POSSIBLY EVERY REAL FUNCTION IS CONTINUOUS
ON A NON-MEAGRE SET

Saharon Shelah

Abstract. We prove consistency of the following sentence: ‘ZFC + every real function is
continuous on a non-meagre set’, answering a question of Fremlin.

0 Introduction

By Abraham, Rubin and Shelah [ARSh:153] it is consistent that every func-
tion from $\mathbb{R}$ to $\mathbb{R}$ is continuous when restricted to some uncountable set (and more
...). We may consider strengthening this statement, by demanding the subset on
which the function is continuous to be large in a stronger sense. In [Fe94, Problem
AR(b)], David Fremlin asked exactly this, namely, is it consistent that every func-
tion from $\mathbb{R}$ to $\mathbb{R}$ is continuous when restricted to some set which is non-meagre (i.e.
not countable union of nowhere dense sets). We answer it (positively) here. We use
$\omega^2$ for reals, and for $B \subseteq \omega^2$ we say $f : B \rightarrow \omega^2$ is continuous if $f(\eta_0) = \eta_1, n < \omega$
implies that for some $m$ we have

$$(\forall \eta)[\eta_0 \mid m < \eta \in B \Rightarrow \eta_n \mid n < f(\eta)],$$

where for sequences $\nu, \eta, \nu < \eta$ means “$\eta$ (properly) extends $\nu$”.

The non-meagre sets here are of cardinality $\aleph_1$ and we may wonder whether
we can get them of higher cardinality. From the point of view of those who asked
the original question, maybe this does not add much (but I think it does make

\textit{AMS Subject Classification} (1991): Primary 03E35, Secondary 26A15

Partially supported by the Edmund Landau Center for Research in Mathematical Ana-
lysis (supported by the Minerva Foundation, Germany) and the “The Israel Science Foundation”
administered by the Israeli Academy of Sciences and Humanities. Publication 473.
We would like to thank Alice Leonhardt for the excellent typing.
for much more interesting partition relations). Another interesting generalization, asked by Heinrich von Weizsäcker (see [Fe94, Problem AR(a)]), is:

\((*)\) is it consistent to have that every function from \(\mathbb{R}\) to \(\mathbb{R}\) is continuous when restricted to some non-null set?

We may ask about 2-place functions; Sierpinski colouring implies we cannot ask for one colour, but we may want the consistency of:

\((**)\) for every 2-place function \(f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) there is a non-meagre set (non-null) \(A \subseteq \mathbb{R}\) and 2-place continuous functions \(f_0, f_1 : A \times A \rightarrow \mathbb{R}\) such that

\[(\forall x, y \in A)(f(x, y) = f_0(x, y) \lor f(x, y) = f_1(x, y))\]

(similarly for larger \(n\) or all \(n\) simultaneously). It is not clear to us how much this interests non-logicians, but to us it seems to be the right question and we shall deal with \((***)\) and the cardinality in Rabus Shelah [RaSh:585].

We can also generalize the proof replacing \(\aleph_0\) by \(\mu = \mu^{<\mu}\) (as done here). On consistent partition relations see [Sh:276], [Sh:288], [Sh:546] and [RaSh:585]. Another related theorem, proved in [Sh:481], is the consistency of

\((***)\) if \(B_1\) is a \(2^{\aleph_0}\)-c.c. Boolean algebra and \(B_2\) is a c.c.c. Boolean algebra, then \(B_1 \ast B_2\) is a \(2^{\aleph_0}\)-c.c. Boolean algebra.

Originally the proof goes through “meagre preserving” and iterations as in section 2 of [Sh:276], but the proof was cumbersome, had flaws and proper version was not manufactured. Here we present a simpler though a “degenerated” variant (i.e. use only the forcing we have to and so we get \(2^{\aleph_0} = \aleph_2\) rather than \(2^{\aleph_0} = \aleph_3\)).

We thank Andrzej Roslanowski and Zoran Spasojević for much help in proof-reading.

1 Continuity on a Non-Meagre Set

1.1 Definition. 1) Cohen\(_\mu\)(\(\alpha\)) = \{\(f : f\) a partial function from \(\alpha\) to \(\{0, 1\}\) of cardinality < \(\mu\)\}

ordered by inclusion.

2) Cohen\(_\mu\) = Cohen\(_\mu\)(\(\mu\)).

3) A \(\mu\)-Cohen forcing \(P\) means Cohen\(_\mu\)(\(\alpha\)) for some ordinal \(\alpha\) (or at least the set \(\{p \in P : P \upharpoonright \{q : q \geq p\}\}\) is equivalent to some Cohen\(_\mu\)(\(\alpha\)))

is a dense subset of \(P\)).
1.2 Definition. A set $A \subseteq \mu^2$ is $\mu$-meagre if it is the union of $\leq \mu$ nowhere dense subsets of $\mu^2$.

1.3 Main Lemma. Let $\mu$ be a regular cardinal such that $\mu = \mu^{<\mu}$. For $\ell < 2$ and $\alpha < \mu^+$, let $Q_{\alpha, \ell}$ be $(\mu^2, \mathcal{A})$ (i.e. Cohen$_{\mu}$ forcing) and for $I \subseteq \mu^+ \times 2$ let $P_I$ be the product $\prod_{i \in I} Q_i$ with $(\mu)$-support and $P = P_{\mu^+ \times 2}$. Let $\eta_{\alpha, \ell}$ be the $Q_{\alpha, \ell}$-name of the generic function from $\mu$ to $2 = \{0, 1\}$. In

$$V^P = V[\{\eta_{\alpha, \ell} : \alpha < \mu^+, \ell < 2\}]$$

let $R$ be a forcing notion defined by:

$$R = \{(u_0, v, \nu) : (a) \text{ } u_0 \text{ is a subset of } \mu^+ \text{ of cardinality } < \mu, \text{ (b) } v \text{ is a subset of } \mu^2 \text{ of cardinality } < \mu \text{ (c) if } \rho \in v \text{ then } \ell g(\rho) \text{ is a successor ordinal and } \rho(\ell g(\rho) - 1) = 1, \text{ (d) } \bar{v} = (\nu_{\rho} : \rho \in v) \text{ with } \nu_{\rho} \in \mu^2, \text{ (e) if } \rho_1 < \rho_2 \text{ are from } v \text{ then } \nu_{\rho_1} \neq \nu_{\rho_2} \text{ (hence they are not equal), (f) for every } \rho \in v \text{ there is } \alpha \in u_0 \text{ such that } \rho < \eta_{\alpha, 0}, \text{ (g) if } \alpha \in u_0, \rho < \eta_{\alpha, 0} \text{ and } \rho \in v \text{ then } \nu_\rho < \eta_{\alpha, 1}\}.$$

The order is natural:

$$(u_0, v, \bar{v}) \leq (u_0', v', \bar{v}') \text{ if and only if } u_0 \subseteq u_0', v \subseteq v', \text{ and } \bar{v} = \bar{v}' \restriction v \text{ (and both are in } R)\).$$

Let $\mathcal{U}$ be an $R$-name such that

$$\models_R \mathcal{U} = \bigcup\{u_0 : \text{there are } v, \bar{v} \text{ such that } (u_0, v, \bar{v}) \in \mathcal{G}_R\},$$

where $\mathcal{G}_R$ is the canonical $R$-name for the generic filter on $R$.

Then:

(a) In $V^{P_{\alpha, 0}}$, the function $\eta_{\alpha, 0} \mapsto \eta_{\alpha, 1}$ from $\{\eta_{\alpha, 0} : \alpha \in \mathcal{U}\}$ to $\{\eta_{\alpha, 1} : \alpha \in \mathcal{U}\}$ is continuous.
(b) In $V^{P_{\alpha, 0}}$, the set $\{\eta_{\alpha, 0} : \alpha \in \mathcal{U}\}$ is a non-$\mu$-meagre subset of $\mu^2$, in fact everywhere non-meagre (i.e. its intersection with any non-empty open set is non-meagre).
(c) $P \ast R$ is a $\mu$-Cohen forcing, moreover $P_{\alpha \times 2} \ast P$ (i.e. $P_{\alpha \times 2}$ is a complete suborder of $P$) and $P \ast R/P_{\alpha \times 2}$ is a $\mu$-Cohen forcing for every $\alpha < \mu^+$, $\alpha \geq \mu$. 

1.3A Remark. We can get slightly more than continuity for \( \mu > \aleph_0 \) (a slight change of the forcing notion), namely, for some club \( W \subseteq \mu \), for every \( i \in W \) we have that \( \eta_{\alpha,0} \upharpoonright i \) determines \( \eta_{\alpha,1} \upharpoonright i \) (and so can get this in the theorem).

Proof. For \( \alpha \leq \mu^+ \), \( \alpha \geq \mu \) let

\[
A'_\alpha = \{ (p,r) : (\alpha) \quad p \in P_{\alpha \times 2} \text{ and } \\
(\beta) \quad r = (u_0,v,p) \text{ satisfies clauses } (a),(b),(c),(d),(e) \quad \\
(\gamma) \quad u_0 \subseteq u^p. \}
\]

and partially order \( A'_\alpha \) in the natural way.

Also let

\[
A_\alpha = \{ (p,r) : p,r \text{ satisfy clauses } (\alpha),(\beta),(\gamma) \text{ above and } \\
(\delta) \quad \text{if } \beta \in u_0 \text{ and } \rho \in v \text{ then } \neg(\eta^p_{\beta,0} < \rho), \\
(\varepsilon) \quad \text{for every } \beta \in u_0 \text{ there is } \gamma \in u_0 \cap \mu \text{ such that } \\
(\eta^p_{\beta,0},\eta^p_{\beta,1}) = (\eta^p_{\gamma,0},\eta^p_{\gamma,1}) \}. 
\]

Note that in \( A'_\alpha \) any increasing sequence of length \( < \mu \) has a least upper bound, i.e., if a sequence \( \{q_i : i < \delta\} \subseteq A'_\alpha \) is increasing, \( \delta < \mu \) then there is a condition \( q \in A'_\alpha \) such that \( q_i \leq q \) for all \( i < \delta \) and if \( \bigwedge_{i<\delta} q_i \in A'_\alpha \) then \( q \leq q' \).

Now:

(A) \( A_\alpha \) is a dense subset of \( A'_\alpha \). If \( (p,r) \in A_\alpha \) and \( \mu \leq \alpha \leq \mu^+ \) then \( (p,r) \in P \ast R \) (i.e., \( p \vdash p \quad \text{"}r \in R\text{"} \)).

[Why? The second phrase should be clear. Trivially \( A_\alpha \subseteq A'_\alpha \). Suppose \( (p,r) \in A'_\alpha \) and we shall find \( (p',r) \in A_\alpha \) such that \( (p,r) \leq (p',r) \). Let \( \beta^* = \sup(u^p \cap \mu) + 1 \) and \( \varepsilon = \sup\{\ell_0(\rho) + 1 : \rho \in v\} \) and let \( \delta_{\varepsilon} \) be the sequence of zeroes of length \( \varepsilon \) and let \( u^p = \{\gamma_\zeta : \zeta < |u^p|\} \) be an enumeration.]
Let \( u^\ell = u^p \cup \{ \beta^* + \zeta : \zeta < |u^p| \} \) and
\[
\eta^p_{\gamma, \ell} = \begin{cases}
\eta^p_{\gamma, \ell} \cdot 0 & \text{if } \gamma \in u^p, \ell = 0, \\
\eta^p_{\gamma, \ell} & \text{if } \gamma \in u^p, \ell = 1, \\
\eta^p_{\beta^* + \zeta, \ell} & \text{if } \gamma = \beta^* + \zeta, \ell = 0, \\
\eta^p_{\beta^* + \zeta, \ell} & \text{if } \gamma = \beta^* + \zeta, \ell = 1.
\end{cases}
\]

Note: if \( \rho \in v \) then \( \rho \preceq \eta^{-1} 0 \iff \rho \preceq \eta \) (because by clause (c) in the definition of \( R \), the last element in \( \rho \) is one).]

(B) \( A_{\mu^+} \) is a dense subset of \( P \ast R \).

[Why? Since \( P \) is \( \mu \)-closed, for each \( (p, r) \in P \ast R \) we can find \( p_1 \in P \) and \( r \) such that \( p \leq p_1 \in P \) and \( p_1 \uplus P \quad \tau = r \). Next we choose \( p_2 \) such that \( p_1 \leq p_2 \in P \) and clauses (a)–(c) in the definition of \( A_{\mu^+} \) hold. So \( (p, r) \leq (p_2, r) \) (in \( P \ast R \)) and \( (p_2, r) \in A_{\mu^+} \).

If \( \mu \leq \alpha \leq \beta \leq \mu^+ \) and \( (p, r) \in A_\beta \) then we let \( (p, r) \limp \alpha = (p \limp \alpha, r \limp \alpha) \) where \( p \limp \alpha = (\eta^p_{\gamma, \ell} : (\gamma, \ell) \in (u^p \cap \alpha) \times 2) \) and \( r \limp \alpha = (u^p_0 \cap \alpha, v^p, p^\alpha) \). It is easy to check that:

(C) If \( \mu \leq \alpha \leq \beta \leq \mu^+ \) and \( (p, r) \in A_\alpha \) then \( (p, r) \limp \alpha \in A_\beta \) and \( (p, r) \limp \alpha \in A_\alpha \) (note that clause (f) of the definition of \( A_\alpha \) follows from clause (e) in the definition of \( A_\alpha \)).

(D) If \( \mu \leq \alpha \leq \beta \leq \mu^+ \) and \( (p_1, r_1) \in A_\beta \) and \( (p_1, r_1) \limp \alpha \leq (p_2, r_2) \in A_\alpha \) then \( (p_1, r_1), (p_2, r_2) \) are compatible (in \( A_\beta \)); moreover, they have the natural least upper bound
\[
(p_2 \uplus p^*, (u_0^p \cup u_0^p, v^p, p^\alpha))
\]
in \( A_\beta \), where \( p^* \in P_{\beta \times 2} \) is such that \( u^p = u^{p_1} \setminus \alpha \), and for \( \gamma \in u^p \),
\[
\eta^p_{\gamma, \ell} = \begin{cases}
\eta^p_{\gamma, \ell} & \text{if } \gamma \in u^{p_1} \text{ and } \ell = 0, \\
\eta^p_{\gamma, \ell} \cup \bigcup \{ \eta^p_{\rho, \ell} : \rho \leq \eta^p_{\gamma, \ell} \} & \text{if } \gamma \in u^{p_1} \text{ and } \ell = 1.
\end{cases}
\]

Moreover:

\[\begin{align*}
\text{(D) }^+ & \text{ if } ([p_{1,i}, r_{1,i}) : i \leq \delta) \text{ is an increasing continuous sequence of conditions in } A_{\beta}' \text{ such that } \delta < \mu \text{ and } (p_{1,i}, r_{1,i}) \in A_\beta \text{ for each non-limit } i \leq \delta \text{ and } (p_{1,i}, r_{1,i}) \limp \alpha \leq (p_2, r_2) \text{ for } i < \delta \\
& \text{then } (p_1, r_1, \delta), (p_2, r_2) \text{ are compatible in } A_{\beta}' \text{ and } (p_1, r_1, \delta) \limp \alpha \leq (p_2, r_2).
\end{align*}\]

Hence

\[\begin{align*}
\text{(D) }^{++} & \text{ if } \mu \leq \alpha \leq \beta \leq \mu^+ \text{ then } A_\alpha \subseteq A_\beta, \text{ moreover } A_\alpha \ll A_\beta \text{ (by (D)) and the quotient } A_\beta/A_\alpha \text{ is } \mu \text{-closed (i.e. increasing chains of length } \mu \text{ have upper bound).}
\end{align*}\]
Remark. For better understanding note that if $Q_0$ is adding a $\mu$-Souslin tree, $Q_1$ is the forcing determined by this tree, then the composition $Q_0 * Q_1$ is not even $\mu$-closed. The point is that increasing sequences in $Q_0 * Q_1$ of length $< \mu$ have an upper bound, but not necessarily least upper bound.

Now (note the support)

(E) the sequence $\langle A_\alpha : \alpha \in [\mu, \mu^+] \rangle$ is increasing, and for limit $\alpha \in (\mu, \mu^+)$, $A_\alpha$ is the inverse limit of $\langle A_\beta : \beta < \alpha \rangle$ if $\text{cf}(\alpha) < \mu$, and direct limit of it if $\text{cf}(\alpha) \geq \mu$. Hence $\mu \leq \alpha < \beta \leq \mu^+ \Rightarrow A_\beta / A_\alpha$ is a $\mu$-Cohen forcing (remember that atomless $\mu$-closed forcing notion of size $\mu$ is $\mu$-Cohen).

It is also clear that

(F) $P_{\alpha \times 2} \bowtie P_{\beta \times 2}$ if $\alpha < \beta \leq \mu^+$. 

Lastly note that

(G) for $\alpha \in [\mu, \mu^+]$, $A_{\alpha} / P_{\alpha \times 2}$ is $\mu$-closed and hence it is $\mu$-Cohen.

From the desirable conclusions, we have gotten clause (c) (by combining clauses (E) and (G)).

(H) If $(p_1, r_1) \in A_{\mu^+}$, $\beta^* \in \mu^+ \setminus u^{p_1}$ and $\nu^* \in \mu > 2$ then there is $(p_2, r_2) \in A_{\mu^+}$ such that

$$(p_1, r_1) \leq (p_2, r_2), \quad \beta^* \in u_0^{p_2^*} \quad \text{and} \quad \nu^* < \eta_{p_2, 0}^{p_2^*}.$$ 

[Why? Let $\beta^* \in \mu^+ \setminus u^{p_1}$, $\beta^{**} \in \mu \setminus (u^{p_1} \cup \{\beta^*\})$ and let $\nu^0 = \bigcup \{\nu^*_\rho : \rho \leq \nu^* \text{ and } \rho \in v^{r_1}\}$ and $\gamma = \sup \{\ell g(\nu) + 1 : \nu \in v^{r_1} \cup \{\nu^*\}\}$. 

Now define $(p_2, r_2)$ as follows:

$$u^{p_2} = u^{p_1} \cup \{\beta^*, \beta^{**}\}$$

$$\eta_{p_2, \ell}^{p_2^*} = \begin{cases} \eta_{p_1, \ell}^{p_1} & \text{if } \beta \in u^{p_1}, \ell < 2, \\ \nu^*, 0 & \text{if } \beta \in \{\beta^*, \beta^{**}\}, \ell = 0, \\ \nu^0 & \text{if } \beta \in \{\beta^*, \beta^{**}\}, \ell = 1, \end{cases}$$

and $r_2 = (u_0^{r_2}, v^{r_2}, \overline{v}^{r_2})$ is defined by $u_0^{r_2} = u_0^{p_1} \cup \{\beta^*, \beta^{**}\}$, $v^{r_2} = v^{r_1}$, $\overline{v}^{r_2} = \overline{v}^{r_1}$. 

Easily $(p_2, r_2)$ is as required.]

(I) If $(p_1, r_1) \in A_{\mu^+}$, $\beta^* \in u_0^{p_1}$ then there is $(p_2, r_2) \in A_{\mu^+}$ such that

$$(p_1, r_1) \leq (p_2, r_2), \quad \eta_{p_2, 0}^{p_2^*} \in v^{r_2} \quad \text{and} \quad \eta_{p_2, 1}^{p_2^*} = (\nu_{\eta_{p_2, 0}^{p_2^*}}^{p_2^*})^{r_2}.$$
Possibly every real function is continuous on a non-meagre set

Why? By clause (A) above it is enough to find such \((p_2, r_2)\) in \(A_\mu^+\). Take \(\beta^{**} \in u_0^\mu \cap \mu\) such that \((\eta^\mu_{\beta^*, 0}, \eta^\mu_{\beta^{**}, 1}) = (\eta^\mu_{\beta^*, 0}, \eta^\mu_{\beta^{**}, 1})\), and let

\[
\gamma = \text{sup}\{\ell g(\nu) + 1 : \nu \in v_{r_1} \cup \{\eta^\mu_{\beta^{**}, \ell} : \beta \in u^{p_1}, \ell < 2\}\}.
\]

Define condition \((p_2, r_2)\) by:

\[
u^{p_2} = u^{p_1}, \text{ and} \]

\[
\eta^{p_2}_{\beta, \ell} = \begin{cases} 
\eta^\mu_{\beta^*, 0} \tilde{\beta}_{\gamma}(1) & \text{if } \beta \in \{\beta^*, \beta^{**}\}, \ell = 0, \\
\eta^\mu_{\beta^{**}, 1} \tilde{\gamma}(1) & \text{if } \beta \in \{\beta^*, \beta^{**}\}, \ell = 1, \\
\eta^\mu_{\beta^*, \ell} \tilde{\beta}_{2\gamma + 1} & \text{if } \beta \in u^{p_1} \setminus \{\beta^*, \beta^{**}\}, \ell < 2,
\end{cases}
\]

\[u_0^{r_2} = u_0^{r_1}, \quad v^{r_2} = v^{r_1} \cup \{\eta^\mu_{\beta^*, 0} \tilde{\beta}_{\gamma}(1)\}, \quad \text{and} \]

\[
\rho^{p_2} = \begin{cases} 
\nu^\mu_{\rho} & \text{if } \rho \in v^{r_1}, \\
\eta^\mu_{\beta^*, 1} \tilde{\gamma}(1) & \text{if } \rho = \eta^\mu_{\beta^{**}, 0} \tilde{\beta}_{2\gamma + 1}.
\end{cases}
\]

It is easy to check that \((p_2, r_2) \in A_\mu^+\) as required.

Due to clause (I) above we immediately get assertion (a) of 1.3, i.e. the function \(\eta \mapsto \bigcup\{u^\rho : \rho < \eta, \rho \in \bigcup\rho \in \mathcal{U}\}\) is trivially continuous and its range is contained in \(\mu \geq 2\), but clause (I) implies that for each \(\alpha \in \mathcal{U}\) the image of \(\eta_{\alpha, 0}\) is \(\eta_{\alpha, 1} \in \mu^2\) (and not a proper initial segment of it).

For clause (b) assume that

\[(*) \quad \nu^* \in \mu^2 \text{ and } (B_i : i < \mu) \text{ is a sequence of } (P \ast R)-\text{names,} \]

\[\Vdash_{P \ast R} \text{"}B_i \subseteq \mu^2 \text{ is nowhere dense and (for simplicity) with no isolated points".} \]

It suffices to prove

\[\Vdash_{P \ast R} \{\eta_{\alpha, 0} : \alpha \in \mathcal{U} \text{ and } \nu^* \not< \eta_{\alpha, 0}\} \notin \bigcup_{i < \mu} B_i.\]

Let \(T_i = \{\eta \upharpoonright \gamma : \eta \in B_i \text{ and } \gamma < \mu\}\), so \(\Vdash_{P \ast R} \text{"}T_i \subseteq \mu^2 \text{ is a nowhere dense subtree and } (\forall \eta \in \mu^2)(\eta \in B_i \Rightarrow \bigwedge_{\gamma < \mu} \eta \upharpoonright \gamma \in T_i)\)\), where \(\text{"}T \subseteq \mu^2 \text{ is a nowhere dense subtree" means: } \langle \gamma \rangle \in T_i, (\forall \eta \in T)(\forall \gamma < \ell g(\eta))(\eta \upharpoonright \gamma \in T) \text{ and for every } \eta \in T \text{ and } \gamma < \mu \text{ there are two } \lessdot\text{-incomparable sequences } \nu_0, \nu_1 \text{ such that} \]

\[\eta \lessdot \nu_0 \in T \quad \eta \lessdot \nu_1 \in \mu^2 \backslash T \quad \ell g(\nu_0) \geq \gamma \quad \ell g(\nu_1) \geq \gamma.\]

We can find \(\alpha < \mu^+\) such that \(\alpha > \mu\) and for every \(i < \mu\), \(T_i\) is an \(A_\alpha\)-name (remember \(\mu^{< \mu} = \mu\)). Let \((p_0, r_0) \in P \ast R\). By the statement \((B)\) above there is
(p_1, r_1) such that (p_0, r_0) ≤ (p_1, r_1) ∈ A_{μ^+}. We can easily find (p_2, r_2) such that (p_1, r_1) ≤ (p_2, r_2) ∈ A_{μ^+} and β^* ∈ (α, μ^+) such that β^* ∈ u_i^2 and ν^* ≺ η_{β^*,0}^p ∈ v_i^2, (ν_{η_{β^*,0}^p})_r^2 = η_{β^*,1}^p (apply clauses (H) and (I)).

It suffices to prove that for each i < μ, (p_2, r_2) \vDash "η_{β^*,0} \notin B_i. So let i < μ and (p_2, r_2) ≤ (p_3, r_3) ∈ A_{μ^+}. Since (p_3, r_3) \models α \in A_α, T_4 is an A_α-name and \vDash_{A_α} "T_4 \subseteq μ^> 2 is nowhere dense", there is a condition (p_4, r_4) ∈ A_α such that (p_3, r_3) \models α ≤ (p_4, r_4) and for some ν^* we have

η_{β^*,0}^p < ν^* ∈ μ^> 2 and (p_4, r_4) \vDash "ν^* \notin T_i."

Lastly we define (p_5, r_5) ∈ A_{μ^+} such that (p_5, r_5) is above (p_3, r_3) and above (p_4, r_4) and (p_5, r_5) \vDash "ν^* < η_{β^*,0}^p. So let

u^{p_5} = u^{p_3} \cup u^{p_4},

η_{β^*, \ell}^{p_5} = \begin{cases} η_{β^*, \ell}^{p_3} & \text{if } β \in u^{p_3}, \\ η_{β^*, \ell}^{p_4} & \text{if } β \in u^{p_4} \setminus (u^{p_3} \cup \{β^*\}), \\ ν^* & \text{if } β = β^*, \ell = 0, \\ \bigcup \{ν_ρ^{r_4} : ρ ∈ v^{r_4} \text{ and } ρ \leq ν^*\} & \text{if } β = β^*, \ell = 1, \end{cases}

r_5 = (u^{r_4}_0 \cup u^{r_4}_3, v^{r_4}, p^{r_4}).

By clause (A) there is (p_6, r_6) such that (p_6, r_6) ≤ (p_5, r_5) ∈ A_{μ^+}. This clearly suffices. Thus we finish proving clause (b) of the conclusion of 1.3.

Together we get all the required conclusions. □_{1.3}

1.4 Theorem. Assume that μ = μ^{<μ}, 2^μ = μ^+, 2^{μ^+} = μ^{+2} and S is a stationary subset of

S_μ^{μ^+} =: \{δ < μ^{+2} : \text{cf}(δ) = μ^+\}.

Suppose that ◊_S holds.

Then there is a μ-closed, μ^+ -c.e. forcing notion P of cardinality μ^{+2} such that, in V_P,

(*) for every function f : μ^2 → μ^2, there is a non-μ-meagre set A ⊆ μ^2 such that f \upharpoonright A is continuous.

Proof. Without loss of generality we may also assume that S_μ^{μ^+} \setminus S is stationary.

For α ≤ μ^{+2} let A_α be the family of sequences Q = (P_i, Q_i : i < α) such that

(a) Q is iteration with support < μ and P_α is the limit,
(b) for $i \in \alpha \cap \mu^{+2} \setminus S$, $Q_i = Q_i$ is $(\mu^2, \prec)$ (i.e. a $\mu$-Cohen forcing notion) and we denote by $\eta_{\beta_{i+1}}^{\beta_i}$ the name of a Cohen generic element of $\mu^2$ adjoined by $Q_i$,

(c) if $\beta_1 < \beta_2 \leq \alpha$, $\beta_2 < \mu^{+2}$, $\beta_1 \notin S$ then $P_{\beta_2}/P_{\beta_1}$ and $P_{\beta_1}/(P_{\beta_1} * Q_{\beta_1})$ are $\mu$-Cohen forcing notions (and for clarity we fix representation of $P_{\beta_1}$ as $P_{\beta_1} = Q_{\beta_1} \times \prod\{Q_{\beta_1}^{\beta_i} : j < j(\beta_1, \beta_2)\}$

with support $< \mu$,

(d) we may assume that each $Q_i$ (for $i < \alpha$) is $\mu$-closed, $\mu^{+}$-cc and of cardinality $\leq \mu^+$.

Note that by clause (c), each $P_\beta$ is a $\mu$-closed forcing notion satisfying the $\mu^{+}$-cc.

Let $\mathcal{R} = \bigcup_{\alpha < \mu^{+2}} \mathcal{R}_\alpha$.

There is a natural ordering of $\mathcal{R}$:

$$Q^1 \leq Q^2 \quad \text{if and only if} \quad Q^1 = Q^2 \upharpoonright \ell g(Q^1).$$

This partial order is $\mu^{+2}$-complete and every (strictly) increasing chain of length $\mu^{+2}$ has a limit in $\mathcal{R}_{\mu^{+2}}$.

Note that by clause (c) we have that

\(\exists \, Q \in \mathcal{R}_\alpha, \beta < \alpha, \beta \notin S \) and $\Vdash_{P_\beta}$ "$B \subseteq \mu$ is not $\mu$-meagre"

then $\Vdash_{P_{\beta_1}}$ "$B \subseteq \mu$ is not $\mu$-meagre".

Because of this, and by $\Diamond_S$, it is enough to show that

\(\bigotimes \) if $Q \in \mathcal{R}_{\mu^{+2}}$, $P = P_{\mu^{+2}} = \lim(Q)$, and $f$ is a $P$-name of a function from $\mu^2$ to $\mu^2$ then for some club $E$ of $\mu^{+2}$, for every $\delta \in S \cap E$ we have

(a) $i < \delta \Rightarrow f(\eta_{i+1}^Q)$ is a $P_\delta$-name,

(b) there is a $Q^{\delta+1} \in \mathcal{R}_{\delta+1}$ such that $Q^{\delta+1} \upharpoonright \delta = Q \upharpoonright \delta$ and

$\Vdash_{\text{lim}(Q^{\delta+1})}$ "for some $A \subseteq \delta, |A| = \mu^+$, we have that

(i) $\{\eta_{i+1}^Q : i \in A\}$ is not meagre,

(ii) the function $\{(\eta_{i+1}^Q, f(\eta_{i+1}^Q)) : i \in A\}$ is continuous".

(Note: $\eta_{i+1}^Q$ is a $P_\delta$-name (as $i+1 < \delta$) and also $f(\eta_{i+1}^Q)$ is a $P_\delta$-name (by clause (a))).
Proof of (⊗). For each $\delta \in S^{\mu+2}_{\mu^+} \setminus S$, $f(\eta^{Q}_{j,\delta+1})$ is a $P_{\mu^2}$-name of a member of $\mu^2$, so for some $\gamma_\delta > \delta + 1$, it is a $P_{\delta}$-name and we can demand that $\gamma_\delta$ is a successor ordinal. As

$$P_{\gamma_\delta} = P_\delta \times Q_\delta \times \prod_{j < j(\delta, \gamma_\delta)} Q_j^{\delta, \gamma_\delta}$$

(with support $< \mu$), we find a set $w_\delta \subseteq j(\delta, \gamma_\delta)$ of cardinality $\leq \mu$ such that $f(\eta^{Q}_{j,\delta+1})$ is a $P_\delta \times Q_\delta \times \prod_{j \in w_\delta} Q_j^{\delta, \gamma_\delta}$-name. Also as $cf(\delta) = \mu^+$ there is $\beta_\delta < \delta$ such that $f(\eta^{Q}_{j,\delta+1})$ is a $P_{\beta_\delta} \times Q_\delta \times \prod_{j \in w_\delta} Q_j^{\delta, \gamma_\delta}$-name and we can demand that $\beta_\delta$ is a successor ordinal.

By Fodor’s lemma (as $2^\mu = \mu^+$) for some stationary $S_1 \subseteq S^{\mu+2}_{\mu^+} \setminus S$ (and $\zeta < \mu^+$, $\beta^* < \mu^{+2}$) we have that

1. $\delta \in S_1 \Rightarrow \beta_\delta = \beta^*$
2. $\delta \in S_1 \Rightarrow \text{otp}(w_\delta) = \zeta$
3. For $\delta_1, \delta_2 \in S_1$ we demand $f_{\delta_1, \delta_2}(f(\eta^{Q}_{\delta_1+1})) = f(\eta^{Q}_{\delta_2+1})$, where $f_{\delta_1, \delta_2}$ is the natural isomorphism

from $P_{\beta^*} \times Q_{\delta_1} \times \prod_{j \in w_{\delta_1}} Q_j^{\delta_1, \gamma_{\delta_1}}$ onto $P_{\beta^*} \times Q_{\delta_2} \times \prod_{j \in w_{\delta_2}} Q_j^{\delta_2, \gamma_{\delta_2}}$.

Clearly $Q^\delta = \prod_{j \in w_\delta} Q_j^{\delta, \gamma_\delta}$ is isomorphic to Cohen forcing.

Lastly, let

$$E = \left\{ \delta < \mu^{+2} : \delta = \text{sup}(S_1 \cap \delta) \text{ and } \delta_1 \in S_1 \cap \delta \Rightarrow \gamma_{\delta_1} < \delta \right\}.$$ 

Clearly the set $E$ is a club of $\mu^{+2}$. We are going to show that for each $\delta^* \in E \cap S$ there is $Q^\delta^{\delta^*}$ as required in $(\otimes)$.

Let $\delta^* \in E \cap S$. We can find an increasing continuous sequence $\langle \beta_\varepsilon : \varepsilon < \mu^+ \rangle$ with limit $\delta^*$ and a sequence $\langle \delta_\varepsilon : \varepsilon < \mu^+ \rangle$ such that $\beta_0 = \beta^*$, $\beta_\varepsilon$ a successor ordinal if $\varepsilon$ is a non-limit, $\beta_\varepsilon \notin S_1$, $\delta_\varepsilon \in S_1$, and $\beta_\varepsilon < \delta_\varepsilon < \gamma_{\delta_\varepsilon} < \beta_{\varepsilon+1}$. Let $\eta^*_{\varepsilon, 0}$ and $\eta^*_{\varepsilon, 1}$ be $P_{\delta}/P_{\beta_\varepsilon}$-names such that $\eta^*_{\varepsilon, 0}$ is $\eta^{Q}_{\delta_{\varepsilon}+1}$ and $\eta^*_{\varepsilon, 1}$ is a name for the Cohen subset of $\mu$ added by $Q^\delta_{\varepsilon}$. Now let

$$R^0_{\delta^*} := \prod \left[ \{ Q_{\delta_\varepsilon} : \varepsilon < \mu^+ \} \cup \{ Q^\delta_{\varepsilon} : \varepsilon < \mu^+ \} \right]$$

and note that $R^0_{\delta^*} \preceq P_{\delta^*}/P_{\beta^*}$. The forcing notion $R^0_{\delta^*}$ is naturally isomorphic to $P$ from 1.3 with $\eta^*_{\varepsilon, t}$ corresponding to $\eta_{\varepsilon, t}$. Moreover the quotient $P_{\delta^*}/(P_{\beta^*} \times R^0_{\delta^*})$
is a $\mu$-Cohen forcing notion. [Why? For each $\varepsilon < \mu^+$, $\delta_\varepsilon, \gamma_\delta, \beta_\gamma \notin S$ and hence we may write $P_{\beta_\lambda + 1}$ as

$$P_{\beta_\lambda} \times Q_{\beta_\gamma} \times \prod_{j < j(\beta_\gamma, \delta_\gamma)} Q_j^{\gamma_j, \beta_\lambda} \times Q_{\delta_\gamma} \times \prod_{j < j(\delta_\gamma, \gamma_\delta)} Q_j^{\gamma_j, \beta_\lambda} \times Q_{\gamma_\delta} \times \prod_{j < j(\gamma_\delta, \beta_\lambda + 1)} Q_j^{\gamma_j, \beta_\lambda + 1}.$$ 

But

$$\prod_{j < j(\delta_\gamma, \gamma_\delta)} Q_j^{\gamma_j, \beta_\lambda} = Q_{\beta_\lambda} \times \prod_{j < j(\delta_\gamma, \gamma_\delta) \setminus \{w_\delta\}} Q_j^{\gamma_j, \beta_\lambda}.$$ 

So we may represent $P_{\beta_\lambda + 1}$ as

$$P_{\beta_\lambda + 1} = P_{\beta_\lambda} \times Q_{\delta_\gamma} \times Q_{\beta_\lambda} \times R_{\varepsilon}^*,$$

where $R_{\varepsilon}^*$ is a product (with support $< \mu$) of $\mu$-Cohen forcing notions. Now, since the sequence $\langle \beta_\varepsilon : \varepsilon < \mu^+ \rangle$ is increasing and continuous with limit $\delta^*$ we may write

$$P_{\delta^*} = P_{\beta_\lambda} \times \prod_{\varepsilon < \mu^+} Q_{\delta_\varepsilon} \times \prod_{\varepsilon < \mu^+} R_{\varepsilon}^* \times \prod_{\varepsilon < \mu^+} R_{\varepsilon}^*$$

(all products with $< \mu$ support).]

Let $Q_{\delta^*}$ be $R[\langle \eta_{\varepsilon, 0}^*, \eta_{\varepsilon, 1}^* : \varepsilon < \mu^+ \rangle]$ (from 1.3), $Q_{\delta^* + 1} = (Q \upharpoonright \delta^*) \upharpoonright (P_{\delta^*}, Q_{\delta^*})$. 

To check that $Q_{\delta^* + 1}$ is as desired we use 1.3. Thus $P_{\delta^*} / P_{\delta^*}$ (for $\beta < \delta$, $\beta \notin S$) is $\mu$-Cohen by 1.3(c) and because, as said above, $P_{\delta^*} / (P_{\beta^*} \times R_{\varepsilon}^*)$ is $\mu$-Cohen. It follows from 1.3(a) that the function $\eta_{\varepsilon, 0}^* \rightarrow \eta_{\varepsilon, 1}^*$ (for $\varepsilon \in \mathcal{U}$) is continuous and hence, as the names $f(\eta_{\delta_\varepsilon}^*)$ are isomorphic, the function $\eta_{\delta_\varepsilon + 1}^* \rightarrow f(\eta_{\delta_\varepsilon + 1}^*)$ (for $\varepsilon \in \mathcal{U}$) is continuous (it is obviously Borel, which actually suffices, but in fact, as the $\eta_{\varepsilon, 0}^*$ are Cohen over the definition of $f$ it is continuous). Finally, by 1.3(b),

$$\vDash_P P_{\delta + 1} \{ \eta_{\delta_\varepsilon + 1}^* : \varepsilon \in \mathcal{U} \} \text{ is not meagre}.$$ 

This finishes the proof. \hfill \square_{1.4}

1.5 Remark 1. Alternatively, one may force the iteration instead of using $\diamondsuit_S$.

2. The forcing notion from 1.4 is constructed like “$\lambda$-free not free (Boolean) algebra of cardinality $\lambda$”, with free interpreted as $\mu$-Cohen.
2 Final remarks

Haim Judah asked us whether adding Cohen reals is enough.

2.1 Observation. If $P$ is the forcing notion for adding $\mathfrak{P}$ Cohen real, then in $V^P$ there is $f : \mathbb{R} \to \mathbb{R}$ which is not continuous on any uncountable set.

Proof. Let $\lambda = \mathfrak{P}$, $P = \{ f : f$ is a finite function from $\lambda$ to $\{0, 1\} \}$. For any set $w \subseteq \lambda$ let $P_w = \{ f \in P : \text{Dom}(f) \subseteq w \}$. It is known that $P_w \leq P = P_\lambda$ and for any $P$-name $\dot{\eta}$ of a real (i.e. $\Vdash_P \dot{\eta} \in \mathcal{P}$) for some countable $w \subseteq \lambda$, $\dot{\eta}$ is a $P_w$-name. For every ordinal $\alpha < \lambda$ let $r_\alpha \in \omega_2$ be defined by:

$$r_\alpha(n) = \ell \text{ if and only if } \text{ for some } f \in G_P, f(\omega \alpha + n) = \ell.$$

Clearly $r_\alpha$ is a Cohen real. Let $\langle \dot{\xi}_\zeta : \zeta < \lambda \rangle$ list all $P$-names of reals, so for some countable $w_\zeta \subseteq \lambda$, $\dot{\xi}_\zeta$ is a $P_{w_\zeta}$-name. Without loss of generality

$$\alpha + \omega = \beta + \omega \& \alpha \in w_\zeta \implies \beta \in w_\zeta.$$

We now define by induction on $\zeta < \lambda$ an ordinal $\alpha_\zeta < \lambda$ as follows:

$\alpha_\zeta$ is the first ordinal $\alpha$ such that

$$\alpha \notin \{ \alpha_\zeta : \varepsilon < \zeta \} \cup \{ \beta : \omega + k \beta \in w_\zeta \text{ for some } k \in \omega \}.$$

Let $G \subseteq P$ be a generic filter over $V$. In $V[G]$ we define a function $F : \omega_2 \to \omega_2$ as follows:

for $s \in \omega_2$ let $\gamma(s) = \min \{ \zeta : s = \dot{\xi}_\zeta[G] \}$ and $F(s) = r_{\alpha_{\gamma(s)}}$.

Suppose now that, in $V[G]$, a set $A \subseteq \omega_2$ is uncountable and $F \restriction A$ is continuous. Thus we have a Borel function $F' : \omega_2 \to \omega_2$ such that $F \restriction A = F' \restriction A$. The function $F'$ is coded by a single real $s \in \omega_2$, $\gamma(s) = \zeta$.

Let $p_0 \in G, A, F, F', s$ and $\zeta < \lambda$ be such that

$$p_0 \Vdash_P "A, F, F', s = \dot{\xi}_\zeta, \zeta = \gamma(s) \text{ are as above}".$$

The set $\{ \xi < \lambda : [\omega \alpha_\xi, \omega \alpha_\xi + \omega) \cap w_\zeta \neq \emptyset \}$ is countable, $p \Vdash_P "A \text{ is uncountable}"$, so we find $\xi < \lambda$ and $p_1 \supseteq p_0, p_1 \in G$ such that

$$p_1 \Vdash_P "\dot{\xi}_\zeta \in A \text{ and } \gamma(\dot{\xi}_\zeta) = \xi"$$

and $[\omega \alpha_\xi, \omega \alpha_\xi + \omega) \cap w_\zeta = \emptyset$. Now

$$p_1 \Vdash_P "F'(\dot{\xi}_\zeta) = F(\dot{\xi}_\zeta) = r_{\alpha_\xi}".$$
But we can compute $F'(s_\xi)$ in $V[G \cap P_{\omega_1 \cup \omega}]$ and (by the choice of $\xi$) $r_\omega$ is Cohen over $V[G \cap P_{\omega_1 \cup \omega}]$ (remember the choice of $\alpha_\xi$’s), which is a contradiction. \( \square \)

2.2 **Remark.** Instead of forcing it suffices to assume the existence of a Luzin set in a strong sense.

* * *

We call $A \subseteq \omega_2$ a Luzin set if it is uncountable and non-meagre and any uncountable $B \subseteq A$ is non-meagre.

2.3 **Definition.** A forcing notion $Q$ is Luzin-preserving if it satisfies the c.c.c. and

(*) for any Luzin set $A \subseteq \omega_2$ and a sequence $\langle p_\eta : \eta \in A \rangle$ of members of $Q$, we have that

(**) for some $\eta \in A$

$p_\eta \Vdash Q$ “for some $\rho \in \omega_2$,

$(\forall \nu)[\rho \leq \nu \in \omega_2 \Rightarrow \text{for uncountably many } \eta_1 \in A, \nu < \eta_1 \text{ and } p_\eta \in G_Q]”$.

2.4 **Claim.** 0) Assume there are Luzin sets, then in Definition 2.3 “it satisfies the c.c.c.” follows from (*$\ast$) of 2.3.

1) A forcing notion $Q$ is Luzin-preserving iif it satisfies the c.c.c. and for any Luzin $A \subseteq \omega_2$ we have, $\Vdash Q$ “$A$ is Luzin”.

2) Being Luzin-preserving is preserved by composition.

3) If $\{P_i : i < \delta\}$ is an increasing continuous sequence of Luzin-preserving forcing notions then $\bigcup_{i < \delta} P_i$ is a Luzin-preserving forcing notions.

4) The statement “a forcing notion $P$ is Luzin preserving” depend on $P$ only up to equivalence of forcing notions.

5) In Definition 2.3 we can weaken the demand to:

$(*')$ for some $\eta \in A$ there is $q$ such that $p_\eta \leq q \in Q$ and $q \Vdash Q$ “…”.

**Proof.** 0) Should be clear.

1) $\Rightarrow$ (the “only if” implication).

By definition, $Q$ satisfies the c.c.c. Assume $A$ is Luzin but $p \Vdash \text{“$A$ is not Luzin”}$.

Then for some $Q$-names $\eta_i$ (for $i < \omega_1$), we have

$p \Vdash \text{“$\eta_i \in A$ for $i < \omega_1$, } \bigwedge_{i < j} \eta_i \neq \eta_j \text{ and } \{\eta_i : i < \omega_1\} \text{ is not meagre”}$.

Thus changing the $\eta_j$’s we can get

$(\Theta) \ p \Vdash \text{“$\eta_i \in A$ for $i < \omega_1$, } \bigwedge_{i < j} \eta_i \neq \eta_j \text{ and } \{\eta_i : i < \omega_1\} \text{ is nowhere dense”}$.
Let \( p \leq p_i \in Q \), \( p_i \models \text{"} \eta_i = \nu_i \text{"} \) so \( \nu_i \in (\omega^2)_V \) and necessarily \( \nu_i \in A \). For no \( \nu \) the set \( \{ i : \nu_i = \nu \} \) is uncountable as then \( Q \) fails the c.c.c. So without loss of generality \( i \neq j \Rightarrow \nu_i \neq \nu_j \). Let \( A' = \{ \nu_i : i < \omega \} \). It is an uncountable subset of \( A \) and hence it is Luzin. Let \( p_\nu = p_i \) if \( \nu = \nu_i \). Apply (\( \ast \)) of Definition 2.3 to \( \langle p_\nu : \nu \in A' \rangle \) and get contradiction to (\( \oplus \)) above.

\( \Leftarrow \) (the "if" implication). Of course \( Q \) satisfies the c.c.c. Let \( \langle p_\eta : \eta \in A \rangle \) be given, \( A \subseteq \omega^2 \) a Luzin set and \( p_\eta \in Q \) (for \( \eta \in A \)) and we should show that (***) of Definition 2.3 holds. Then \( A = \{ \eta \in A : p_\eta \in G_Q \} \) is a \( Q \)-name of a subset of \( A \). It is known that for all but countably many \( \eta \in A \), we have

\[ p_\eta \models \text{"} A \subseteq A \text{ is uncountable"} \]

so that (by the assumption of the implication we are proving)

\[ p_\eta \models \text{"} A \text{ is non-meagre, hence nowhere meagre above some } \nu_\eta \in (\omega^2)_V \text{"}. \]

We have really finished, but we can elaborate. As we can replace \( A \) by \( A \setminus B \) for any countable \( B \subseteq A \). Without loss of generality this holds for every \( \eta \in A \). Now for \( \eta \in A \) and \( p_\eta, q_\eta \leq p'_\eta \in P \) there are \( \nu_\eta \) and \( q_\eta, p'_\eta \leq q_\eta \in Q \) such that

\[ q_\eta \models \text{"} \{ \nu \in A : p_\nu \in G_Q \} \text{ is uncountable and everywhere non-meagre above } \nu_\eta \text{"}. \]

As this holds for any \( p'_\eta, p_\eta \leq p'_\eta \in P \) we are done.

2) Follow by 2.4(1) (and as c.c.c. is preserved).

3) Like the preservation of c.c.c.

4) Easy (e.g. use 2.4(1)).

5) Left to the reader.

\[ \square \]

REFERENCES


Institute of Mathematics
The Hebrew University
Jerusalem, Israel
Rutgers University
Mathematics Department
New Brunswick, NJ USA

(Received 26 08 1995)
(Revised 28 11 1995)