SOME REMARKS ON GENERALIZED MARTIN’S AXIOM

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Abstract. Let $GMA$ denote that if $\mathbb{P}$ is well-met, strongly $\omega_1$-closed and $\omega_1$-centered partial order and $\mathcal{D}$ a family of $< 2^{\omega_1}$ dense subsets of $\mathbb{P}$ then there is a filter $G \subseteq \mathbb{P}$ which meets every member of $\mathcal{D}$. The consistency of $2^{\omega_1} = \omega_1 + 2^{\omega_1} > \omega_2 + GMA$ was proved by Baumgartner [1] and in [13] many of its consequences were considered. In this paper we give a consequence and present an independence result. Namely, we prove that, as a consequence of $2^{\omega_1} = \omega_1 + 2^{\omega_1} > \omega_2 + GMA$, every strictly increasing $\omega_2$-sequence in $(\omega_1^{\omega_1}, \leq^*)$ is a lower half of some $(\omega_2, \omega_2)$-gap and show that the existence of an $\omega_2$-Kurepa tree is consistent with and independent of $2^{\omega_1} = \omega_1 + 2^{\omega_1} > \omega_2 + GMA$.

1. Introduction. With the discovery of Martin’s Axiom [8] and its many consequences a number of set-theorists considered the problem of generalizing Martin’s Axiom to higher cardinals. Their aim actually was to generalize the consequences of $MA$ to higher cardinals. One of the first generalizations of Martin’s Axiom is due to Baumgartner [1] and one of the strongest generalizations is due to Shelah [9]. We will return to Shelah’s version in the last section but now we state Baumgartner’s result. A partial order $\mathbb{P}$ is well-met if any two compatible elements in $\mathbb{P}$ have the greatest lower bound. We denote compatibility of $p, q \in \mathbb{P}$ by $p \not\perp q$ and their incompatibility by $p \perp q$. $\mathbb{P}$ is $\omega_1$-closed if any decreasing $\omega_1$-sequence in $\mathbb{P}$ has a lower bound and it is strongly $\omega_1$-closed if the greatest lower bound exists for any such sequence. $\mathbb{P}$ is centered if any finite sub-collection of $\mathbb{P}$ has a lower bound and it is $\omega_1$-centered if it is a union of $\omega_1$ many centered partial orders. Baumgartner [1] constructed a model for

$$(BA)\quad 2^{\omega_1} = \omega_1 + 2^{\omega_1} > \omega_2 + GMA$$

and thus obtained the consistency of one of the first versions of Generalized Martin’s Axiom. In fact, Baumgartner considered a somewhat bigger class of partial orders, but in this paper we will only consider partial orders which are well-met, strongly $\omega_1$-closed, $\omega_1$-centered and of size $< 2^{\omega_1}$, where $2^{\omega_1}$ is computed in the final model.

Many consequences of $(BA)$ were considered in [13]. The object of this paper is to present one more consequence and an independence result. We show that every

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\(\leq^*\)-increasing \(\omega_2\)-sequence in \((\omega_1^{\omega_1}, \leq^*)\) is a lower half of some \((\omega_2, \omega_2)\)-gap (see \(\S 2\) for notation and terminology). As usual, this result will be obtained by applying \(GMA\) to suitably chosen partial orders. It will be fairly straightforward to show that this partial order is well-met and strongly \(\omega_1\)-closed. Somewhat harder will be to show that it is \(\omega_1\)-centered. For this we first need to recall the notion of a complete embedding.

**Definition 1.1.** Let \(P\) and \(Q\) be partial orders. An \(i: P \to Q\) is a complete embedding if

\[
\begin{align*}
(1) & \; \forall p, p' \in P (p' \preceq p \implies i(p') \preceq i(p)), \\
(2) & \; \forall p, p' \in P (p' \perp p \implies i(p') \perp i(p)), \\
(3) & \; \forall q \in Q \exists p \in P \forall p' \in P (p' \preceq p \implies i(p') \not\in i(p)).
\end{align*}
\]

We also recall a result from [13].

**Proposition 1.2.** Assume \(2^\omega = \omega_1\). Then any countable support iteration of length \(\leq 2^{\omega_1}\) with well-met, strongly \(\omega_1\)-closed and \(\omega_1\)-centered partial orders yields an \(\omega_1\)-centered partial order.

At this stage we also point out that \(2^\omega = \omega_1\) is assumed throughout this paper. Now, let \(P\) be a partial order which is well-met and strongly \(\omega_1\)-closed and suppose that all the conditions in \(P\) are countable. To show that \(P\) is \(\omega_1\)-centered, it suffices to exhibit a sequence \(\langle P_\xi : \xi \leq \alpha \leq 2^{\omega_1} \rangle\) of sub-orders of \(P\) such that \(P_\alpha = P\) and each \(P_\xi\), for \(\xi < \alpha\), is well-met, strongly \(\omega_1\)-closed and \(\omega_1\)-centered, as well as a sequence \(\langle i_{\xi \eta} : \xi \leq \eta \leq \alpha \rangle\), with \(i_{\xi \eta} : P_\xi \to P_{\eta}\), of complete embeddings such that \(\forall \xi, \eta, \theta (\xi \leq \eta \leq \theta \implies i_{\xi \theta} = i_{\eta \theta} \circ i_{\xi \eta})\). Then \(P\) can be viewed as a countable support iteration of length \(\alpha \leq 2^{\omega_1}\) with well-met, strongly \(\omega_1\)-closed and \(\omega_1\)-centered partial orders so that by Proposition 1.2 \(P\) is also \(\omega_1\)-centered.

It is well known that \(2^\omega = \omega_1\) implies the existence of an \(\omega_2\)-Aronszajn tree (see [6]). The results of Laver and Shelah [7] and Shelah and Stanley [10] show that the existence of an \(\omega_2\)-Suslin tree is consistent with and independent of \((BA)\). In the final section we consider the influence of \((BA)\) on the existence of \(\omega_2\)-Kurepa trees. Our result is that the existence of such trees is consistent with and independent of \((BA)\).

2. Gaps. Let \(\kappa^\kappa\) be the set of all function from \(\kappa\) to \(\kappa\). If \(f, g \in \kappa^\kappa\) then \(f \leq^* g\) if and only if \(\forall n < \kappa \forall i \geq n \to f(i) \leq g(i)\) and \(f(i) < g(i)\) on a set of size \(\kappa\). A \((\kappa^+, \kappa^+)\)-pregap in \((\kappa^\kappa, \leq^*)\) is a pair \((a, b)\) where \(a = (a_\xi : \xi < \kappa^+)\) and \(b = (b_\zeta : \xi < \kappa^+)\) are subsets of \(\kappa^\kappa\) such that \(\forall \xi, \eta < \kappa^+ (a_\xi \leq^* b_\eta)\) and \(\forall \xi < \eta < \kappa^+ (a_\xi \leq^* a_\eta \land b_\xi \leq^* b_\eta)\). If there is a \(c \in \kappa^\kappa\) such that \(\forall \xi, \eta < \kappa^+ (a_\xi \leq^* c \leq^* b_\eta)\) then \(c\) splits the pregap \((a, b)\). If no such \(c\) exists then \((a, b)\) is a \((\kappa^+, \kappa^+)\)-gap.

Hausdorff [4] showed (in ZFC) that \((\omega^\omega, \leq^*)\) contain a \((\omega_1, \omega_1)\)-gap. Herin [5] and independently Blaszczyk and Szymanski [2] generalized Hausdorff’s result to higher cardinals by proving that if \(\kappa\) is a regular cardinal then \((\kappa^\kappa, \leq^*)\) contains a \((\kappa^+, \kappa^+)\)-gap. Hausdorff’s result was refined in [11] by showing that \(MA\) implies that every \(\leq^*\)-increasing \(\omega_1\)-sequence in \((\omega^\omega, \leq^*)\) is a lower half of some \((\omega_1, \omega_1)\)-gap. And this last result was further improved in [12] by establishing that \(t > \omega_1\) is
in fact equivalent to the statement that every $\leq^*$-increasing $\omega_1$-sequence in $(\omega^\omega, \leq^*)$ is a lower half of some $(\omega_1, \omega_1)$-gap. The goal of this section is to show that (BA) implies that every $\leq^*$-increasing $\omega_2$-sequence in $(\omega_1^{\omega_1}, \leq^*)$ is a lower half of some $(\omega_2, \omega_2)$-gap and thus refine the results of Herink [5] and Blaszczyk and Szymanki [2].

Let $a = \langle a_\xi; \xi < \omega_2 \rangle$ be an $\leq^*$-increasing $\omega_2$-sequence in $(\omega_1^{\omega_1}, \leq^*)$. A $\leq^*$-decreasing $\omega_2$-sequence $b = \langle b_\xi; \xi < \omega_2 \rangle$ on top of $a$, such that $(a,b)$ is an $(\omega_2, \omega_2)$-gap, will be obtained from an application of GMA to a suitably defined partial order $\mathbb{P}_a$. In order to guarantee that $(a,b)$ is in fact a gap, the elements of the sequences $a$ and $b$ have to satisfy the following condition:

$$(\star) \quad \forall \xi < \omega_2 \forall i < \omega_1(a_\xi(i) \leq b_\xi(i)) \land \forall \xi, \eta < \omega_2(\xi < \eta \rightarrow \exists i < \omega_1(b_\xi(i) < a_\eta(i))).$$

This condition is a refinement of the following condition due to Kunen for $(\omega_1, \omega_1)$-gaps in $(\omega^\omega, \leq^*)$ (unpublished work):

$$\forall \xi < \omega_1 \forall i < \omega(a_\xi(i) \leq b_\xi(i)) \quad \text{and} \quad \forall \xi, \eta < \omega_1(\xi \neq \eta \rightarrow \exists i < \omega(a_\xi(i) \geq b_\eta(i) \lor a_\eta(i) > b_\xi(i))).$$

Now we show that if $2^\omega = \omega_1$ then every $(\omega_2, \omega_2)$-pregap in $(\omega_1^{\omega_1}, \leq^*)$ satisfying $(\star)$ is in fact a gap.

Lemma 2.1. Assume $2^\omega = \omega_1$ and let $(a,b) = \langle a_\xi, b_\xi; \xi < \omega_2 \rangle$ be an $(\omega_2, \omega_2)$-pregap in $(\omega_1^{\omega_1}, \leq^*)$ whose elements satisfy $(\star)$. Then $(a,b)$ is a gap.

Proof. By way of contradiction, assume $(a,b)$ is split by $c: \omega_1 \to \omega_1$. Then

$$(\circ) \quad \forall \xi < \omega_2 \exists n_\xi < \omega_1 \forall n \geq n_\xi(a_\xi(n) \leq c(n) \leq b_\xi(n)).$$

By a first thinning process we may assume that $\forall \xi < \omega_2(n_\xi = m)$, for some fixed $m < \omega_1$. Since $2^\omega = \omega_1$ and $m$ is a countable ordinal, we have $|\omega_1^m| = \omega_1$. Hence, by another thinning process we may assume that

$$(\bullet) \quad \forall \xi, \eta < \omega_2(a_\xi \upharpoonright m = a_\eta \upharpoonright m \land b_\xi \upharpoonright m = b_\eta \upharpoonright m).$$

But then $(\circ), (\bullet)$ and the first clause of $(\star)$ imply that $\forall \xi, \eta < \omega_2 \forall i < \omega_1(a_\xi(i) \leq b_\eta(i))$, which contradicts the second clause of $(\star)$. Hence, $(a,b)$ is a gap and the Lemma is proved. \qed

Therefore, the definition of $\mathbb{P}_a$ has to incorporate the requirements in $(\star)$.

Definition 2.2. Let $a = \langle a_\xi; \xi < \omega_2 \rangle$ be an $\leq^*$-increasing $\omega_2$-sequence in $(\omega_1^{\omega_1}, \leq^*)$.

$$\mathbb{P}_a = \{(x,y,n,s); f,x,y \in [\omega^\omega]^{<\omega_1} \land n < \omega_1 \land s: y \to \omega_1^s \land f\forall \xi \in y((\xi \in x \rightarrow \forall i < n(a_\xi(i) \leq s(\xi)(i))) \land f\forall \eta \in x(\eta > \xi \rightarrow \exists i < n(s(\xi)(i) < a_\eta(i))))\}$$

where $\langle x_2, y_2, n_2, s_2 \rangle \leq \langle x_1, y_1, n_1, s_1 \rangle$ if and only if

1. $x_1 \subseteq x_2, y_1 \subseteq y_2, n_1 \leq n_2$, \}
(2) \( \forall \xi \in y_1(s_2(\xi) \upharpoonright n_1 = s_1(\xi)) \),

(3) \( \forall \xi, \eta \in y_1 \forall i < \omega_1(\xi \leq \eta \land n_1 \leq i < n_2 \rightarrow s_2(\eta)(i) \leq s_2(\xi)(i)) \),

(4) \( \forall \xi \in x_1 \forall i < y_1 \forall \eta < \omega_1(n_1 \leq i < n_2 \rightarrow a_\xi(i) \leq s_2(\eta)(i)) \).

Clearly \( P_a \) is a partial order and the next step is to show that \( P_a \) is well-met, strongly \( \omega_1 \)-closed and \( \omega_1 \)-centered so that \( GMA \) can be applied to it.

So let \( \langle x_1, y_1, n_1, s_1 \rangle, \langle x_2, y_2, n_2, s_2 \rangle \in P_a \) and suppose \( \langle u, v, k, t \rangle \in P_a \) is their lower bound. We may assume that \( u = x_1 \cup x_2 \) and \( v = y_1 \cup y_2 \). Then there is the least \( m \) such that \( \max(n_1, n_2) \leq m < k \) and \( \langle u, v, m, t \mid m \rangle \in P_a \), where \( t \mid m \) is a function with domain \( v \) such that \( \forall \xi \in v(t(\xi) = t(\xi) \mid m) \). Then it is easily seen that \( \langle u, v, m, t \mid m \rangle \) is the greatest lower bound of \( \langle x_1, y_1, n_1, s_1 \rangle \) and \( \langle x_2, y_2, n_2, s_2 \rangle \) so that \( P_a \) is well-met.

Now let \( \langle x_0, y_0, n_0, s_0 \rangle \geq \langle x_1, y_1, n_1, s_1 \rangle \geq \cdots \) be a decreasing \( \omega \)-sequence in \( P_a \). Let \( u = \bigcup_{i < \omega} x_i, v = \bigcup_{i < \omega} y_i, m = \sup_{i < \omega}(n_i) \) and let \( t \) be a function with domain \( v \) such that \( \forall \xi \in v(t(\xi) = \bigcup_{i < \omega} s_i(\xi) : \xi \in y_i) \). Then \( \langle u, v, m, t \rangle \) is the greatest lower bound in \( P_a \) of the above sequence so that \( P_a \) is strongly \( \omega_1 \)-closed.

As indicated in §1, to show that \( P_a \) is \( \omega_1 \)-centered it suffices to show that there is a sequence \( \langle P_\alpha : \alpha \leq \omega_2 \rangle \) of sub-orders of \( P_a \) such that \( P_\omega = P_a \) and a sequence \( \langle \iota_\alpha : \alpha \leq \beta \leq \omega_2 \rangle \), with \( \iota_\alpha : P_a \rightarrow P_\beta \), of complete embeddings such that \( \forall \alpha, \beta, \gamma (\alpha \leq \beta \leq \omega_2 \rightarrow \iota_\alpha \gamma = \iota_\beta \gamma \circ \iota_\alpha) \) and such that each \( P_\alpha \), for \( \alpha < \omega_2 \), is well-met, strongly \( \omega_1 \)-closed and \( \omega_1 \)-centered. Then \( P_a \) can be viewed as a countable support iteration of length \( \omega_2 \) with well-met, strongly \( \omega_1 \)-closed and \( \omega_1 \)-centered partial orders, since \( P_a \) consists of countable conditions. Then, by Proposition 1.2, \( P_a \) is also \( \omega_1 \)-centered.

For each \( \alpha \leq \omega_2 \) let \( P_\alpha = \{ \langle x, y, n, s \rangle \in P_a : y \subseteq \alpha \} \) and for each \( \alpha \leq \beta \leq \omega_2 \) let \( \iota_\alpha : \omega_2 \rightarrow \omega_2 \) be the inclusion map \( \iota(p) = p \). Then \( P_\alpha = P_\omega \), each \( P_\alpha \) is a sub-order of \( \omega_2 \) and the ordering relation inherited from \( P_a \) and \( \forall \alpha \leq \beta \leq \gamma \leq \omega_2 \iota_\alpha \gamma = \iota_\beta \gamma \circ \iota_\alpha \). Analogous proof can be used to show that each \( P_\alpha \), for \( \alpha < \omega_2 \), is well-met and strongly \( \omega_1 \)-closed as the one used to show that \( P_a \) has these properties.

**Lemma 2.3.** For each \( \alpha \leq \beta \leq \omega_2 \), \( \iota_\alpha \beta \) is a complete embedding.

**Proof.** Properties (a) and (b) of Definition 1.1 are satisfied in a trivial way. For (c), let \( q = \langle x_q, y_q, n_q, s_q \rangle \in P_\beta \). Then \( p = \langle x_q, y_q \cap \alpha, n_q, s_q \upharpoonright (y_q \cap \alpha) \rangle \) has the required property. \( \square \)

**Lemma 2.4.** Assume \( 2^{\omega_1} = \omega_1 \). Then for each \( \alpha < \omega_2 \), \( P_\alpha \) is \( \omega_1 \)-centered.

**Proof.** Let \( \alpha < \omega_2 \) and for each \( y \in [\alpha]^{\omega_1} \), \( n < \omega_1 \), \( s \in (\omega_1)^\nu \) let \( \mathcal{P}_{\text{args}} = \{ \langle x, z, m, t \rangle \in \mathcal{P}_a : z = y \land m = n \land t = s \} \). Then \( \mathcal{P}_a = \bigcup \mathcal{P}_{\text{args}}: y \in [\alpha]^{\omega_1} \land n < \omega_1 \land s \in (\omega_1)^\nu \) and since \( 2^{\omega_1} = \omega_1 \), hence \( \omega_1^\nu = \omega_1 \), we have that \( \mathcal{P}_a \) is a union of \( \omega_1 \) many sub-orders. Furthermore, if \( \langle x_1, y, n, s \rangle, \ldots, \langle x_k, y, n, s \rangle \in \mathcal{P}_{\text{args}} \) then \( \langle x_1 \cup \ldots \cup x_k, y, n, s \rangle \in \mathcal{P}_{\text{args}} \) and \( (x_1 \cup \ldots \cup x_k, y, n, s) \leq (x_1, y, n, s), \ldots, (x_k, y, n, s) \). Thus, each \( \mathcal{P}_{\text{args}} \) is centered so that \( \mathcal{P}_a \) is \( \omega_1 \)-centered. \( \square \)
Therefore, Lemmas 2.3 and 2.4 imply that $\mathbb{P}_a$ can be viewed as a countable support iteration of length $\omega_2$ with partial orders which are well-met, strongly $\omega_1$-closed and $\omega_1$-centered. Thus, by Proposition 1.2, $\mathbb{P}_a$ is also $\omega_1$-centered.

Now we come to the main result of this section.

**Theorem 2.5**  Assume $(BA)$. Then every $\leq^*$-increasing $\omega_2$-sequence in $(\omega_1^{\omega_1}, \leq^*)$ is a lower half of some $(\omega_2, \omega_2)$-gap.

**Proof.** Let $a = \langle a_\xi : \xi < \omega_2 \rangle$ be an $\leq^*$-increasing $\omega_2$-sequence in $(\omega_1^{\omega_1}, \leq^*)$. Then by the previous results, $\mathbb{P}_a$ is well-met, strongly $\omega_1$-closed and $\omega_1$-centered. Let $G$ be a filter in $\mathbb{P}_a$ and for each $\eta < \omega_2$ let

$$b_\eta = \bigcup \{ s(\eta) : \exists p \in G \langle p = \langle x_p, y_p, n, s \rangle \wedge s = s_p \rangle \}.$$  

Condition (4) of Definition 2.2 together with the requirement that for each $\xi, \eta < \omega_2$ and each $m < \omega_1$ the filter $G$ has a nonempty intersection with the following dense sets

$$D_{\xi m} = \{ (x, y, n, s) \in \mathbb{P}_a : \xi \in x \wedge \eta \in y \wedge n \geq m \}$$

will guarantee that $\forall \xi, \eta < \omega_2 (a_\xi \leq^* b_\eta)$. In addition, condition (3) of Definition 2.2 together with the requirement that for each $\xi < \eta < \omega_2$ and each $m < \omega_1$ the filter $G$ has a nonempty intersection with the following dense sets

$$E_{\xi m} = \{ (x, y, n, s) \in \mathbb{P}_a : \xi \in y \wedge \{ i : s(\eta)(i) < s(\xi)(i) \} \geq m \}$$

will guarantee that $\forall \xi < \eta < \omega_2 (b_\eta \leq^* b_\xi)$. Then the total number of these dense sets $D_{\xi m}$ and $E_{\xi m}$ is $\omega_2$. Therefore, to satisfy the requirements that $\forall \xi, \eta < \omega_2 (a_\xi \leq^* b_\eta)$ and $\forall \xi < \eta < \omega_2 (b_\eta \leq^* b_\xi)$ the filter $G$ needs to intersect $\omega_2$ dense subsets of $\mathbb{P}_a$ and by $(BA)$ there is one such filter. In addition, the definition of $\mathbb{P}_a$ implies that $\forall \xi < \omega_2 \forall i < \omega_1 (a_\xi(i) \leq b_\xi(i))$ and $\forall \xi, \eta < \omega_2 (\xi < \eta \rightarrow \exists i < \omega_1 (b_\xi(i) < a_\eta(i)))$ so that $(a, b)$ is in fact an $(\omega_2, \omega_2)$-gap in $(\omega_1^{\omega_1}, \leq^*)$. \[\Box\]

3. **Trees.** It is well known that $2^{\omega_1} = \omega_1$ implies the existence of an $\omega_2$-Aronszajn tree (see [6]). Since $2^{\omega_1} = \omega_1$ is a part of $(BA)$, it follows that $(BA)$ settles the existence of an $\omega_2$-Aronszajn tree. By the results of Laver and Shelah [7] and Shelah and Stanley [10] the existence of an $\omega_2$-Suslin tree is consistent with and independent of $(BA)$. In this section we consider the influence of $(BA)$ on the existence of $\omega_2$-Kurepa trees. We show that the existence of such trees is consistent with and independent of a stronger version of Generalized Martin’s Axiom, due to Shelah [9], than the one we have considered so far. Recall that an $\omega_2$-Kurepa tree is a tree $T = (T, \leq_T)$ of height $\omega_2$ such that any level of $T$ is of size $< \omega_2$. If $x \in T$ let $\hat{x} = \{ y \in T : y \not< T x \}$. We also assume that $T = \omega_2$ and that all our trees have the following properties:

1. $|\operatorname{Lev}_0(T)| = 1$,
2. $\forall \alpha < \beta < \operatorname{hight}(T) \forall x \in \operatorname{Lev}_\alpha(T) \exists y_1, y_2 \in \operatorname{Lev}_\beta(T) (y_1 \neq y_2 \land x \not< T y_1, y_2)$,
3. $\forall \alpha < \operatorname{hight}(T) \forall x, y \in \operatorname{Lev}_\alpha(T) (\lim \alpha \rightarrow (x = y \leftrightarrow \hat{x} = \hat{y}))$.  

We begin by formulating Shelah’s version of Generalized Martin’s Axiom. A partial order \(P\) is \(\omega_2\)-normal if \(\{p_\alpha : \alpha < \omega_2\} \subseteq P\) then there is a club \(C \subseteq \omega_2\) and a regressive function \(f: \omega_2 \to \omega_2\) such that for \(\alpha, \beta \in C\) if \(cf(\alpha) = cf(\beta) = \omega_1\) and \(f(\alpha) = f(\beta)\) then \(p_\alpha\) and \(p_\beta\) are compatible. Note that \(\omega_2\)-normality is a strengthening of \(\omega_2\)-Knaster condition, which states that if \(\{p_\alpha : \alpha < \omega_2\} \subseteq P\) then there is an \(A \in [\omega_2]^{\omega_2}\) such that any two elements in \(\{p_\alpha : \alpha \in A\}\) are compatible. Let \(GMA^*\) denote the statement that if \(P\) is a partial order such that \(|P| < 2^{\omega_1}\), it is well-met, strongly \(\omega_1\)-closed and \(\omega_2\)-normal and \(D\) is a family of \(< 2^{\omega_1}\) dense subsets of \(P\) then there is a filter \(G \subseteq P\) meeting all the elements of \(D\). The following Lemma is due to Shelah [9].

**Lemma 3.1.** Suppose \(2^\omega = \omega_1\), \(2^{< \kappa} = \kappa\) and \(\kappa\) is a regular cardinal. Let \(\langle P_\alpha : \alpha \leq \kappa \rangle, \langle Q_\alpha : \alpha < \kappa \rangle\) be a countable support iteration such that

\[1 \Vdash_{P_\alpha} \text{“ } Q_\alpha \text{ is well-met, strongly } \omega_1\text{-closed and } \omega_2\text{-normal \”}.

Then \(P_\kappa\) is strongly \(\omega_1\)-closed and \(\omega_2\)-normal.

This Lemma is essentially all that is needed in Shelah’s proof [9] of the consistency of

\[(SA) \quad 2^\omega = \omega_1 + 2^{\omega_1} > \omega_2 + GMA^*\]

This Lemma will also play a role in the analysis below.

To obtain a model for \(SA\) in which there is an \(\omega_2\)-Kurepa tree, we start with a ground model \(V\) for \(ZFC + GCH\) in which there is an \(\omega_2\)-Kurepa tree. For example, the constructible universe \(L\) has this property. Then iterate, as in [9], to obtain a model for \(SA\). By Lemma 3.1 cofinalities and hence cardinals are preserved by the iteration so that any \(\omega_2\)-Kurepa tree in the ground model remains an \(\omega_2\)-Kurepa tree in the extension. Thus, the existence of an \(\omega_2\)-Kurepa tree is consistent with \(SA\).

The construction of a model for \(2^{\omega_2} = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^*\) in which there are no \(\omega_2\)-Kurepa trees requires the existence of a strongly inaccessible cardinal and it is analogous to Devlin’s construction [3] of a model for \(2^\omega = \omega_2 + MA\) in which there are no \(\omega_1\)-Kurepa trees. Therefore we only present an outline of our construction.

The construction will proceed as follows. Start with a model for \(ZFC + GCH\) in which \(\kappa\) is a strongly inaccessible cardinal. Then collapse \(\kappa\) to \(\omega_3\) by the Levy collapse \(L_\kappa\) (see below). In the extension, there are no \(\omega_2\)-Kurepa trees. Then iterate, as in [9], to obtain a model for \(2^{\omega_2} = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^*\). We use Lemma 3.1 to show that in the final model there are no \(\omega_2\)-Kurepa trees.

Now, we define the Levy collapse \(L_\kappa\) and present some of its properties whose proofs are standard.

**Definition 3.2.** \(L_\kappa = \{p : p \leq \omega_1 \land p \text{ is a function} \land \text{dom}(p) \subseteq \kappa \times \omega_2 \land \forall (\alpha, \xi) \in \text{dom}(p)(p(\alpha, \xi) \in \alpha)\}\), where \(p \leq q\) if and only if \(p \supseteq q\).

For \(\lambda < \kappa\) let \(L_\lambda = \{p \in L_\kappa : \text{dom}(p) \subseteq \lambda \times \omega_2\}\) and \(L^\lambda = \{p \in L_\kappa : \text{dom}(p) \subseteq (\kappa \setminus \lambda) \times \omega_2\}\). Then \(L_\lambda \times L^\lambda\) is isomorphic to \(L_\kappa\).
LEMMA 3.3. \( L_\kappa \) is \( \omega_2 \)-closed. If \( \kappa \) is strongly inaccessible, then \( L_\kappa \) has the \( \kappa \)-cc.

LEMMA 3.4. Let \( M \) be a countable transitive model (c.t.m.) for \( ZFC + GCH \) and suppose \( \kappa \) is strongly inaccessible in \( M \) and \( G \) is \( L_\kappa \)-generic over \( M \). Then \( \omega^M[G] = \omega^M, \omega_2^M[G] = \omega_2^M, \omega_3^M[G] = \kappa \) and, in \( M[G] \), there are no \( \omega_2 \)-Kurepa trees.

So, by extending with \( L_\kappa \), \( \omega_1 \) and \( \omega_2 \) remain unchanged and \( \kappa \) gets collapsed to \( \omega_3 \) and if \( GCH \) holds in \( M \) it also holds in \( M[G] \).

The idea now is to start with a model \( M[G] \), as above, and iterate, as in \([9]\), \( \omega_3 \) times to obtain a model for \( 2^\omega = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^* \). But we need to know that the iteration does not introduce any new \( \omega_2 \)-Kurepa trees. The next two Lemmas are toward this end. We omit the proofs as the Lemmas and their proofs are the analogues of the corresponding Lemmas for \( \omega_1 \)-trees. The first one is the analogue of Lemma 3.6 in \([3]\) and the second one is the analogue of Theorem 8.5 in \([1]\).

LEMMA 3.5. Let \( M \) be a c.t.m. for \( ZFC \) and suppose that, in \( M \), \( P \) and \( Q \) are partial orders such that \( P \) is \( \omega_1 \)-closed and \( \omega_2 \)-normal and \( Q \) is \( \omega_2 \)-closed. Let \( G \) be \( P \times Q \)-generic over \( M \). Let \( G_P = \{ p \in P : (p, 1) \in G \} \) and \( G_Q = \{ q \in Q : (1, q) \in G \} \). Then if \( T \) is an \( \omega_2 \)-tree in \( M[G_P] \) and \( b \) is an \( \omega_2 \)-branch of \( T \) in \( M[G] \), then \( b \in M[G_P] \). In addition \( \omega^M[G] = \omega^M_1 \) and \( \omega^M_2[G] = \omega^M_2 \).

LEMMA 3.6. Suppose \( T \) is an \( \omega_2 \)-tree and \( P \) is strongly \( \omega_1 \)-closed and \( \omega_2 \)-normal partial order. Then forcing with \( P \) adds no new \( \omega_2 \)-branches through \( T \).

Now we state and prove the main result of this section.

THEOREM 3.7. Let \( M \) be a c.t.m. for \( ZFC + GCH \) and \( \kappa \) strongly inaccessible in \( M \). Then there is an extension of \( M \) which is a model for \( 2^\omega = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^* \) in which there are no \( \omega_2 \)-Kurepa trees.

**Proof.** Let \( M \) be as above and \( G \) \( L_\kappa \)-generic over \( M \). Then, by Lemma 3.4, in \( N = M[G] \) there are no \( \omega_2 \)-Kurepa trees and \( GCH \) still holds. In \( N \), we perform a countable support iteration of length \( \omega_3 \), as in \([9]\), to obtain a model for \( 2^\omega = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^* \). Let \( \langle \langle P_\alpha : \alpha \leq \omega_3 \rangle, \langle Q_\alpha : \alpha < \omega_3 \rangle \rangle \) be such iteration and \( H \) \( P_{\omega_3} \)-generic over \( N \). Then \( N[H] \) is a model for \( 2^\omega = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^* \) and we now show that there are no \( \omega_2 \)-Kurepa trees in \( N[H] \). In \( N \), let \( \tau \) be a nice \( P_{\omega_3} \)-name for an \( \omega_2 \)-tree in \( N[H] \) (see \([6]\) for the definition of a nice name). Since, by Lemma 3.1, \( P_{\omega_3} \) has \( \omega_2 \)-cc there is an \( \alpha < \omega_3 \) such that \( \tau \) is in fact a \( P_{\alpha} \)-name. Then \( H_\alpha \), the restriction of \( H \) to \( P_\alpha \), is \( P_\alpha \)-generic over \( N \). Since \( \alpha < \omega_3 \), the iteration is with countable supports, we are considering only partial orders of size \( < \omega_3 \) (i.e. \( 1 \vdash_{P_\alpha} " \langle Q_\alpha \downarrow \omega_3 \rangle \) \( GCH \) holds in \( M[G_\lambda] \), hence the density of \( P_\alpha \) is \( < \omega_3 \), we may assume that \( | P_\alpha | < \omega_3 \). Now, in \( M \), \( L_\kappa \) has the \( \kappa \)-cc (by Lemma 3.3), so there is some \( \lambda < \kappa \) such that if \( G_\lambda \) is the restriction of \( G \) to \( L_\lambda \), then \( P_\alpha \in M[G_\lambda] \) and \( H_\alpha \) is \( P_{\alpha} \)-generic over \( M[G_\lambda] \). Now \( T = \tau[G] \in M[G_\lambda][H_\alpha] \) and, by Lemma 3.5, any \( \omega_2 \)-branch of \( T \) which is in \( M[G_\lambda][H_\alpha][G_\lambda] \) is already in...
$M[G][H_\alpha]$. So, in $M[G][H_\alpha]$, $T$ has at most $2^{\omega_2} = \theta$ such branches and since $\kappa$ is still strongly inaccessible we have that $\theta < \kappa$. But, in $M[G][H_\alpha][G^\lambda]$, $\kappa$ is collapsed to $\omega_2$. So $T$ can have at most $\aleph_2$ many $\omega_2$-branches in $M[G][H_\alpha][G^\lambda]$. But $M[G][H_\alpha][G^\lambda] = M[G][G^\lambda][H_\alpha] = M[G][H_\alpha] = N[H_\alpha]$. So $T$ has at most $\aleph_2$ many $\omega_2$-branches in $N[H_\alpha]$. However, by Lemma 3.1, $P^{\alpha}$ is $\omega_2$-normal so that, by Lemma 3.6, $T$ does not obtain any new $\omega_2$-branches in the extension $N[H_\alpha][H^\alpha]$. But $N[H_\alpha][H^\alpha] = N[H]$. So $T$ can not be an $\omega_2$-Kurepa tree in $N[H]$ which proves that in the model $N[H]$ there are no $\omega_2$-Kurepa trees. This finishes the proof of the Theorem. \[ \square \]

Therefore, the existence of an $\omega_2$-Kurepa tree is consistent with and independent of (S.A) and hence consistent with and independent of (B.A).

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