak{A} = \langle A, R_i, c_j, \mu_i^k, \mu^m_i \rangle_{i \in I, j \in J, n \geq 1}$ such that:

1. Each $\mu_i^k$ is a countably additive probability measure on $A^n$.
2. Each $n$-ary relation $R_i$ is $\mu_i^k$-measurable and the identity relation is $\mu_{I_2}^k$-measurable.
3. $\mu_i^k \times \mu_i^m \leq \mu_{n+m}^k$.
4. Each $\mu_i^k$ is preserved under permutation of $\{1, 2, \ldots, n\}$.
5. $\langle \mu_i^k : n \in \mathbb{N} \rangle$ has the Fubini property: If $B$ is $\mu_{m+n}^k$-measurable, then
   (a) For each $\bar{x} \in A^m$, the section $B_{\bar{x}} = \{ \bar{y} : B(\bar{x}, \bar{y}) \}$ is $\mu_n^k$-measurable.
   (b) The function $f(\bar{x}) = \mu_i^k(B_{\bar{x}})$ is $\mu_n^k$-measurable.
   (c) $\int f(\bar{x}) d\mu_m = \mu_m^{n+m}(B)$. \hfill \square

Definition 3. (a) A graded biprobability model for $L^{a}_{A I_1I_2}$ is a graded biprobability model $\mathfrak{A}$ such that $\mu_i^k \ll \mu_{n}^k$ for each $n \in \mathbb{N}$.

(b) A graded biprobability model for $L^{a}_{A I_1I_2}$ is a graded biprobability model $\mathfrak{A}$ such that $\mu_i^k \perp \mu_{n}^k$ for each $n \in \mathbb{N}$. \hfill \square

1. The logic $L^{g}_{A I_1I_2}$. Axioms and rules of inference for $L^{a}_{A I_1I_2}$ are those for $L_{A I}$, as listed in [3] with both $\int_1$ and $\int_2$ playing the role of $\int$, together with the following axioms:

(A1) Axioms of continuity of integral operators: $(i, j = 1, 2)$

(a) $\bigwedge_n \bigvee_m \bigwedge_k \int_i F_k \left( \int_j \tau(\bar{x}, \bar{y}) \, d\bar{x} \right) d\bar{y} < \frac{1}{n}$,

where $F_k(s) = \begin{cases} 1, & \text{if } r - 1/m + 1/k \leq s \leq r - 2/k, \\ 0, & \text{if } s \leq r - 1/m \text{ or } s \geq r - 1/k \\ \text{linear, for other cases} \end{cases}$

is a continuous real function such that $F_k \upharpoonright \mathbb{Q} \in A$.

(b) $\bigwedge_n \bigvee_m \bigwedge_k \int_i G_k \left( \int_j \tau(\bar{x}, \bar{y}) \, d\bar{x} \right) d\bar{y} < \frac{1}{n}$,

where $G_k(s) = \begin{cases} 1, & \text{if } r + 2/k \leq s \leq r + 1/m - 1/k, \\ 0, & \text{if } s \leq r + 1/k \text{ or } s \geq r + 1/m \\ \text{linear, for other cases} \end{cases}$

(A2) Axiom of absolute continuity:

$\bigwedge_{\epsilon \in \mathbb{Q}_+} \bigvee_{\delta \in \mathbb{Q}_+} \bigwedge_n \bigvee_{\tau \in T_n} (| \int_2 \tau(\bar{x}) \, d\bar{x} | < \delta \Rightarrow | \int_1 \tau(\bar{x}) \, d\bar{x} | < \epsilon)$,

where $T = \bigcup_n T_n$, $T_n$ is a set of terms with $n$ free variables and $T, T_n \in A$.

(A3) $\int_1 (\int_2 \tau \, d\bar{x}) \, d\bar{y} = \int_2 (\int_1 \tau \, dx) \, dy$.

Now we introduce two sorts of auxiliary models.

Definition 4. (a) A weak model for $L^{g}_{A I_1I_2}$ is a model $\langle \mathfrak{A}, I_1, I_2 \rangle$ where $\mathfrak{A}$ is a first-order model and $I_1$ is what may be called an $A$-Daniell integral on $A$, that is, $I_k$ is a positive linear real function on the set of terms with at most one free
variable \( x \) and parameters from \( A \), i.e.

\[
I_k(r) = r, \quad k = 1, 2
\]

\[
I_k(r \cdot \sigma + s \cdot \tau) = r \cdot I_k(\sigma) + s \cdot I_k(\tau),
\]

if \( \tau(b, \tilde{a}) \geq 0 \) for all \( b \in A \), then \( I_k(\tau(x, \tilde{a})) \geq 0 \).

(b) A middle model for \( L^\sigma_{A \models I_1 I_2} \) is a weak model \( \mathfrak{A} \) such that for each \( \varepsilon > 0 \) there is \( \delta > 0 \) such that for each term \( \tau(x, \tilde{y}) \) and \( \tilde{a} \in A^n \), if \( |I_2(\tau(x, \tilde{a}))| < \delta \) then \( |I_1(\tau(x, \tilde{a}))| < \varepsilon \).

In both cases, for \( \tau \) a term, define \( \tau^3 \) inductively as for biprobability models, except that at the integral step, we define

\[
(f_x(\tau(x, \tilde{a}) \, dz)^3 = I_k(\tau(x, \tilde{a})).
\]

Lemma 1. (Middle Completeness Theorem for \( L^\sigma_{A \models I_1 I_2} \)) Let \( T \) be a set of sentences of \( L^\sigma_{A \models I_1 I_2} \) such that \( T \) is \( \Sigma_1 \)-definable over \( A \). Then \( T \) is consistent with the axioms of this logic iff it has a middle model in which each theorem of \( L^\sigma_{A \models I_1 I_2} \) is true.

Proof. The soundness is easy to prove because all the axioms represent known properties of integrals (the Generalized Radon-Nikodym Theorem and the Fubini Theorem prove that each function \( \tau(x, \tilde{y})^3 \): \( A \times A \to \mathbb{R} \) is compatible with absolutely continuous measures \( \mu_1 \) and \( \mu_2 \), i.e.

\[
\int \int \tau(x, y)^3 \, d\mu_1(x) \, d\mu_2(y) = \int \int \tau(x, y)^3 \, d\mu_2(y) \, d\mu_1(x).
\]

A Henkin argument is used to construct a weak model \( \langle \mathfrak{A}, I_1, I_2 \rangle \) of \( T \) in which each theorem of \( L^\sigma_{A \models I_1 I_2} \) is true. Let \( K = L \cup C \) be the language introduced in this construction, where \( C \) is a set of new constant symbols and \( C \in A \). We wish the axiom \( A_2 \) to hold for all the terms and that is done by the following construction (see [9]).

Let \( K' \) be a language with four kinds of variables: \( X, Y, Z, \ldots \) are variables for sets, \( x, y, z, \ldots \) are variables for urelements, \( r, s, t, \ldots \) are variables for reals from \([0, 1] \cap A\), and \( U, V, W, \ldots \) are variables for functions \( A^n \to \mathbb{R}, n \geq 0 \). Predicates are: \( E_n^m(x, \tilde{x}) \) for \( n \geq 1, E_n^{n+1}(\tilde{x}, r, U) \) for terms, \( n \geq 0 \); \( I_k(U, r) \) for \( U: A^0 \to \mathbb{R} \) or \( U: A^1 \to \mathbb{R}, k = 1, 2 \); and \( \leq \) for reals. Function symbols are \( f, g, h, \ldots \) for each continuous real functions \( F: \mathbb{R}^n \to \mathbb{R} \) such that \( F \upharpoonright \mathbb{Q}^n \in A \). Constant symbols are: \( X_{\varphi} \) for each formula \( \varphi; \) \( U_\tau \) for each term \( \tau; \) and \( \tau \) for each real number \( r \in [0, 1] \cap A \).

Let \( S \) be the following theory of \( K'_{A} \):

1. Axioms of validity:
   1.1. \( \forall X \) \( \bigwedge_{n < m} \neg(\exists \tilde{z}, \tilde{y}) E_n^m(\tilde{x}, \tilde{y}, X) \land E_n^2(\tilde{x}, X), \) where \( \{ \tilde{x} \} \cap \{ \tilde{y} \} = \emptyset; \)
   1.2. \( \forall U \) \( \bigwedge_{n < m} \neg(\exists \tilde{x}, \tilde{y}, r, s) E_n^{m+1}(\tilde{x}, \tilde{y}, r, U) \land E_n^{n+1}(\tilde{x}, s, U); \)
   1.3. \( \forall \tilde{x}, r, s \) \( (E_{n+1}^n(\tilde{x}, r, U) \land E_{n+1}^n(\tilde{x}, s, U)) \implies r = s; \)
2. Axioms of extensionality:
2.1 $(\forall \bar{x})(E^0_n(\bar{x}, X) \iff E^0_n(\bar{x}, Y)) \iff X = Y$;
2.2 $(\forall \bar{x}, r)(E^1_{n+1}(\bar{x}, r, U) \iff E^1_{n+1}(\bar{x}, r, V)) \iff U = V$;

3. Axioms of terms:
3.1 $(\forall \bar{x})(E^1_{n+1}(\bar{x}, 0, U_r) \lor E^1_{n+1}(\bar{x}, 1, U_r))$ if $\tau$ is $1(R(\bar{x}))$;
3.2 $(\forall \bar{x}, y)(E^2_{n+1}(\bar{x}, y, 0, U_r) \lor E^2_{n+1}(\bar{x}, y, 1, U_r))$ if $\tau$ is $1(x = y)$;
3.3 $E^\sigma_{n+1}(\tau, U_r)$ if $\tau$ is $r$;
3.4 $(\forall \bar{x}, r)(E^1_{n+1}(\bar{x}, r, U_r) \iff (\exists s)(\bigwedge_{s=1}^k E^1_{n+1}(\bar{x}, s, U_{r_i}) \land f(s_1, \ldots, s_k) = r))$ if $\tau$ is $\mathcal{F}(t_1, \ldots, t_k)$;
3.5 $(\forall \bar{x}, r)(E^1_{n+1}(\bar{x}, r, U_r) \iff (\exists V)((\forall y, s)(E^1_{n+1}(y, s, V) \iff \bigwedge_{s=1}^k E^1_{n+1}(\bar{x}, y, s, U_r) \land I_k(V, r)))$ if $\tau$ is $\int_k \sigma(v, u) dv_0, k = 1, 2$;

4. Axioms of satisfaction:
4.1 $(\forall \bar{x})(E^0_n(\bar{x}, X_\varphi) \iff (\exists r \geq 0)E^1_{n+1}(\bar{x}, r, U_r))$ if $\varphi$ is $\tau \geq 0$;
4.2 $(\forall \bar{x})(E^0_n(\bar{x}, X_{\varphi_\neg}) \iff \neg E^0_n(\bar{x}, X_{\varphi}))$;
4.3 $(\forall \bar{x})(E^0_n(\bar{x}, X_{\varphi_\land}) \iff \bigwedge_{\varphi \in \Phi} E^0_n(\bar{x}, X_{\varphi}))$;

5. Axioms of integral operators:
5.1 $(\forall U)((\bigwedge_{n \geq 2} \neg (\exists r, r)E^1_{n+1}(\bar{x}, r, U_r)) \iff (\exists s)I_k(U, s), k = 1, 2$;
5.2 $(\forall r)I_k(U, r, r)$;
5.3 $(\forall U, r, s)I_k(\bar{r} \cdot U + s \cdot V) = r \cdot I_k(U) + s \cdot I_k(V)$, where $I_k(U) = r$ iff $I_k(U, r)$;
5.4 $(\forall \bar{x})(\forall U)(\forall \varphi(\exists r \geq 0)E^1_{n+1}(\bar{x}, r, U_r) \iff (\exists r \geq 0)I_k(U, s))$;

6. Axiom of absolute continuity:
$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall U)(|I_2(U)| < \delta \implies |I_1(U)| < \varepsilon)$;

7. Axioms for an Archimedean field:

8. Transformations of axioms of $K^a_{A_1A_2}$:
$(\forall \bar{x})E^a_n(\bar{x}, X_\varphi)$, where $\varphi$ is an axiom of this logic;

9. Axioms of realizability of all sentences $\varphi$ of $T$:
$(\forall x_0)E^0_{1}(x_0, X_\varphi)$.

A weak model $(\mathfrak{A}, I_1, I_2)$ for $K^a_{A_1A_2}$ can be transformed to a standard model $\mathfrak{M}$ for $K^a_A$ by taking:
$E^\mathfrak{M}_n = \{ \bar{a} ∈ A^n : \mathfrak{A} \models \varphi[\bar{a}] \}$,
$U^\mathfrak{M}_n(\bar{a}) = \tau^\mathfrak{A}(\bar{a})$ for $\bar{a} ∈ A^n$ and $I^\mathfrak{M}_{I_k}(U^\mathfrak{M}_n) = I_k(\tau)$ for each term $\tau$ with at most one free variable. By the Barwise Compactness Theorem (see [1]), it can be shown that $S$ has a standard model $\mathfrak{D}$, because $S$ is $\Sigma$-definable over $\mathfrak{A}$ and $A_2$ holds in $\mathfrak{A}$. $\mathfrak{D}$ can be transformed to a middle model $\mathfrak{C}$ of $T$ by taking:

$R^\mathfrak{C} = \{ \bar{x} ∈ D^n : E^0_n(\bar{x}, X_{1(R(\bar{x}))-1}) \}$ and
$I^\mathfrak{C}_k(\tau(\bar{x}, \bar{a})) = I^\mathfrak{C}_k(U_{\tau(\bar{x}, \bar{a})})$ for $\bar{a} ∈ D^n$ and $k = 1, 2$.

This completes the proof of the Middle Completeness Theorem. □
In order to construct an absolutely continuous bprobability model, we need the following lemma.

**Lemma 2.** (Loeb [4]) In an $\omega_1$-saturated nonstandard universe, let $M$ be an internal vector lattice of functions from an internal set $A$ into $^{*}\mathbb{R}$ (the set of hyperreal numbers), and let $I$ be an internal positive linear functional on $M$, such that $1 \in M$ and $I(1) = 1$. Then there is a complete probability measure $\mu$ on $A$ such that for each finitely bounded $\varphi \in M$, the standard part of $\varphi$ is integrable with respect to $\mu$ and its integral is equal to the standard part of $I(\varphi)$.

**Theorem 1.** (Completeness Theorem for $L_{A,1}^{\omega}$) Let $T$ be a set of sentences of $L_{A,1}^{\omega}$ such that $T$ is $\Sigma_1$ on $A$ and consistent. Then there is an absolutely continuous bprobability model of $T$.

**Proof.** Let $(\mathfrak{A}, I_1, I_2)$ be a middle model of $T$ in which each theorem of $L_{A,1}^{\omega}$ is true. The Daniell integral construction from Lemma 2 produces probability measures $\mu_1, \mu_2$ on $^{*}A$ such that for each $*$-term $\tau(x)$, the standard part of $^{*}I_k(\tau)$ is the integral $\int_{^*} \tau(b) d\mu_k(b)$ (we define measures $\mu_n^{*}$ on $^{*}A$ by using iterated integrals). The absolute continuity in the middle model $\mathfrak{A}$ implies the absolute continuity for all measurable sets. Also, using axiom $A_3$, it can be shown that $\mu_1^n \ll \mu_2^n$ for each $n \in \mathbb{N}$. This graded bprobability model $\mathfrak{A} = (^{*}\mathfrak{A}, \mu_1^n, \mu_2^n)$ can be used to produce an absolutely continuous bprobability model of $T$ (see [3]). □

We can look only for a part of $L_{A,1}^{\omega}$ which satisfies the finite compactness property, because this logic cannot satisfy the full compactness (for example, each finite subset of $T = \{ \mu_1(1(R(x))) dx > 0 \} \cup \{ \mu_1(1(R(x))) dx \leq \frac{1}{n} : n \in \mathbb{N} \}$, where $R$ is a unary predicate, has a probability model, but not $T$ itself).

**Theorem 2.** Let $T$ be a set of sentences of $L_{A,1}^{\omega}$ of the form $\tau \in [r,s]$. If every finite subset of $T$ has a graded bprobability model, then $T$ has a graded bprobability model.

**Proof.** Let us suppose that each finite subset $\Psi \subseteq T$ has a model $\mathfrak{A}_\Psi$. By Lemma 1 we can suppose that $\mathfrak{A}_\Psi$ is a middle model. Take an ultraproduct $^{*}\mathfrak{A}$ such that, for each $\varphi \in T$, almost every $\mathfrak{A}_\Psi$ satisfies $\varphi$. Then form a graded bprobability model $\mathfrak{A}$ from $^{*}\mathfrak{A}$ by the Daniell integral construction (Lemma 2). It can be shown by induction that every sentence of $L_{A,1}^{\omega}$ of the form $\tau \in [r,s]$ which is true in almost all $\mathfrak{A}_\Psi$ holds in $\mathfrak{A}$, too. The absolute continuity condition can be expressed in the middle model by the first-order sentence

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall U)(|I_2(U)| < \delta \implies |I_1(U)| < \varepsilon).$$

By Los’s Theorem and Loeb construction the sentence

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall X)(\mu_2(X) < \delta \implies \mu_1(X) < \varepsilon)$$

holds in $\mathfrak{A}$. □
2. The logic $L^*_{A_1 I_2}$. Axioms and rules of inference for the logic $L^*_{A_1 I_2}$ are those of $L_{A I}$ (with both $I_1$ and $I_2$ in place of $I$, see [3]) together with the axioms of continuity $A_1$ and $A_2$:

(A4) Axiom of singularity:
$$\forall x \in A, \exists y \in A : \chi_{x \rightarrow y} \in \text{dom}(I_k)$$
where $H_k(s, t) = \begin{cases} 1, & \text{if } s \geq \frac{1}{k} \text{ and } t \geq \frac{2}{k} \\ 0, & \text{if } s \leq \frac{1}{k} \text{ or } t \leq \frac{1}{k} \\ \text{linear}, & \text{for other cases} \end{cases}$

**Theorem 3.** (Completeness Theorem for $L^*_{A_1 I_2}$) A theory $T$ of $L^*_{A_1 I_2}$ is consistent if and only if $T$ has a singular biprobability model.

**Proof.** The proof of soundness is easy. Let $(\mathcal{A}, I_1, I_2)$ be a weak model of $T$ in which each theorem of $L^*_{A_1 I_2}$ is true. Let $\mathcal{F} = \{ B \subseteq A : \chi_B \in \text{dom}(I_1) = \text{dom}(I_2) \}$ be an algebra of sets in $A$, where $\chi_B = \begin{cases} 1, & \text{if } x \in B \\ 0, & \text{if } x \notin B \end{cases}$.

Define finitely additive probability measures $\nu_1, \nu_2$ on $\mathcal{F}$ by $\nu_k(B) = I_k(\chi_B)$, $B \in \mathcal{F}$ and $k = 1, 2$.

Then, for $a \in A$, the singleton $\{ a \}$ belongs to $\mathcal{F}$ because $\chi_{\{ a \}} = (x = a)^A$, the set $B = \{ a \in A : \nu_1(a) > 0 \}$ belongs to $\mathcal{F}$ and $\nu_1(B) = 0$ $k = 1, 2$ by $A_4$.

By construction from [7], the measures $\nu_1, \nu_2$ can be extended so that $\nu_1 \subseteq \mathcal{P}_1$, $\nu_2 \subseteq \mathcal{P}_2$ and the measures $\mathcal{P}_1, \mathcal{P}_2$ are singular. Then construct a middle biprobability model $(\mathcal{A}, \mathcal{I}_1, \mathcal{I}_2)$ of $T$ by

$$\text{dom}(\mathcal{I}_k) = \text{dom}(I_k) \cup \{ \chi_C : C \in \mathcal{F} \} \quad \text{and} \quad \mathcal{I}_k(\chi_C) = \mathcal{P}_k(C),$$

for each $C$ from the extension $\mathcal{F}$ of $\mathcal{F}$.

By Loeb’s construction (Lemma 2) and the construction of the biprobability model from a graded biprobability model (see [3]), the singularity of finitely additive measures in the middle model will be preserved in the biprobability model. □

Finally, we prove Finite Compactness Theorem for the singular case.

**Theorem 4.** Let $T$ be a set of sentences of $L^*_{A_1 I_2}$ of the form $\tau \in [r, s]$. If every finite subset of $T$ has a graded biprobability model, then $T$ has a graded biprobability model.

**Proof.** As in Theorem 2, our proof is based on the ultraproduct and Daniell integral construction. Now, we can suppose that $\mathcal{A}_\psi$ is a weak model for each finite subset $\Psi \subseteq T$. Let $\mathcal{A}_\psi$ be a middle model as in Theorem 3. Take an ultraproduct $\mathcal{A} = \prod \mathcal{A}_\psi$ such that, for each $\varphi \in T$, almost every $\mathcal{A}_\psi$ satisfies $\varphi$. The condition of singularity can be express in the middle model by the first-order sentence $(\exists f)(I_1(f) = 1 \land I_2 = 0)$. By Los’s Theorem and Loeb’s construction the sentence $(\exists X)(\mu_1(X) = 1 \land \mu_2(X) = 0)$ holds in $\mathcal{A}$. □

**Logics with two types of integral operators.**
REFERENCES


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