ON TOPOLOGICAL SPACES WITH DENSE COMPLETELY METRIZABLE SUBSPACES

T. Nagamizu

Dedicated to Professor Akihiro Okuyama on his 60th birthday

Abstract. We obtain some characterizations for the spaces which have a dense completely metrizable subspace and some results related with these spaces.


We know that for a metrizable space $X$, the following statements are equivalent [7, Proposition 4.4]:

(1) $X$ is a almost complete space,

(2) $X$ has a dense completely metrizable subspace.

The first purpose of this paper is to obtain some characterizations for the spaces which have a dense completely metrizable subspace and some results related with these spaces.

Arhangel'skii and Kocinac asked several questions on weakly perfect spaces and spaces with dense $G_δ$-diagonal [1]. The second purpose of this paper is to give answers to their Questions 8 and 9.

2. Definitions and notations. All considered spaces are completely regular. A sequence $\{U_n| n \in \mathbb{N}\}$ of subsets of a space $X$ is said to be complete if every filter base $F$ on $X$ which is controlled* by $\{U_n| n \in \mathbb{N}\}$ clusters at some $x \in X$.

A sequence $\{U_n| n \in \mathbb{N}\}$ of collections of subsets of $X$ is said to be complete if $\{U_n| n \in \mathbb{N}\}$ is a complete sequence whenever $U_n \in \mathcal{U}_n$ for all $n \in \mathbb{N}$.

A collection $\mathcal{U}$ of subsets of a space $X$ is said to be an almost cover if $\bigcup \mathcal{U}$ is dense in $X$. Let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets of $X$. We say that $\mathcal{V}$ is a strong

* $F$ is controlled by $\{U_n\}$ if each $U_n$ contains some $F \in F$. 
refinement of $\mathcal{U}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$ and for each element $V \in \mathcal{V}$ there exists an element $U \in \mathcal{U}$ with $\text{Cl}(V) \subset U$.

The following lemma is proved in [7, Lemma 4.6].

**Lemma 2.1.** If $X$ has a complete sequence $\{\mathcal{U}_n| n \in \mathbb{N}\}$ of open almost covers, then there exists a complete sequence $\{\mathcal{V}_n| n \in \mathbb{N}\}$ of disjoint open almost covers of $X$ such that $\mathcal{V}_{n+1}$ is a strong refinement of $\mathcal{V}_n$ for each $n \in \mathbb{N}$.

Let $\mathcal{U}$ be a collection of subsets of $X$. $\mathcal{U}$ is said to separate points of $X$ if $x$ and $y$ are distinct points of $X$, then there exists different elements $U_x$ and $U_y$ in $\mathcal{U}$ such that $x \in U_x$ and $y \in U_y$. $\mathcal{U}$ is said to have a finite intersection property (f.i.p.) if every finite subcollection of $\mathcal{U}$ have a nonempty intersection.

3. Characterizations. The main purpose in this section is to prove Theorem 3.4.

**Lemma 3.1.** Let $X$ has a complete sequence $\{\mathcal{U}_n| n \in \mathbb{N}\}$ of open almost covers such that for each sequence $\{U_n| U_n \in \mathcal{U}_n, n \in \mathbb{N}\}$ with f.i.p., the set $\bigcap\{\text{Cl}(U_n)| n \in \mathbb{N}\}$ is a singleton. Then there exists a complete sequence $\{\mathcal{V}_n| n \in \mathbb{N}\}$ of disjoint open almost covers of $X$ such that

(i) $\mathcal{V}_{n+1}$ is a strong refinement of $\mathcal{V}_n$ for each $n \in \mathbb{N}$, and

(ii) for each decreasing sequence $\{V_n| V_n \in \mathcal{V}_n, n \in \mathbb{N}\}$, the set $\bigcap\{\text{Cl}(V_n)| n \in \mathbb{N}\}$ is a singleton.

**Proof.** From Lemma 2.1, there exists a complete sequence $\{\mathcal{V}_n| n \in \mathbb{N}\}$ of disjoint open almost covers of $X$ such that $\mathcal{V}_{n+1}$ is a strong refinement of $\mathcal{V}_n$ and $\mathcal{U}_n$ for each $n \in \mathbb{N}$.

Let $\{V_n| n \in \mathbb{N}\}$ is a decreasing sequence where $V_n \in \mathcal{V}_n$ for each $n \in \mathbb{N}$. By the construction of $\mathcal{V}_n$, for each $n \in \mathbb{N}$, there exists $U_n \in \mathcal{U}_n$ such that $\text{Cl}(V_{n+1}) \subset V_n \cap U_n$. Since $\{V_n| n \in \mathbb{N}\}$ is decreasing, $\{U_n| n \in \mathbb{N}\}$ has f.i.p.

For each $n \in \mathbb{N}$, we put $F_n = \text{Cl}(V_{n+1})$. By completeness of $\{\mathcal{V}_n| n \in \mathbb{N}\}$, we have:

$$\emptyset \neq \bigcap_{n \in \mathbb{N}} F_n = \bigcap_{n \in \mathbb{N}} V_n = \bigcap_{n \in \mathbb{N}} \text{Cl}(V_n) = \bigcap_{n \in \mathbb{N}} \text{Cl}(U_n).$$

Since $\bigcap_{n \in \mathbb{N}} \text{Cl}(U_n)$ is a singleton, $\bigcap_{n \in \mathbb{N}} \text{Cl}(V_n)$ is also a singleton. \(\square\)

**Theorem 3.2.** Let $X$ have a complete sequence $\{\mathcal{U}_n| n \in \mathbb{N}\}$ of disjoint open almost covers such that

(i) $\mathcal{U}_{n+1}$ is a strong refinement of $\mathcal{U}_n$ for each $n \in \mathbb{N}$, and

(ii) the set $\bigcap_{n \in \mathbb{N}} \text{Cl}(U_n)$ is a singleton for each decreasing sequence $\{U_n| U_n \in \mathcal{U}, n \in \mathbb{N}\}$.

Then $X$ has a dense $G_\delta$ completely metrizable subspace.

**Proof.** By [7, Proposition 4.5], $X$ is a Baire space. Since $G_n = \bigcup \mathcal{U}_n$ is an open dense subset in $X$ for each $n \in \mathbb{N}$, then $M = \bigcap_{n \in \mathbb{N}} G_n$ is a dense $G_\delta$ set in $X$. 

By the condition (ii), if $x$ and $y$ are two distinct points of $M$, then there exists $n \in \mathbb{N}$ such that $U_n$ separates $x$ and $y$.

Let us define the metric $\rho$ on $M$ by

$$\rho(x, y) = \begin{cases} 0, & x = y \\ \min\{n \mid U_n \text{ separates } x \text{ and } y\}^{-1}, & \text{otherwise} \end{cases}$$

It is easy to check that $\rho$ is a complete metric on $M$, by the condition (i) and (ii). Moreover, $U_n \cap M$ is a $1/n$-open ball at $x$ for each $U_n \in \mathcal{U}_n$ and $x \in U_n \cap M$. Hence the original topology on $M$ is stronger than $\rho$-topology.

Now we show the next claim.

**Claim.** Let $F$ be a closed subset of $M$ and $x \in M \setminus F$. Then there exist $n \in \mathbb{N}$ and $U_n \in \mathcal{U}_n$ such that $x \in U_n$ and $U_n \cap F = \emptyset$.

Proof of the claim. Suppose that $U_n \cap F \neq \emptyset$ whenever $x \in U_n$ for each $n \in \mathbb{N}$. Pick a point $x_n$ in $U_n \cap F$, and put $F_n = \text{Cl}\{x_m \mid m \geq n + 1\}$ for each $n \in \mathbb{N}$. Then by the condition (i), $\{U_n \mid n \in \mathbb{N}\}$ is a decreasing sequence, and $\{x\} = \bigcap_{n \in \mathbb{N}} U_n$, by the condition (ii). It follows that

$$\emptyset \neq \bigcap_{n \in \mathbb{N}} F_n \subset \bigcap_{n \in \mathbb{N}} U_n = \{x\}.$$  

Hence $x \in F$. This is a contradiction, and the claim is proved.

By the claim, $\rho$-topology is stronger than the original topology. It follows that $M$ is a dense $G_δ$ completely metrizable subspace. The proof is complete. □

**Theorem 3.3.** Let $X$ be a space with a dense completely metrizable subspace. Then there exists a complete sequence $\{U_n \mid n \in \mathbb{N}\}$ of open almost covers of $X$ such that for each sequence $\{U_n \mid U_n \in \mathcal{U}_n, n \in \mathbb{N}\}$ with the f.i.p., the set $\bigcap_{n \in \mathbb{N}} \text{Cl}(U_n)$ is a singleton.

Proof. Let $M$ be a dense completely metrizable subspace of $X$ and $\rho$ a compatible metric on $M$. Let $U(x, n)$ be an open subset of $X$ such that $B(x, 1/n) = U(x, n) \cap M$ for each $x \in M$ and $n \in \mathbb{N}$, where $B(x, 1/n) = \{y \in M \mid \rho(x, y) < 1/n\}$ be a $1/n$-open ball in $M$. Then for each $n \in \mathbb{N}, U_n = \{U(x, n) \mid x \in M\}$ is an open almost cover of $X$.

Now we show that $\{U_n \mid n \in \mathbb{N}\}$ is a complete sequence. Let $\{U_n \mid U_n \in \mathcal{U}_n, n \in \mathbb{N}\}$ be a sequence and $\mathcal{F}$ a filter base on $X$ which is controlled by $\{U_n \mid n \in \mathbb{N}\}$. Then for each $n \in \mathbb{N}$, there exists $F_n \in \mathcal{F}$ such that $F_n \subset U_n$. By the construction of $\mathcal{U}_n$, there exists $x_n$ such that $x_n \in M$, $U_n = U(x_n, n)$ for each $n \in \mathbb{N}$. Since $\{U_n \mid n \in \mathbb{N}\}$ has the f.i.p., it follows that $\{x_n \mid n \in \mathbb{N}\}$ is a $\rho$-Cauchy sequence. Then there exists $x_0 \in M$ such that $\{x_n \mid n \in \mathbb{N}\}$ converges to $x_0$. Therefore we have that $x_0 \in \bigcap_{n \in \mathbb{N}} \text{Cl}(F)$ if $F \in \mathcal{F}$. Hence $\{U_n \mid n \in \mathbb{N}\}$ is a complete sequence.

In the same way, it is easy to see that $\bigcap_{n \in \mathbb{N}} \text{Cl}(U_n) = \{x_0\}$. The proof is complete. □
These results lead to the following theorem.

**Theorem 3.4.** For the space $X$, the following conditions are equivalent.

1. $X$ has a complete sequence $\{U_n\}_{n \in \mathbb{N}}$ of open almost covers such that for each sequence $\{U_n\}_{n \in \mathbb{N}}$ with f.i.p., the set $\bigcap_{n \in \mathbb{N}} \text{Cl}(U_n)$ is a singleton.

2. $X$ has a complete sequence $\{U_n\}_{n \in \mathbb{N}}$ of disjoint open almost covers such that
   
   (i) $U_{n+1}$ is a strong refinement of $U_n$ for each $n \in \mathbb{N}$,
   
   (ii) for each decreasing sequence $\{U_n\}_{n \in \mathbb{N}}$, the set $\bigcap_{n \in \mathbb{N}} \text{Cl}(U_n)$ is a singleton.

3. $X$ has a dense $G_\delta$ completely metrizable subspace.

4. $X$ has a dense completely metrizable subspace.

A space $X$ is said to be a **Namioka space** if the following condition is satisfied:

(5) for any compact space $Y$ and any separately continuous function $f: X \times Y \to \mathbb{R}$, there exists a dense $G_\delta$ subset $A \subset X$ such that $f$ is jointly continuous at each point of $A \times Y$.

Next we consider the following game. Let $\alpha$ and $\beta$ be two players with $\beta$ the first to move. $\beta$ starts by choosing a nonempty open subset $U_1 \subset X$. Then $\alpha$ chooses an open subset $V_1 \subset U_1$ and a point $x_1 \in V_1$, then $\beta$ chooses a nonempty open subset $U_2 \subset V_1$ (he may choose as he wishes but is expected to escape from $x_1$). Next $\alpha$ chooses an open subset $V_2 \subset U_2$ and a point $x_2 \in V_2$, and so on. $\alpha$ wins if any subsequence $\{x_{n_p}\}_{p \in \mathbb{N}}$ of the sequence $\{x_n\}_{n \in \mathbb{N}}$ accumulates to at least one point of the set $\bigcap_{i=1}^{\infty} V_i = \bigcap_{i=1}^{\infty} U_i$. Then $X$ is said to be $\sigma$-well $\alpha$-favorable if $\alpha$ has a winning strategy in the game above.

It is well known that $\sigma$-well $\alpha$-favorable spaces are Namioka [9, Theorem 6.3].

**Theorem 3.5.** Let $X$ be a space with a dense completely metrizable subspace $M$. Then $X$ is a $\sigma$-well $\alpha$-favorable space. Hence $X$ is a Namioka space.

**Proof.** Let $U_1$ be a nonempty open subset of $X$. Since $M$ is a dense subspace, we can pick a point $x_1 \in M \cap U_1$. Then there exists a nonempty open subset $V_1$ of $X$ such that $x_1 \in V_1 \subset \text{Cl}(V_1) \subset U_1$ and $d_M - \text{diam}(V_1 \cap M) \leq 1/2$, where $d_M$ is a compatible metric on $M$. By induction, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$ and sequences $\{V_n\}_{n \in \mathbb{N}}, \{U_n\}_{n \in \mathbb{N}}$ of subsets of $X$ such that

$x_n \in V_n \cap M$, $U_{n+1} \subset V_n \subset \text{Cl}(V_n) \subset U_n$, and $d_M - \text{diam}(V_n \cap M) \leq 1/n + 1$

for each $n \in \mathbb{N}$. Since $\{x_n\}_{n \in \mathbb{N}}$ is a $d_M$-Cauchy sequence in $M$, there exists $x_0$ in $M$ such that $\{x_n\}_{n \in \mathbb{N}}$ converges to $x_0$. By the construction of $\{V_n\}_{n \in \mathbb{N}}$, we have $x_0 \in \bigcap_{n \in \mathbb{N}} V_n = \bigcap_{n \in \mathbb{N}} \text{Cl}(V_n)$. The proof is complete. $\Box$

**Theorem 3.6.** Let $X$ be a space with a dense completely metrizable subspace, $Y$ a space and $f: X \to Y$ an irreducible, closed, continuous and onto map. Then $Y$ has a dense completely metrizable subspace.
Proof. By Theorem 3.4, there exists a complete sequence \( \{ \mathcal{U}_n \mid n \in \mathbb{N} \} \) of disjoint open almost covers of \( X \), which satisfies the conditions (i) and (ii) of (2). For each \( U \in \mathcal{U}_n \), put \( W(U) = Y \setminus f(X \setminus U) \). Then each \( W(U) \) is a nonempty open subset of \( Y \). Now put \( \mathcal{V}_n = \{ W(U) \mid U \in \mathcal{U}_n \} \) for each \( n \in \mathbb{N} \). It is easy to see that \( \{ \mathcal{V}_n \mid n \in \mathbb{N} \} \) is a complete sequence of open almost covers of \( Y \), which satisfies the condition (1) of Theorem 3.4. The proof is complete. \( \square \)

4. countable dense \( \Delta \)-base. Here \( \Delta_X = \{(x,x) \mid x \in X\} \) is the diagonal in \( X \times X \). Arhangel’skiĭ and Kočinac [1] asked the following questions:

Question 1. When there exist a countable family \( \mathcal{U} \) of open sets in \( X \times X \) such that \( \bigcap \mathcal{U} \cap \Delta_X \) is dense in \( \Delta_X \) and for each open neighborhood \( V \) of \( \Delta_X \) in \( X \times X \) one can find \( U \in \mathcal{U} \) such that \( U \subset V \)? Such \( \mathcal{U} \) will be called a dense \( \Delta \)-base of \( X \).

Question 2. Let \( X \) be a compact space with a countable dense \( \Delta \)-base. Does there exist a dense open metrizable subspace \( Y \subset X \)? A dense separable subspace \( Z \subset X \)?

It is clear that if \( X \) has a dense discrete subspace, then \( X \) has a countable dense \( \Delta \)-base. Now we prove the following theorem.

**Theorem 4.1.** Let \( X \) be a compact space. If \( X \) has a dense completely metrizable subspace, then \( X \) has a countable dense \( \Delta \)-base.

**Proof.** Let \( M \) be a completely metrizable subspace of \( X \) and \( \rho \) a compatible metric on \( M \). For each \( n \in \mathbb{N} \), put \( V_n = \{(x,y) \in M \times M \mid \rho(x,y) < 1/n \} \). Since each \( V_n \) is open set in \( M \times M \), there exists an open set \( U_n \) in \( X \) such that \( V_n = U_n \cap (M \times M) \). We show that \( \mathcal{U} = \{ U_n \mid n \in \mathbb{N} \} \) is a countable dense \( \Delta \)-base of \( X \).

Let \( V \) be an open neighborhood of \( \Delta_X \) in \( X \times X \). Then we prove that there exists \( n \in \mathbb{N} \) such that \( U_n \subset V \). By normality of \( X \times X \), it is enough to show that \( U_n \subset \text{Cl}(V) \).

Indeed, suppose that \( U_n \not\subset \text{Cl}(V) \) for each \( n \in \mathbb{N} \). Then there exists \( (x_n,y_n) \in V_n \setminus \text{Cl}(V) \) for each \( n \in \mathbb{N} \). By the definition of \( V_n \), \( \rho(x_n,y_n) < 1/n \) and \( \{(x_n,y_n) \mid N \in \mathbb{N} \} \subset (X \times X) \setminus \text{Cl}(V) \subset (X \times X) \setminus V \). Since \( (X \times X) \setminus V \) is compact, there exists a cluster point \( (x_0,y_0) \) of \( \{(x_n,y_n) \mid n \in \mathbb{N} \} \) such that \( (x_0,y_0) \in (X \times X) \setminus V \). Hence \( x_0 \neq y_0 \). Then there exist open subsets \( V_{x_0} \) and \( V_{y_0} \) such that \( x_0 \in V_{x_0} \), \( y_0 \in V_{y_0} \) and \( \text{Cl}(V_{x_0}) \cap \text{Cl}(V_{y_0}) = \emptyset \). By the completeness, it follows that \( \text{dist}(\text{Cl}(V_{x_0}) \cap M, \text{Cl}(V_{y_0}) \cap M) > 0 \). But \( \rho(x_n,y_n) < 1/n \) for each \( n \in \mathbb{N} \), a contradiction.

Finally, since \( \Delta_M \subset (\bigcap \mathcal{U}) \cap \Delta_X \), the set \( (\bigcap \mathcal{U}) \cap \Delta_X \) is dense in \( \Delta_X \). The proof is complete. \( \square \)

Next we consider Question 2. We remark the following proposition.

**Proposition 4.2.** Let \( X \) be a space and \( M \) a dense completely metrizable subspace of \( X \). Then the following conditions are equivalent.

1. \( X \) is separable.
(2) $M$ is separable.
(3) $X$ satisfies the countable chain condition.

We have the negative answer of the second part of Question 2.

Example 4.3. Let $X$ be the closed ordinal space $[0, \Omega]$, where $\Omega$ is the first uncountable ordinal. Since $X$ is a compact scattered space, it has a dense uncountable discrete subspace. Therefore $X$ has a countable dense $\Delta$-base. But it is clear that $X$ does not have any dense separable metrizable subspaces.

Let us note that if $M$ is a dense open metrizable subspace of a compact space $X$, then $M$ is a completely metrizable.

REFERENCES


