ON THE FOURTH MOMENT OF THE RIEMANN ZETA-FUNCTION

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Dedicated to the memory of Professor Duro Kurepa

Abstract. Atkinson proved in 1941 that $\int_0^\infty e^{-t/T} |t(1/2 + it)|^4 dt = T Q_4(\log T) + O(T^\epsilon)$ with $\epsilon = 8/9 + \epsilon$, where $Q_4(y)$ is a suitable polynomial in $y$ of degree four. We improve Atkinson's result by showing that $\epsilon = 1/2$ is possible, and we provide explicit expressions for all the coefficients of $Q_4(y)$ and the closely related polynomial $P_4(y)$.

1. Introduction

In recent years there has been much progress with problems involving the function $E_2(T)$ (see [7], [8], [9], [10], [12], [13], [15]). This important function, which represents the error term in the asymptotic formula for the fourth moment of the Riemann zeta-function $\zeta(s)$ on the so-called "critical line" $\text{Re} s = \frac{1}{2}$, is defined by the relation

$$ \int_0^T |\zeta \left( \frac{1}{2} + it \right) |^4 \, dt = T \sum_{j=0}^4 a_j \log^j T + E_2(T). \quad (1.1) $$

In 1926 Ingham [4] proved that $a_4 = 1/(2\pi^2)$. Much later in 1979 Heath-Brown [3] proved that $E_2(T) \ll T^{7/6 + \epsilon}$ ($f \ll g$ and $f = O(g)$ both mean that $|f(x)| \leq Cg(x)$ for $x \geq x_0, C > 0$ and $g(x) > 0$), and calculated

$$ a_3 = 2(4\gamma - 1 - \log(2\pi) - 12\zeta'(2)\pi^{-2})\pi^{-2}, \quad (1.2) $$

where as usual $\gamma = 0.577215 \ldots$ is Euler's constant. The constants $a_2, a_1$ and $a_0$ are more complicated, and were not stated explicitly in [3]. Heath-Brown's bound for $E_2(T)$ was improved to

$$ E_2(T) = O(T^{2/3} \log^{C} T) \quad (C > 0) \quad (1.3) $$

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in [9] by Motohashi and the author (see also [7]), where it was also proved that
\[ E_2(T) = \Omega(T^{1/5}), \]
and as usual \( f = \Omega(g) \), \( g > 0 \) means that \( \lim_{x \to \infty} f(x)/g(x) \neq 0 \). Therein it was also shown that
\[ \int_0^T E_2(t) dt = O(T^{3/2}). \]

Recently Motohashi [15] improved (1.4) to \( E_2(T) = \Omega_\pm (T^{1/5}) \), and in [10] Motohashi and the author established that, with some \( C > 0 \),
\[ \int_0^T E_2(t) dt = O(T^2 \log^C T). \]

In general, one can define the error term function \( E_k(T) \) by the relation
\[ \int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^{2k} dt = TP_k(\log T) + E_k(T), \]
where \( k \geq 1 \) is a fixed integer, and \( P_k(y) \) is a suitable polynomial in \( y \) of degree \( k^2 \).

Apart from the classical case \( k = 1 \) (see [6] and [7] for an extensive discussion) and \( k = 2 \), our knowledge about the general \( E_k(T) \) (see Ch. 4 of [7]) is very modest. At present it is not known \( E_k(T) = o(T) \) as \( T \to \infty \) for any \( k \geq 3 \), and in fact it is not clear how to define properly the coefficients of \( P_k(y) \) for \( k \geq 3 \).

Instead at (1.7) one may look at the related formula of the Laplace transform (see Ch. 7 of Titchmarsh [16])
\[ \int_0^\infty e^{-\delta t} \left| \zeta \left( \frac{1}{2} + it \right) \right|^{2k} dt \quad (\delta \to 0+, \ k \geq 1 \text{ an integer}), \]

since Laplace transforms of many functions are easier to handle than the original functions. Kober [11] proved that
\[ \int_0^\infty e^{-2\delta t} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt = \frac{\gamma - \log(4\pi \delta)}{2\sin \delta} + \sum_{n=0}^N c_n \delta^n + O(\delta^{N+1}) \]
for \( \delta \to 0+, \) any fixed integer \( N \geq 1 \) and suitable constants \( c_n \). This is much sharper than the corresponding asymptotic formula (1.7) when \( k = 1 \). Atkinson [1] obtained
\[ \int_0^\infty e^{-\delta t} \left| \zeta \left( \frac{1}{2} + it \right) \right|^4 dt \]
\[ = \frac{1}{\delta} (A \log^4 \delta^{-1} + B \log^3 \delta^{-1} + C \log^2 \delta^{-1} + D \log \delta^{-1} + E) + L_2(\delta^{-1}) \]
as $\delta \to 0+$ with

$$A = 1/(2\pi^2), \quad B = \pi^{-2}(2 \log(2\pi) - 6\gamma + 24\zeta'(2)\pi^{-2})$$  \hspace{1cm} (1.11)$$

and $L_2(1/\delta) \ll (1/\delta)^{13/14+\varepsilon}$ for any given $\varepsilon > 0$. Atkinson’s proof used bounds for the function $E(x,r)$ in (4.6). In his work it was indicated on p. 185 how a better bound for $E(x,r)$, which depends on bounds for Kloosterman sums, will lead to the better bound

$$L_2\left(\frac{1}{\delta}\right) \ll \left(\frac{1}{\delta}\right)^{8/9+\varepsilon} \quad (\delta \to 0+).$$  \hspace{1cm} (1.12)$$

In analogy with (1.7) we define (writing in (1.8) $T = 1/\delta$ with $T \to \infty$) the function $L_k(T)$ by the relation

$$\int_0^\infty e^{-t/T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt = TQ_{k^2}(\log T) + L_k(T),$$  \hspace{1cm} (1.13)$$

where $k \geq 1$ is a fixed integer, $Q_{k^2}(y)$ is a suitable polynomial in $y$ of degree $k^2$, and one should have $L_k(T) = o(T)$ as $T \to \infty$. At present (analogously to $E_k(T) = o(T)$) the relation $L_k(T) = o(T)$ is not known to hold for $k \geq 3$.

2. Statement of results

A comparison of the asymptotic formulas (1.1) and (1.10) shows that $a_4 = A = 1/(2\pi^2)$, but that $a_3 \neq B$. One actually has

$$B = a_3 + 2\pi^{-2}(1 - \gamma),$$  \hspace{1cm} (2.1)$$

the reason for which will become clear later. Recent advances concerning $E_k(T)$ make it possible to improve (1.12), and we have

**Theorem 1.** If $L_2$ is defined by (1.10), then as $T \to \infty$ one has

$$L_2(T) = O(T^{1/2}).$$  \hspace{1cm} (2.2)$$

This result gives a substantial improvement over Atkinson’s exponent $8/9 + \varepsilon$ in (1.12). The exponent $1/2$ in (2.2) is the limit of the method of proof, which is based on the use of (1.5). One can prove, more generally, the following

**Theorem 2.** Suppose $k \geq 2$ is a fixed integer and

$$\int_0^T E_k(t)dt = O(T^{c_k})$$  \hspace{1cm} (2.3)$$

holds for some $c_k > 0$. If $L_k$ is defined by (1.13), then as $T \to \infty$

$$L_k(T) = O(T^{c_k-1}).$$  \hspace{1cm} (2.4)$$
Moreover the coefficients of \( Q_{k^2}(y) \) can be expressed as linear combinations of the coefficients of \( P_{k^2}(y) \), defined by (1.7).

The remaining aim of this paper is to provide explicit expressions for the coefficients \( a_2, a_1 \) and \( a_0 \) in (1.1) that were not stated explicitly by Heath-Brown [3]. From the nature of the problem it is clear that the expressions for these coefficients will be more complicated than the expression (1.2) for \( a_2 \). For this reason they will not be stated here as a theorem, but will be dealt with in section 4, where all the appropriate notation will be introduced. From the proof of Theorem 2 in section 3 it will be clear that we can evaluate explicitly \( C, D \) and \( E \) in (1.10) as linear combinations of the \( a_j \)'s. Conversely, if we know explicitly the coefficients of \( Q_{k^2}(y) \), then it is not difficult to see that the coefficients of \( P_{k^2}(y) \) can be written as linear combinations of the coefficients of \( Q_{k^2}(y) \).

3. The Laplace transform of the \( 2k \)-th moment

It is enough to prove Theorem 2, since Theorem 1 is its consequence because, by (1.5), we have \( c_2 = 3/2 \) in (2.3) for \( k = 2 \). Let

\[
I_k(T) = \int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^{2k} dt = T P_{k^2}(\log T) + E_k(T)
\]

with

\[
P_{k^2}(y) = \sum_{j=0}^{k^2} a_j y^j, \quad a_j = a_j(k).
\]

Then integration by parts gives

\[
\int_0^\infty e^{-t/T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^{2k} dt = T^{-1} \int_0^\infty e^{-t/T} I_k(t) dt
\]

\[
= T^{-1} \int_0^\infty e^{-t/T} t P_{k^2}(\log t) dt + T^{-1} \int_0^\infty e^{-t/T} E_k(t) dt
\]

\[
= T \int_0^\infty e^{-t/T} x P_{k^2}(\log x + \log T) dx + T^{-2} \int_0^\infty e^{-t/T} \left( \int_0^t E_k(y) dy \right) dt = I' + I'',
\]

say. Inserting (3.1) in the expression for \( I' \) we obtain

\[
I' = T \int_0^\infty e^{-t/T} x \sum_{j=0}^{k^2} a_j \left( \log x + \log T \right)^j dx
\]

\[
= T \int_0^\infty e^{-t/T} x \sum_{j=0}^{k^2} \sum_{i=0}^j \frac{j!}{i!} \log^i T \cdot \log^{j-i} x \cdot dx
\]
On the fourth moment of the Riemann zeta-function

\[ I = T \sum_{j=0}^{k^2} a_j \sum_{i=0}^{j-1} \left( \frac{j}{i} \right) \log^i T \left( \int_{0}^{\infty} e^{-x} x \cdot \log^{j-i} x \cdot dx \right). \]

But \( \Gamma^{(k)}(z) = \int_{0}^{\infty} e^{-t_t^{z-1}} (\log t)^k dt \), for \( \Re z > 0 \) and \( k \geq 0 \) which gives

\[ I' = T \sum_{j=0}^{k^2} a_j \sum_{i=0}^{j-1} \left( \frac{j}{i} \right) \Gamma^{(j-i)}(2) \cdot \log^i T = T \sum_{i=0}^{k^2} b_i \log^i T \quad (3.3) \]

with

\[ b_i = b_i(k) = \sum_{j=i}^{k^2} \left( \frac{j}{i} \right) a_j \Gamma^{(j-i)}(2) \quad (i = 0, 1, \ldots, k^2), \quad (3.4) \]

so that the coefficients of \( Q_{k^2} \) are linear combinations of the coefficients of \( P_{k^2} \). By using (2.3) we obtain

\[ I'' = T^{-2} \int_{0}^{\infty} \left( \int_{0}^{t} E_k(y)dy \right) dt \ll T^{-2} \int_{0}^{\infty} e^{-t/T} \Gamma^2 dt \quad (3.5) \]

so that Theorem 2 follows from (1.13) and (3.2)–(3.5). Note that in the particular case \( k = 2 \) (3.4) yields

\[ A = b_4 = a_4 = 1/(2\pi^2), \quad B = b_3 = 4a_4 \Gamma'(2) + a_3 \Gamma(2), \]
\[ C = b_2 = 6a_4 \Gamma''(2) + 3a_3 \Gamma'(2) + a_2 \Gamma(2), \]
\[ D = b_1 = 4a_4 (3\Gamma''(2) + 2a_3 \Gamma'(2) + a_2 \Gamma(2), \]
\[ E = b_0 = a_4 (4\Gamma''(2) + 3a_3 (2\Gamma''(2) + 2a_2 \Gamma(2)) + a_1 \Gamma(2), \quad (3.6) \]

Since \( \gamma = -\int_{0}^{\infty} e^{-x} \log x dx = -\Gamma'(1) \), it follows by an integration by parts that

\[-\gamma = -\int_{0}^{\infty} x (e^{-x} \log x)' dx = \Gamma'(2) - 1. \quad \text{Thus } \Gamma'(2) = 1 - \gamma, \quad \text{and (3.6) yields}
\]
\[ B = b_3 = 4 \cdot \frac{1}{\gamma^2} (1 - \gamma) + a_3, \quad \text{which is (2.1). Conversely, if the } b_j \text{s are known, then (3.4) is a system of } k^2 + 1 \text{ linear equations in } k^2 + 1 \text{ unknowns } a_0, a_1, \ldots, a_{k^2} \]

with a triangular determinant whose value is \((\Gamma(2))^{k^2+1} = 1\), so that the \( a_j \)'s can be uniquely expressed as linear combinations of the \( b_j \)’s.

4. The coefficients of the main term in the fourth moment formula

There are several ways to obtain explicitly the coefficients \( a_j \) in (1.1). This can be achieved by following the proofs of the fourth moment formula (see Ch. 4 of [7] or [13]). Here we shall follow the method of Heath-Brown [3], who showed that the main term \( T P_{k_2}(\log T) \) in (1.1) consists of two parts: the part coming from the “diagonal” terms, and the part coming from the “non-diagonal” terms of a sum involving the number of divisors function \( d(n) \).
The diagonal terms furnish an expression of the form \( TR_4(\log T) \), and the non-diagonal terms the expression \( TQ_2(\log T) \). Here \( R_4(x) \) and \( Q_2(x) \) denote suitable polynomials of degree four and two, respectively. It will turn out that the coefficients of \( Q_2(x) \) have a more complex form than those of \( R_4(x) \). Thus we shall have

\[
P_4(x) = R_4(x) + Q_2(x).
\]

(4.1)

As on p. 403 of [3] the diagonal terms make a contribution which is equal to

\[
2 \sum_{m \leq T/(2\pi)} d^2(m)m^{-1}(T - 2\pi m)
\]

\[
= \frac{1}{2\pi^2} \left\{ \int_{1-i\infty}^{1+i\infty} \frac{4\pi^{s+1}(s+1)\zeta^{-1}(2s+2)}{(2\pi)^s} \frac{ds}{s(s+1)} \right\}.
\]

The term \( TR_4(\log T) \) will be the residue of the simple pole (of the integrand in curly brackets) at \( s = 0 \). Near \( s = 0 \) we have the expansions

\[
\zeta^{-1}(2s+2) = \zeta^{-1}(2) - \frac{2s\zeta'(2)}{\zeta^2(2)} + c_2 s^2 + c_3 s^3 + \ldots
\]

with

\[
c_k = \frac{2^k}{k!} \sum_{n=1}^{\infty} \mu(n)(-\log n)^{n-2} = \frac{1}{k!} \left\{ \frac{d^k}{ds^k}(\zeta^{-1}(2s+2)) \right\} \bigg|_{s=0},
\]

\[
\frac{1}{s+1} = 1 - s + s^2 - s^3 + \ldots,
\]

\[
\left( \frac{T}{2\pi} \right)^{s+1} = \frac{T}{2\pi} \left\{ 1 + s \log \left( \frac{T}{2\pi} \right) + \frac{s^2}{2!} \log^2 \left( \frac{T}{2\pi} \right) + \frac{s^3}{3!} \log^3 \left( \frac{T}{2\pi} \right) + \ldots \right\},
\]

\[
\zeta^4(s+1) = \frac{1}{s^4} + \frac{4\gamma}{s^3} + \frac{b_{-2}}{s^2} + \frac{b_{-1}}{s} + b_0 + b_1 s + b_2 s^2 + \ldots.
\]

The coefficients \( b_k (k \geq -2) \) may be found from the relation

\[
\zeta^4(s+1) = \left( \frac{1}{s} + \gamma_0 + \gamma_1 s + \gamma_2 s^2 + \gamma_3 s^3 + \ldots \right)^4,
\]

(4.3)

where one has (see Theorem 1.3 of [6])

\[
\gamma_0 = \gamma \gamma_k = \frac{(-1)^k}{k!} \lim_{N \to \infty} \left( \sum_{m \leq N} \frac{\log^k m}{m} - \frac{\log^{k+1} N}{k+1} \right).
\]

Israilov [5] calculated

\[
\gamma_1 = 0.072815846 \ldots, \quad \gamma_2 = -0.00485182 \ldots, \quad \gamma_3 = -0.000342305 \ldots,
\]

and Euler’s constant \( \gamma = \gamma_0 \) is of course known with much greater accuracy,

\[
\gamma = 0.5772 15664 90153 28606 06512 \ldots.
\]
From (4.2) and (4.3) we obtain by comparing the coefficients
\[
\begin{align*}
  b_{-2} &= 4\gamma_1 + 6\gamma^2, \\
  b_{-1} &= 4\gamma_2 + 12\gamma\gamma_1 + 4\gamma^3, \\
  b_0 &= 4\gamma_3 + 12\gamma\gamma_2 + 6\gamma^2 + 12\gamma^2\gamma_1 + \gamma^4.
\end{align*}
\]
Since the residue is the coefficient of \( s^{-1} \), we obtain
\[
TR_4(\log T) = \frac{2T}{\zeta(2)} \left\{ \frac{1}{24} \log^4 \left( \frac{T}{2\pi} \right) + \frac{a}{6} \log^3 \left( \frac{T}{2\pi} \right) + \frac{b}{2} \log^2 \left( \frac{T}{2\pi} \right) + c \log \left( \frac{T}{2\pi} \right) + d \right\}
\]
with
\[
\begin{align*}
  a &= 4\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)}, \\
  b &= 1 + \frac{2\zeta'(2)}{\zeta(2)} + c_2 \zeta(2) + 4\gamma \left( -1 - \frac{2\zeta'(2)}{\zeta(2)} \right) + b_{-2}, \\
  c &= -1 - \frac{2\zeta'(2)}{\zeta(2)} + (c_3 - c_2) \zeta(2) + 4\gamma \left( 1 + c_2 \zeta(2) + \frac{2\zeta'(2)}{\zeta(2)} \right) + b_{-2} \left( -1 - \frac{2\zeta'(2)}{\zeta(2)} \right) + b_{-1}, \\
  d &= 1 + \frac{2\zeta'(2)}{\zeta(2)} + (c_2 - c_3 + c_4) \zeta(2) + 4\gamma \left( -1 - \frac{2\zeta'(2)}{\zeta(2)} + (c_3 - c_2) \zeta(2) \right) + b_{-2} \left( 1 + \frac{2\zeta'(2)}{\zeta(2)} + c_2 \zeta(2) \right) + b_{-1} \left( -1 - \frac{2\zeta'(2)}{\zeta(2)} \right) + b_0.
\end{align*}
\]
Expanding \( \log^2(T/2\pi) = (\log T - \log 2\pi)^2 \) by the binomial theorem we obtain \( TR_4(\log T) = \zeta(2) (\log T) \) in Heath-Brown’s notation
\[
TR_4(\log T) = \frac{T}{\pi^2} \left\{ \frac{1}{2} \log^4 T + (2a - 2\log(2\pi)) \log^3 T + (3 \log^2(2\pi) - 6a \log(2\pi) + 6b) \log^2 T + (-2\log^3(2\pi) + 6a \log^2(2\pi) - 12b \log(2\pi) + 12c) \log T + \left( \frac{1}{2} \log^4(2\pi) - 2a \log^3(2\pi) + 6b \log^2(2\pi) - 12c \log(2\pi) + 12d \right) \right\}.
\]
The coefficients \( a_4 \) and \( a_5 \) in (1.1) are the coefficients of \( \log^4 T \) and \( \log^3 T \), respectively, and so by the first formula in (4.5) they are
\[
a_4 = \frac{1}{2\pi^2}, \quad a_5 = \frac{2a - 2\log(2\pi)}{\pi^2} = \frac{8\gamma - 2 - 24\zeta'(2)\pi^2 - 2\log(2\pi)}{\pi^2}.
\]
These are the same values that were obtained by Ingham and Heath-Brown.

For the remaining part \( \zeta(2)(\log T) \) in the main term one has (see p. 404 of [3]) that it is the main term in the asymptotic formula for
\[
f_0(T) \sim 2\Re \left\{ \sum_{r=1}^{R} (ir)^{-1} S_r \right\} \quad (R \to \infty),
\]
where $S_r \sim \int_0^{r/(2\pi)} m'(x,r) e^{irx} \, dx$. Here $m(x,r)$ stands for the main term in the asymptotic formula for the so-called binary additive divisor problem (see Motohashi [14] for an extensive discussion), namely

$$\sum_{n \leq x} d(n)d(n + r) = m(x,r) + E(x,r), m(x,r) = x \sum_{j=0}^2 c_j(r) \log^j x.$$  \hspace{1cm} (4.6)

We have $m'(x,r) = d_0(r) \log^2 x + d_1(r) \log x + d_2(r)$ with

$$d_0(r) = c_1(r), d_1(r) = c_2(r) + 2c_1(r), d_2(r) = c_2(r) + c_0(r).$$  \hspace{1cm} (4.7)

In Theorem 2 of [3] Heath-Brown evaluated the constants $c_d(r)$ in (4.6), but his expressions are cumbersome. We find it more expedient to use the expressions given by Balakrishnan and Sengupta [2], and from these expressions and (4.7) we obtain, with the notation

$$\sigma_z(n) = \sum_{d|n} d^z, \quad \sigma'_z(n) = \frac{d}{dz}(\sigma_z(n)) = \sum_{d|n} d^z \log d, \quad \sigma''_z(n) = \sum_{d|n} d^z \log^2 d,$$

the following:

$$d_0(r) = \frac{\sigma_{-1}(r)}{\zeta(2)}, \quad d_1(r) = d_0(r) \left\{ 4\gamma - 4 \frac{c'_r}{\zeta(2)} - 4 \frac{c''_r}{\sigma_{-1}(r)} \right\},$$

$$d_2(r) = d_0(r) \left\{ 4\gamma - 1 - 4 \frac{c'_r}{\zeta(2)} - 4 \frac{c''_r}{\sigma_{-1}(r)} - 4 \left( \frac{c_r}{\zeta(2)} \right)^2 + \frac{4}{\sigma_{-1}(r)} - 4 \left( \frac{\sigma''_{-1}}{\sigma_{-1}(r)} \right)^2 \right\}.$$  \hspace{1cm} (4.8)

Further, after a change of variable, we have

$$S_r \sim \int_0^{r/(2\pi)} \{d_0(r) \log^2 x + d_1(r) \log x + d_2(r)\} e^{irx} \, dx$$

$$= T \int_{2\pi r}^\infty \{d_0(r) \log^2 \left( \frac{T}{y} \right) + d_1(r) \log \left( \frac{T}{y} \right) + d_2(r)\} e^{iy} \, dy.$$

Therefore it follows that

$$TQ_2(\log T) = 2 \text{Re} \left\{ \sum_{r=1}^\infty (ir)^{-1} S_r \right\} = T(e_0 \log^2 T + e_1 \log T + e_2)$$

with

$$e_0 = 2 \sum_{r=1}^\infty d_0(r) \int_{2\pi r}^\infty \frac{\sin x}{x^2} \, dx,$$

$$e_1 = \sum_{r=1}^\infty \left\{ d_0(r) \int_{2\pi r}^\infty \frac{(2 \log r - 2 \log x) \sin x}{x^2} \, dx + d_1(r) \int_{2\pi r}^\infty \frac{\sin x}{x^2} \, dx \right\}.$$  \hspace{1cm} (4.9)
On the fourth moment of the Riemann zeta-function

\[
e_2 = \sum_{r=1}^{\infty} \int_{2\pi r}^{\infty} \left\{ d_0(r) \log^2 x - (2d_0(r) \log r + d_1(r)) \log x \\
+ d_0(r) \log^2 r + d_1(r) \log r + d_2(r) \right\} \frac{\sin x}{x^2} \, dx.
\]

If, for \( \text{Re} \alpha < 1 \) and \( z > 0 \) we introduce the standard notation

\[
C(x, \alpha) = \int_{z}^{\infty} t^{\alpha-1} \cos t \, dt, \quad S(x, \alpha) = \int_{z}^{\infty} t^{\alpha-1} \sin t \, dt,
\]

then we have

\[
\frac{\partial S(x, \alpha)}{\partial \alpha} = \int_{z}^{\infty} t^{\alpha-1} \log t \sin t \, dt, \quad \frac{\partial^2 S(x, \alpha)}{\partial \alpha^2} = \int_{z}^{\infty} t^{\alpha-1} \log^2 t \sin t \, dt,
\]

and

\[
e^{-\frac{1}{2} \pi \iota} \Gamma(a, iz) = C(x, a) - iS(x, a), \quad S(x, a) = -\text{Im} \{ e^{-\frac{1}{2} \pi \iota} \Gamma(a, iz) \},
\]

\[
\Gamma(a, x) = \Gamma(a) - \gamma(a, x) = \int_{x}^{\infty} e^{-t} t^{a-1} \, dt,
\]

where \( \gamma(a, x) = \int_{0}^{x} e^{-t} t^{a-1} \, dt \) is the incomplete gamma-function. With this notation we have

\[
e_0 = 2 \sum_{r=1}^{\infty} \frac{1}{\alpha} S(2\pi r, -1),
\]

\[
e_1 = 2 \sum_{r=1}^{\infty} \left\{ (2d_0(r) \log r + d_1(r)) S(2\pi r, -1) - 2d_0(r) \frac{\partial S}{\partial \alpha}(2\pi r, -1) \right\},
\]

\[
e_2 = 2 \sum_{r=1}^{\infty} \left\{ (d_0(r) \frac{\partial^2 S}{\partial \alpha^2}(2\pi r, -1) - (2d_0(r) \log r + d_1(r)) \frac{\partial S}{\partial \alpha}(2\pi r, -1) \\
+ (d_0(r) \log^2 r + d_1(r) \log r + d_2(r)) S(2\pi r, -1) \right\}.
\]

Hence finally from (4.1) and the above expressions we obtain

**Theorem 3.** For the coefficients \( a_j \) in (1.1) we have

\[
a_4 = \frac{1}{2\pi^2}, \quad a_3 = 2(4\gamma - 1 - \log(2\pi) - 12\zeta'(2)\pi^{-2})\pi^{-2},
\]

\[
a_2 = (3 \log^2(2\pi) - 6a \log(2\pi) + 6b) \pi^{-2} + e_0,
\]

\[
a_1 = (-2 \log^3(2\pi) + 6a \log^2(2\pi) - 12b \log(2\pi) + 12c) \pi^{-2} + e_1,
\]

\[
a_0 = \left( \frac{1}{2} \log^4(2\pi) - 2a \log^3(2\pi) + 6b \log^2(2\pi) - 12c \log(2\pi) + 12d \right) \pi^{-2} + e_2,
\]
where \( a, b, c, d \) are given by (4.5) and \( e_0, e_1, e_2 \) by (4.10).

REFERENCES


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