TRACE FORMULA FOR NONNUCLEAR PERTURBATIONS
OF SELFADJOINT OPERATORS

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Abstract. The trace formulas for the operator \( \varphi(H_1) - \varphi(H_0) \) are deduced when \( H_1 - H_0 \)

is a nonnuclear operator and \( \varphi \) is an enough wide class of functions.

1. Introduction. Suppose \( B(\mathcal{H}) \) is the algebra of all bounded operators over the Hilbert space \( \mathcal{H} \). Denote by \( C_p \) and \( \| \cdot \|_p \) the Neumann-Schatten class of
operators and their norm [2]. For an operator \( W \in C_2 \) by \( \text{det}_2(I + W) \) we denote
its regularized determinant.

Let \( H_1 \) and \( H_0 \) be selfadjoint operators (possibly unbounded) on a Hilbert
space \( \mathcal{H} \). If \( H_1 - H_0 = V \) is a nuclear operator and \( \varphi \) is an element in a sufficiently
large class of functions, then Krein [3,4] proved that \( \varphi(H_1) - \varphi(H_0) \) is a nuclear
operator and that

\[
\text{trace } (\varphi(H_1) - \varphi(H_0)) = \int_R \xi(\lambda)\varphi'(\lambda)d\lambda, \quad (1.1)
\]

where \( \xi \) is the real function in \( L^1(R) \) uniquely determined by \( H_0 \) and \( H_1 \). Usually,
the relation (1.1) is called the trace formula.

Krein also proved the following relation between the function and the perturbation
determinant:

\[
\det (I + V(H_0 - z)^{-1}) = \exp \left( \int_R \frac{\xi(\lambda)}{\lambda - z} \, d\lambda \right), \quad \text{Im} z \neq 0.
\]

Koplienko [5] extended the trace formula (1.1) to the case when \( H_1 - H_0 \)
is not a nuclear operator. The trace formula was deduced when \( \varphi \) is a rational
function with the poles in \( C \backslash R \) and \( |\varphi(\infty)| < \infty \).

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In this paper we prove regularized trace formulas for a nonnuclear perturbation of selfadjoint operators in the case when $\varphi$ is not an analytic function. Our method is different from the method given in [6]. The following theorem is a result of Koplinko:

**Theorem 1.1** Let $H_0, V$ be selfadjoint operators, $V \in C_2$ and let $\varphi$ be a rational function with poles lying in $C \setminus R$ and $|\varphi(\infty)| < \infty$. Then

1. $R_2 = \varphi(H_0 + V) - \varphi(H_0) - \frac{d}{dx}\varphi(H_0 + xV)|_{x=0} \in C_1$
2. There exists a real function $\sigma$ (depending only on $H_0$ and $V$) of bounded variation such that
   \[
   \text{trace } R_2 = \int_R \varphi''(\lambda) d\sigma(\lambda)
   \]
3. $V_{-\infty}^\infty \leq |V|^2 / 2!$ (\(V_{-\infty}^\infty f\) is the variation of $f$ on \((-\infty, \infty)\)).
4. $\det_2 \left( I + V(H_0 - z)^{-1} \right) = \exp \left( -\int R \frac{d\sigma(\lambda)}{|\lambda - z|^2} \right)$.

**2. Results.** Let $\mathcal{M}_p$ be the set of all the functions $\varphi$ of the form $\varphi(x) = \int_R e^{itg(t)}dt$ where $g$ is a measurable function such that $\int_R |t|^{p} |g(t)|dt < \infty$ for $\nu = 0, 1, 2, \ldots, p$ ($p \geq 2$). The functions from $\mathcal{M}_p$ are not necessarily analytic, and the set $\mathcal{M}_p$ contains the class of functions from Theorem 1.

Recall that if $A, B \in B(\mathcal{H})$, then [1] we have

\[
e^{(A + B)t} = e^{At} + \sum_{n=1}^{\infty} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} e^{A(t - s_1)} Be^{A(s_1 - s_2)} B \cdots Be^{A(s_{n-1} - s_n)} B e^{A s_n} ds_1 ds_2 \ldots ds_n
\]

(The series converges in $B(\mathcal{H})$).

**Lemma 2.1** If $H_0 = H_0^*$, $V = V^* \in B(\mathcal{H})$, $t, x \in R$, then

\[
e^{it(H_0 + xV)} = e^{itH_0} + \sum_{n=1}^{\infty} \frac{i^n x^n B_n(t)}{n!},
\]

where

\[
B_n(t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} e^{itH_0} V e^{itH_0(s_1 - \varepsilon)} V \cdots V e^{itH_0(s_{n-1} - \varepsilon)} V e^{itH_0 s_n} ds_1 ds_2 \ldots ds_n
\]

**Proof.** If $H_0 \in B(\mathcal{H})$, then (2.11) and (2.12) follow directly from (2.01). So, suppose that $H_0$ is an unbounded operator. Set $H^{(k)} = H_0 E(-k, k)$, where $E$ is the spectral measure of $H_0$. The operator $H^{(k)}$ is bounded and we have

\[
\lim_{k \to \infty} \left( \lambda - H^{(k)} \right)^{-1} = \left( \lambda - H_0 \right)^{-1} h \quad (h \in \mathcal{H}, \ \Im \lambda \neq 0)
\]
i.e. the sequence $H^{(k)}$ converges to $H_0$ in the strong resolvent sense. By the Trotter Theorem [7] we have

$$\lim_{k \to \infty} e^{itH^{(k)}} h = e^{itH_0} h, \ h \in \mathcal{H}, \ t \in \mathbb{R}. \quad (2.1.3)$$

Setting $H^{(k)}$ instead of $H_0$ in (2.1.1), (2.1.2) and letting $k$ tend to infinity, we complete the proof of Lemma 1.

**Lemma 2.2** If $H_0 = H_0^*$, $V = V^* \in B(\mathcal{H})$, $x,t \in \mathbb{R}$, then for $k = 0, 1, 2, \ldots$ the following inequality holds

$$\left\| \frac{d^k}{dx^k} e^{it(H_0 + xV)} \right\| \leq |t|^k \|V\|^k. \quad (2.2.1)$$

**Proof.** Since

$$\frac{d^k}{dx^k} e^{it(H_0 + xV)} = \frac{d^k}{d\xi^k} e^{i(H_1 + \xi V)}|_{\xi = 0} (H_1 = H_0 + xV),$$

by Lemma 2.1, it follows

$$\frac{d^k}{d\xi^k} e^{i(H_1 + \xi V)}|_{\xi = 0} = i^k k! \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-1}} e^{iH_1(t-s_1)} V \ldots V e^{iH_1(s_n-t)} ds_1 ds_2 \ldots ds_n$$

From the previous equality we have

$$\left\| \frac{d^k}{dx^k} e^{it(H_0 + xV)} \right\| \leq k! \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-1}} \|V\|^k ds_1 ds_2 \ldots ds_n = |t|^k \|V\|^k, \ (t > 0)$$

and the proof is complete.

**Lemma 2.3** If $H_0 = H_0^*$, $x,t \in \mathbb{R}$, $V = V^* \in B(\mathcal{H})$ and $\varphi \in \mathcal{M}_p$ (i.e. $\varphi(x) = \int_\mathbb{R} e^{ixg(t)} dt$, $g$ is a measurable function), then

$$\frac{d^k}{dx^k} \varphi(H_0 + xV) = \int_\mathbb{R} \frac{d^k}{dx^k} e^{i(H_0 + xV)} g(t) dt. \quad (2.3.1)$$

**Proof.** For $k = 0$, (2.3.1) follows from the spectral theorem. If $k = 1$ we have

$$\frac{d}{dx} \varphi(H_0 + xV) = \lim_{\xi \to \infty} \int_\mathbb{R} \left( e^{i(H_0 + (x + \xi)V)} - e^{i(H_0 + xV)} \right) \frac{g(t)}{\xi} dt. \quad (2.3.2)$$

Since

$$e^{i(H_0 + xV)} - e^{i(H_0 + x_1V)} = \int_{x_1}^{x_2} \frac{\partial}{\partial x} e^{i(H_0 + xV)} dx$$
by Lemma 2.2 we obtain
\[ \left\| e^{it(H_0 + x_2 V)} - e^{it(H_0 + x_1 V)} \right\| \leq |t||V||x_2 - x_1|. \] (2.3.3)

From (2.3.2), (2.3.3) and \( \int_R |t||g(t)| dt < \infty \), by the Theorem of Dominated Convergence, we conclude that
\[
\frac{d}{dx} \varphi (H_0 + x V) = \int_R \frac{d}{dx} e^{it(H_0 + x V)} g(t) dt.
\]

Repeating this procedure several times we prove Lemma 2.3.

From now on we assume \( V = V^* \in C_p \ (p \geq 2, \ p \in \mathbb{N}) \).

**Lemma 2.4** If \( x, t \in R, \ H_0 = H_0^* \), then \( \frac{d^p}{dx^p} e^{it(H_0 + x V)} \) is a nuclear operator and
\[
\left| \frac{d^p}{dx^p} e^{it(H_0 + x V)} \right| \leq |t|^p |V|^p.
\]

**Proof.** Since
\[
\frac{d^p}{dx^p} e^{it(H_0 + x V)} = i^p p! \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{p-1}} e^{it_1 H_1(t-s_1) V} \cdots V e^{it_{p-1} s_{p-1} - V} V e^{it_1 s_1} ds_1 ds_2 \cdots ds_n
\]
where \( H_1 = H_0 + x V \) and \( V \in C_p \) we get \( \frac{d^p}{dx^p} e^{it(H_0 + x V)} \in C_1 \) and
\[
\left| \frac{d^p}{dx^p} e^{it(H_0 + x V)} \right| \leq p! |V|^p \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{p-1}} ds_1 ds_2 \cdots ds_n = t^p |V|^p \ (t > 0).
\]

Set
\[
U_p(t) = e^{it(H_0 + V)} - \sum_{k=0}^{p-1} \frac{1}{k!} \frac{d^k}{dx^k} e^{it(H_0 + x V)} \bigg|_{x=0}
\]

Now, by Lemma 2.3
\[
R_p \stackrel{def}{=} \varphi (H_0 + V) - \sum_{k=0}^{p-1} \frac{1}{k!} \frac{d^k}{dx^k} \varphi (H_0 + x V) \bigg|_{x=0} = \int_R U_p(t) g(t) dt.
\]

**Lemma 2.5** If \( H_0 = H_0^* \), \( V = V^* \in C_p, \ t \in R, \) then \( U_p(t) \in C_1, \ R_p \in C_1 \) and
\[
\text{trace } R_p = \int_R g(t) \text{ trace } U_p(t) dt \quad (2.5.1)
\]
Proof. Let \( m(x) = e^{it(H_0 + xV)} \), (where \( t \) is a fixed real number and \( x \in \mathbb{R} \)). By the Taylor theorem we obtain
\[
m(1) = \sum_{\nu=0}^{p-1} \frac{m^{(\nu)}(0)}{\nu!} + \frac{1}{(p-1)!} \int_0^1 m^{(p)}(x)(1-x)^{p-1} dx
\]
i.e.
\[
U_p(t) = \frac{1}{(p-1)!} \int_0^1 m^{(p)}(x)(1-x)^{p-1} dx
\]
Since \( m^{(p)}(x) \in C^1 \) (by Lemma 2.4) and \( m^{(p)} \) is a continuous function in \( C_1 \) norm, we have \( U_p(t) \in C_1 \) (\( \forall t \in \mathbb{R} \)). By Lemma 2.4 we get
\[
|U_p(t)|_1 \leq \frac{1}{(p-1)!} \int_0^1 |m^{(p)}(x)|_1 (1-x)^{p-1} dx \leq \frac{1}{(p-1)!} \int_0^1 |t|^p |V_p|^p(1-x)^{p-1} dx
\]
i.e.
\[
|U_p(t)|_1 \leq |V_p|^p |t|^p / p!.
\] (2.5.2)
Hence \( R_p \) is a compact operator.

Let \( S \) be a unitary operator and suppose \( \{e_i\} \) is an orthonormal set. Then
\[
\left| \sum_{i=1}^r (SR_p e_i, e_i) \right| = \left| \sum_{i=1}^r \left( \int_R g(t)SU_p(t)e_i, e_i \right) \right| = \left| \int_R g(t) \sum_{i=1}^r (SU_p(t)e_i, e_i) dt \right|
\]
\[
\leq \left( \int_R |g(t)| \right) \sum_{i=1}^r |s_i(U_p(t))| dt \leq \int_R |g(t)||U_p(t)|_1 dt
\]
\[
\leq \int_R |g(t)||t|^p dt \cdot \frac{|V_p|^p}{p!}. \quad \text{(by 2.5.2)}
\]
So,
\[
\left| \sum_{i=1}^r (SR_p e_i, e_i) \right| \leq \frac{|V_p|^p}{p!} \int_R |t|^p |g(t)| dt.
\]
When we take the supremum over all unitary operators \( S \) and over all orthonormal sets \( \{e_i\} \), we obtain
\[
\sum_{i=1}^r s_i(R_p) \leq \frac{|V_p|^p}{p!} \int_R |t|^p |g(t)| dt,
\]
i.e.
\[
R_p \in C_1, \ |R_p|_1 \leq \frac{|V_p|^p}{p!} \int_R |t|^p |g(t)| dt \quad \text{and \ trace } R_p = \int_R g(t) \ \text{trace } U_p(t) dt.
\]
This completes the proof.

**Lemma 2.6** If $H_0 = H_0^* \in B(H)$, $V = V^* \in C_p$ and $\Gamma = \{ \lambda : |\lambda| = 1 + \|H_0\| + \|V\| \}$, then
\[
U_p(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} G_\lambda(\lambda) d\lambda
\]
where $G_\lambda(\lambda) = (\lambda - H_0)^{-1} (V(\lambda - H_0)^{-1})^p (I - V(\lambda - H_0)^{-1})^{-1}$.

**Proof.** From $H_0 = H_0^*$ we have $\|V(\lambda - H_0)^{-1}\| \leq \|V\|(1 + \|V\|)^{-1} < 1$ for every $\lambda \in \Gamma$. Hence $(\lambda - H_0 - xV)^{-1} = \sum_{k=0}^{\infty} x^k(\lambda - H_0)^{-1}(V(\lambda - H_0)^{-1})^k$; it follows that
\[
\frac{d^k}{dx^k} (\lambda - H_0 - xV)^{-1} \big|_{x=0} = k!(\lambda - H_0)^{-1} \left( V(\lambda - H_0)^{-1} \right)^k
\]
and
\[
\frac{1}{k!} \frac{d^k}{dx^k} e^{it(H_0+xV)} \big|_{x=0} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda - H_0)^{-1} \left( V(\lambda - H_0)^{-1} \right)^k d\lambda \tag{2.6.2}
\]
Since $e^{it(H_0+V)} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda - H_0 - V)^{-1} d\lambda$ from (2.6.2) it follows
\[
U_p(t) = \frac{1}{2\pi i} \int_\Gamma \frac{e^{\lambda t}}{\lambda - H_0} \sum_{k=p}^{\infty} (\lambda - H_0)^{-1} \left( V(\lambda - H_0)^{-1} \right)^k d\lambda = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} G_\lambda(\lambda) d\lambda
\]

**Lemma 2.7** Let $H_0 = H_0^*$ be an unbounded operator and $V = V^* \in C_p$. If $H_0^{(n)} = H_0 E((-n,n))$ ($E$ is the spectral measure of $H_0$) and
\[
U_p^{(n)}(t) = e^{it(H_0^{(n)}+V)} - \sum_{k=0}^{p-1} \frac{1}{k!} \frac{d^k}{dx^k} e^{it(H_0^{(n)}+xV)} \big|_{x=0}
\]
then for every $t \in R$
\[
\lim_{n \to \infty} \left| U_p(t) - U_p^{(n)}(t) \right|_1 = 0. \tag{2.7.1}
\]

**Proof.** Since
\[
U_p(t) = \frac{1}{(p-1)!} \int_0^1 m^{(p)}(x)(1-x)^{p-1} dx
\]
and
\[
U_p^{(n)}(t) = \frac{1}{(p-1)!} \int_0^1 m^{(p)}_n(x)(1-x)^{p-1} dx
\]
where \( m(x) = e^{it(H_0 + xV)} \) and \( m_n(x) = e^{it(H_0^n + xV)} \), we obtain
\[
\left| U_p(t) - U_p^{(n)}(t) \right|_1 \leq \frac{1}{(p-1)!} \int_0^1 \left| m^{(p)}(x) - m_n^{(p)}(x) \right| (1 - x)^{p-1} dx
\]

From (2.1.1), (2.1.2) and [2, p. 119], Theorem 6.3 we have
\[
\lim_{n \to \infty} \left| m^{(p)}(x) - m_n^{(p)}(x) \right|_1 = 0, \ x \in (0, 1).
\]

Now, since \( |m^{(p)}(x)|_1 \leq \|t\|^p \|V\|^p_p \) and \( |m_n^{(p)}(x)|_1 \leq \|t\|^p \|V\|^p_p \), (2.7.1) follows from the Lebesgue Dominated Convergence Theorem.

**Theorem 2.8** If \( V \in \mathcal{C}_2 \), \( H_0 = H_0^* \) and \( \varphi \in \mathcal{M}_2 \), then

1. \( R_2 = \varphi(H_0 + V) - \varphi(H_0) - \frac{d}{dx} \varphi(H_0 + xV) |_{x=0} \in \mathcal{C}_1 \)
2. There exists a real function \( \sigma \) of bounded variation (\( \sigma \) depends only on \( H_0 \) and \( V \)) such that \( \text{trace} \ R_2 = \int_R \sigma(\lambda) d\sigma(\lambda) \)
3. \( V^{-\infty}_{-\infty} \sigma \leq \|V\|_{\mathcal{L}}^2 \)
4. \( \det_2 \left( I + V (H_0 - z)^{-1} \right) = \exp \left( -\int_R \frac{d\sigma(\lambda)}{(\lambda - z)^2} \right) \).

**Proof.** (1) follows from Lemma 2.5. Let \( H_0 \) be a bounded operator, \( \lambda_0 \in \mathcal{C} \setminus \mathcal{R} \) and \( |\lambda_0| > 1 + \|H_0\| + \|V\| \). Now, by Runge’s Theorem there exists a sequence of polynomials \( P_n \) such that \( r_n(\lambda) = P_n(1/(\lambda - \lambda_0)) \Rightarrow e^{i\lambda t} \) on \( D = \{\lambda : |\lambda| \leq 1 + \|H_0\| + \|V\|\} \). Hence
\[
r_n^{(\nu)}(\nu) = (it)^\nu e^{i\nu} (\nu = 0, 1, 2, \ldots) \text{ on } D. \quad (2.8.1)
\]

Let \( R^{(n)}_2 = r_n(H_0 + V) - r_n(H_0) - \frac{d}{dx} r_n(H_0 + xV) |_{x=0} \). As in Lemma 2.6, we get
\[
R^{(n)}_2 = \frac{1}{2\pi i} \int_R r_n(\lambda) G_2(\lambda) d\lambda.
\]

Now, by Lemma 2.6 we have
\[
\left| U_2(t) - R^{(n)}_2(t) \right| \leq \frac{1}{2\pi} \int_\Gamma \left| e^{i\lambda t} - r_n(\lambda) \right| G_2(\lambda) d\lambda
\]
and
\[
\left| U_2(t) - R^{(n)}_2(t) \right|_1 \leq \frac{1}{2\pi} \int_\Gamma \left| e^{i\lambda t} - r_n(\lambda) \right| \max_{\lambda \in \Gamma} |G_2(\lambda)|_1 \cdot |d\lambda| \to 0 \ (n \to \infty).
\]

Hence \( \lim_{n \to \infty} \text{trace} \ R^{(n)}_2 = \text{trace} \ U_2(t) \). By Theorem 1, there exists a function \( \sigma \) of bounded variation (depending only on \( H_0 \) and \( V \)) such that \( V^{-\infty}_{-\infty} \sigma \leq \|V\|_{\mathcal{L}}^2/2! \) and
\[
\text{trace} \ R^{(n)}_2 = \int_R r''(\sigma) d\sigma(s) \quad (2.8.2)
\]
From (2.8.1) and (2.8.2) it follows
\[ \text{trace } U_2(t) = \int_R (it)^2 e^{it^2} d\sigma(s). \quad (2.8.3) \]

From (2.5.1) and (2.8.3) we conclude trace \( R_2 = \int \varphi^p(s) d\sigma(s). \)

Now consider the case when \( H_0 \) is not a bounded operator. Lemma 2.7 gives
\[ \lim_{n \to \infty} \text{trace } U_2^{(n)}(t) = \text{trace } U_2(t). \quad (2.8.4) \]

From (2.8.3) it follows
\[ \text{trace } U_2^{(n)}(t) = \int_R (it)^2 e^{it^2} d\sigma_n(s) \text{ and } V_{-\infty}^\infty \sigma_n \leq \frac{|V_2|^2}{2}. \]

By (2.8.4) and the Helly election Theorem there exists a function \( \sigma \) of bounded variation \( (V_{-\infty}^\infty \sigma \leq |V_2|^2/2!) \) such that
\[ \text{trace } U_2(t) = \int_R (it)^2 e^{it^2} d\sigma(s) \text{ and } \text{trace } R_2 = \int_R \varphi^p(s) d\sigma(s). \]

The property (4) we obtain similarly as in [5].

**Theorem 2.9** If \( H_0 = H_0^* \in B(\mathcal{H}), V = V^* \in C_p \ (p \geq 3, \ p \in N) \) and \( \varphi \in \mathcal{M}_{p+1}, \) then
\[ \text{trace } R_p = \frac{\text{trace } V^p}{p!} \varphi^{(p)}(0) + \int_R \varphi^{(p+1)}(x) \gamma(x) dx \]

where \( \gamma \in L^2(R) \) is a function which depends only on \( V, H_0 \) and \( p. \)

**Proof.** From Lemma 2.6 it follows that \( f(t) = \text{trace } U_p(t) \) is an entire function of the exponential type. On the other hand from (2.5.2) we get \( |f(t)| \leq |t|^p |V|^p_p / p! \) for every \( t \in R. \) Hence \( f(0) = f'(0) = \cdots = f^{(p-1)}(0) = 0. \) From (2.6.1) we obtain
\[ \lim_{t \to 0} \frac{f(t)}{(it)^p} = \frac{\text{trace } V^p}{p!} \]

So, \( \frac{1}{t} \left( \frac{f(t)}{(it)^p} - \frac{\text{trace } V^p}{p!} \right) \in L^2(R) \) is an entire function of the exponential type. Now, by the Paley-Wiener Theorem we have
\[ \frac{f(t)}{(it)^p} - \frac{\text{trace } V^p}{p!} = it \int R e^{it^2} \gamma(s) ds \]

for some \( \gamma \in L^2(R) \) and
\[ f(t) = (it)^{p+1} \int R e^{it^2} \gamma(s) ds + (it)^p \frac{\text{trace } V^p}{p!}. \quad (2.9.1) \]
The proof of Theorem 2.9 follows from (2.5.1) and (2.9.1).

**Corollary 2.10.** If \( H_0 = H_0^* \) is an unbounded operator, \( V = V^* \in C_p \) (\( p \geq 3 \)), then there exists a sequence \( \gamma_n \in L^2(R) \) such that for every \( \varphi \in \mathcal{M}_{p+1} \)

\[
\text{trace } R_p = \frac{\text{trace } V^p}{p!} \varphi^{(p)}(0) + \lim_{n \to \infty} \int_R \varphi^{(p+1)}(x) \gamma_n(x) dx.
\]

**Proof.** Since

\[
R_p^{(n)} = \varphi \left( H_0^{(n)} + V \right) - \sum_{k=0}^{p-1} \frac{1}{k!} \frac{d^k}{dx^k} \varphi \left( H_0^{(n)} + xV \right) \bigg|_{x=0} \left( H_0^{(n)} = H_0 E(-n,n) \right)
\]

\[= \int_R g(t) U_p^{(n)}(t) dt,
\]

by Lemma 2.7 we get

\[
\lim_{n \to \infty} \text{trace } R_p^{(n)} = \text{trace } R_p. \tag{2.10.1}
\]

By Theorem 2.9 there exists a sequence \( \gamma_n \in L^2(R) \) such that

\[
\text{trace } R_p^{(n)} = \frac{\text{trace } V^p}{p!} \varphi^{(p)}(0) + \int_R \varphi^{(p+1)}(x) \gamma_n(x) dx
\]

and the proof follows by (2.10.1).

**REFERENCES**


