IDENTITY AND PERMUTATION

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Abstract. It is known that in the purely implicational fragment of the system $\textbf{TW}_ \rightarrow$ if both $(A \to B)$ and $(B \to A)$ are theorems, then $A$ and $B$ are the same formula (the Anderson-Belnap conjecture). This property is equivalent to NOID (no identity!): if the axiom-schema $(A \to A)$ is omitted from $\textbf{TW}_ \rightarrow$ and the system $\textbf{TW}_ \rightarrow \text{ID}$ is obtained, then there is no theorem of the form $(A \to A)$.

A Gentzen-style purely implicational system $\textbf{J}$ is here constructed such that NOID holds for $\textbf{J}$. NOID is proved to be equivalent to NOE; there no theorem of $\textbf{J}$ of the form $((A \to A) \to B) \to B$, i.e., of the form of the characteristic axiom of the implicational system $\textbf{E}_ \rightarrow$ of entailment.

If $(p \to p)$ is adjoined to $\textbf{J}$ as an axiom-schema (ID), then there are theorems $(A \to B)$ and $(B \to A)$ such that $A$ and $B$ are distinct formulas, which shows that for $\textbf{J}$ the Anderson-Belnap conjecture is not equivalent to NOID.

The system $\textbf{J} + \text{ID}$ is equivalent to $\textbf{RW}_ \rightarrow$ of relevance logic.

Introduction

By $\textbf{TW}_ \rightarrow$ we understand the system of propositional relevance logic defined in the language with $\rightarrow$ as the sole connective, by the following axiom-schemata:

\[
\begin{align*}
\text{ID} & : (A \to A) \\
\text{ASU} & : ((A \to B) \to ((B \to C) \to (A \to C))) \\
\text{APR} & : ((B \to C) \to ((A \to B) \to (A \to C))).
\end{align*}
\]

The only rule of $\textbf{TW}_ \rightarrow$ is modus ponens.

By $\textbf{TW}_ \rightarrow - \text{ID}$ we understand the system obtained from $\textbf{TW}_ \rightarrow$ by deleting the schema ID.

It has been shown that the following propositions were equivalent (Dwyer-Powers theorem):

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\end{center}

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if both \((A \rightarrow B)\) and \((B \rightarrow A)\) are provable in \(\textbf{TW}_\rightarrow\), then \(A\) and \(B\) are the
same formula (Anderson-Belnap's conjecture).

For no formula \(A\) is \((A \rightarrow A)\) provable in \(\textbf{TW}_\rightarrow\text{-ID}\) (NOID).

Anderson-Belnap's conjecture is about an interesting property. Let us write
\(A \equiv B\) iff both \((A \rightarrow B)\) and \((B \rightarrow A)\) are theorems of \(\textbf{TW}_\rightarrow\); then the axioms
of \(\textbf{TW}_\rightarrow\) and modus ponens are sufficient to show that \((a)\ \equiv 3\) is an equivalence
relation and \((b)\) that it is a congruence with respect to \(\rightarrow\). By Anderson-Belnap's
conjecture (the antisymmetry of \(\rightarrow\)) this congruence is the smallest congruence
relation i.e., equality. Thus, the identity of formulas in the language with \(\rightarrow\) as the
only connective can be characterized exclusively by logical means – by the theory
\(\textbf{TW}_\rightarrow\) of implication.

NOID (and hence the Anderson-Belnap's conjecture) has been proved true
(cf. \([2], [3] \) and \([4]\)).

The proof of NOID in \([3]\) has been obtained for a proper extension \(\textbf{L}\) of
\(\textbf{TW}_\rightarrow\text{-ID}\).

Let \(S\) and \(S'\) be theories of implication and let \(A\)-B and NOID be the following
claims about \(S\) and \(S'\):

\(A\)-B if both \((A \rightarrow B)\) and \((B \rightarrow A)\) are provable in \(S\), then \(A\) and \(B\) are the same
formula,

and

NOID there is no theorem of \(S'\) of the form \((A \rightarrow A)\).

Obviously, if \(S = \textbf{TW}_\rightarrow\) and \(S' = \textbf{TW}_\rightarrow\text{-ID}\), then \(A\)-B and NOID are equiva-
 lent.

In this paper we shall develop a proper extension \(\textbf{J}\) of \(\textbf{L}\) and prove that

(1) NOID holds for \(\textbf{J}\) and \(A\)-B does not hold for \(\textbf{J}+\text{ID}\);

(2) NOID is equivalent to the following proposition: \(((A \rightarrow B) \rightarrow B)\) is a
theorem of \(\textbf{J}\) iff so is \(A\).

The non-equivalence of \(A\)-B and NOID for \(\textbf{J}\) and \(\textbf{J}+\text{ID}\) is due to permutation
present in \(\textbf{J}\) in the form of the rule \(\text{PERM}\).

The claim (2) is interesting because it shows that NOID cannot hold in any
system containing as a theorem any form of the \(E_\rightarrow\) axiom

\(((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B))\).

Also, (2) will enable us to prove that there are in \(\textbf{J}\) some restricted forms
of contraction: any formula \(((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B))\) is a theorem of \(\textbf{J}\) iff
so is \(A\).

(3) We shall show that NOE can be extended to formulas of a certain type.
The system \( J \)

Some of the basic definitions given below are taken from [3].

Let \( p, q, r, \ldots \) stand for propositional variables. The letters \( A, B, C, \ldots \) range over the set of formulas. Instead of \( (A \rightarrow B) \) we shall write \( (AB) \). Also, we omit parentheses, with the association to the left. Thus, \( ABC \) stands for \( (AB)C \).

Let \( R, S, T, U, V, W, X, Y, Z, \ldots \) range over finite (possibly empty) sequences of formulas. If \( X \) consists of a single formula \( A \), we shall write \( A \) for \( X \).

If \( X \) is empty, let \( XB \) denote \( B \). If \( X = \langle A_1, \ldots, A_n \rangle \), \( n \geq 1 \), then \( XB \) denotes the formula

\[
A_1 \rightarrow (A_2 \rightarrow \cdots \rightarrow (A_n \rightarrow B) \ldots).
\]

Notice that any formula is of the form \( Wp \), for some \( W \) and a variable \( p \). Very often we shall write \( W_A p \) for \( A \), for any formula \( A \).

By \( \pi(X) \) we denote any permutation of \( X \), and by \( \pi(X).B \) we denote any formula \( YB \) such that \( Y \) is a permutation of \( X \).

Let \( CDE \) be a subformula of \( A \); suppose that \( B \) is obtained from \( A \) by substitution of \( DCE \) for \( CDE \), at a single occurrence of \( CDE \) in \( A \); then we shall say that \( B \) is obtained from \( A \) by the rule \( \text{PERM} \). Let us write \( A \sim B \) iff \( B \) can be obtained from \( A \) by a finite (possibly zero) number of applications of \( \text{PERM} \). It is clear that \( \sim \) is an equivalence relation. We shall write \( X \sim Y \) iff \( Y \) can be obtained from a permutation \( Z \) of \( X \) by a finite (possibly zero) number of applications of \( \text{PERM} \) to some members of \( Z \). For any \( A \) by \( A^* \) we shall denote any formula \( B \) such that \( A \sim B \). Also, for any \( X \) by \( X^* \) we denote any \( Y \) such that \( X \sim Y \). It is clear that \( (\pi(X))^* \sim \pi(X^*) \).

The axioms of \( J \) are given by the following scheme:

\[
\text{ASU} \quad \pi((AB)^*, Bp, A).p
\]

The rules of \( J \) are:

\[
\begin{align*}
\text{JSU} & \quad \text{From } \pi(X,Y).p \text{ to infer } \pi(X^*, (Y^*).q).q. \\
\text{JPR} & \quad \text{From } \pi(X,B).p \text{ to infer } \pi(X^*, (AB)^*, A).p. \\
\text{JG} & \quad \text{From } \pi(X,Y).p \text{ and } \pi(Z,B).q \text{ to infer } \\
& \quad \pi(X^*, Z^*, ((Y).p(B))^*).q.
\end{align*}
\]

The rule \( \text{JG} \) is to be understood as follows: if there are permutations \( V \) and \( W \) of the sequences \( X, Y \) and \( Z, B \), respectively, such that \( Vp \) and \( Wq \), are derivable in \( J \), so is \( W^t q \), for any permutation \( W' \) of the sequence \( X^*, Z^*( ((Y).p(B))^* \). In a similar way we understand \( \text{JSU} \) and \( \text{JPR} \).

We shall assume that derivations in \( J \) are given in forms of trees, with usual properties. The weight \( w \) of a node in a derivation, derivability with weight \( w \) and the combined weight are defined as in [5, p. 113]. By the degree of \( A \) (of \( X \)) we understand the number of occurrences of \( \rightarrow \) in \( A \) (in \( X \)).

Let us define \( Ap^n \) as follows: \( Ap^0 = A \); \( Ap^{n+1} = (Ap^n)p \).
\textbf{J} is closed under modus ponens

We start with

\textbf{Theorem 1.} If \( A \) is derivable in \( J \) with weight \( w \), so is \( A^* \); if \( X.p \) is derivable in \( J \), so is \( \pi(X).p \), for any permutation \( \pi(X) \) of \( X \).

\textit{Proof.} By an easy induction on the weight of \( A \) in a given derivation of \( A \).

Theorem 1 shows that \( J \) is closed under PERM; it enables us to identify \( A \) and \( A^* \), \( X \) and \( X^* \), and \( X \) and \( \pi(X) \) in derivations in \( J \). In the sequel this identification is assumed.

\textbf{Theorem 2.} If (a) \( \pi(X,Y).p \) is derivable in \( J \), so is (b) \( \pi(X,\pi(Y,p,Z).q,Z).q \).

\textit{Proof.} By JSU we obtain \( \pi(X,(Y.p)q).q \) from (a); hence, (b) is obtained by using JPR.

Theorem 2 and JG show that \( J \) is closed under the following assertion rules:

\begin{itemize}
  \item [ASS1] From \( A \) to infer \( ABB \).
  \item [ASS2] From \( A \) and \( \pi(X,B).p \) to infer \( \pi(X,AB).p \).
\end{itemize}

\textbf{Theorem 3.} (Transitivity, JTR) If (a) \( \pi(X,Y).p \) and (b) \( \pi(Y,p,Z).q \) are derivable in \( J \), so is (c) \( \pi(X,Z).q \).

\textit{Proof.} Proceed by double induction. Suppose that (a) and (b) are derivable with combined weight \( w \) and that \( Y.p \) is of degree \( d \). Our induction hypotheses are:

\begin{itemize}
  \item [Hyp 1] The theorem holds for any \( Y'.p \) of degree \( d' < d \) and any combined weight \( w' \);
  \item [Hyp 2] The theorem holds for \( Y.p \) and any combined weight \( w' < w \).
\end{itemize}

\begin{itemize}
  \item [Case I] (b) is an instance of ASU; hence, \( \pi(Y,p,Z) \sim \pi(AB,Bq,A) \) for some \( A, B \) and \( q \).
    \begin{itemize}
      \item [I.1] \( Y.p \sim AB \sim \pi(A,W_B).p \), and \( Y \sim \pi(A,W_B) \). From (a) we obtain (c) by using JSU.
      \item [I.2] \( Y.p \sim Bq \); hence, \( Y \sim B \) and \( p \sim q \). From (a) we obtain (c) by using JPR.
      \item [I.3] \( Y.p \sim A \); hence, by Theorem 2 we obtain (c).
    \end{itemize}
  \end{itemize}

\begin{itemize}
  \item [Case II] (b) is obtained by JSU from (b1) \( \pi(V,W).r \), where \( \pi(V,(W.r)q) \sim \pi(Y,p,Z) \).
    \begin{itemize}
      \item [II.1] \( V \sim \pi(V',Y,p) \); by (a), (b1) and Hyp 2 \( \pi(X,V',W).r \) is derivable; hence, by using JSU we obtain (c).
      \item [II.2] \( (W.r)q \sim Y.p \) and \( Z \sim V \); hence, \( Y \sim W.r \) and \( p \sim q \). By (a), (b1) and Hyp 1, (c) is derived.
    \end{itemize}
  \end{itemize}

\begin{itemize}
  \item [Case III] (b) is obtained by JPR from (b1) \( \pi(V,B).q \), where \( \pi(V,AB,A) \sim \pi(Y,p,Z) \).
\end{itemize}
III.1 \( V \sim \pi(V', Y, p) \) and \( Z \sim \pi(V', AB, A) \); by (a), (b1), and Hyp 2, we 

obtain \( \pi(X, V', B), q \), and then (c) by using JPR.

III.2 \( AB \sim Y, p \). We have (a) \( \pi(X, A), B \); hence, by (a), (b1), and Hyp 1, (c) 
is derived.

III.3 \( A \sim Y, p \). From (a) and (b1) we obtain (c) by JG.

Case IV \( (b) \) is obtained by JG from (b1) \( \pi(U, V), r \) and (b2) \( \pi(W, A), q \), where

\[ \pi(Y, p, Z) \sim \pi(U, W; (V, r).A). \]

IV.1 \( U \sim \pi(U', Y, p) \) and \( Z \sim \pi(U', W, (V, r).A) \). By (a), (b1) and Hyp 2, 

\( \pi(X, U', V).r \) is derivable. Hence (c), by using (b2) and JG.

IV.2 \( W \sim \pi(W', Y, p) \) and \( Z \sim \pi(U, W', (V, r).A) \). Now \( \pi(X, W', A).q \) is 

derivable by (a), (b2) and Hyp 2; hence (c), by using (b1) and JG.

IV.3 \( (V, r).A \sim Y, p \) and \( Z \sim \pi(U, W) \). It is clear that (a) is \( \pi(X, V, r).A \). By 

(a), (b2) and Hyp 1, (a') \( \pi(X, W, V, r).q \) is derivable. Now by (b1), (a'), and Hyp 1, 
(c) is derivable.

A trivial consequence of this theorem is

**Theorem 4 (Modus Ponens, MP).** If \( A \) and \( AB \) are derivable in \( J \), so is \( B \).

There is a Hilbert style formulation of \( J \). Let \( K \) be the system with MP,
PERM, ASS1 and the axiom-schema \( \pi(AB, BC, A).C \).

**Theorem 5.** \( K \) and \( J \) are equivalent.

**Proof.** It is obvious that \( J \) contains \( K \).

The rules JTR and JPR are easily derivable in \( K \), by using the axioms, MP 

and PERM. In the same way the rules JSU and JG are easily derivable provided 

that \( X \) is nonempty. The rule ASS1 plays the role of JSU when \( X \) is empty. Now 

by using ASS1, JPR and JTR we derive JG when \( X \) is empty (ASS2).

**The system L**

The system \( L \) is obtained from \( J \) by restricting JSU and JG: in JSU and JG 

\( X \) must not be empty. Let \( LSU \) and \( LG \) be JSU and JG, respectively, restricted in 

this way. In [3] it is assumed that \( L \) has a single propositional variable \( p \).

The following theorems were proved in [3].

L1 If \( A \) is derivable in \( L \) with weight \( w \), so is \( A^* \).

L2 \( L \) is closed under the following transitivity rule:

from \( \pi(X, A, Y).p \) and \( \pi(Z, Y^*, p).p \) to infer \( \pi(X^*, Z^*, A^*).p \).

L3 \( L \) contains \( TW_{\rightarrow -ID} \).

L4 There is no theorem of \( L \) of the form \( Ap \).

L5 There is no theorem of \( L \) of the form \( \pi((\pi(X, Y).p)p^{2k}, Y^*).p \), \( k \in \omega \).

L6 There is no theorem of \( L \) of the form \( AA \).
L7  There is no theorem of L of the form ABB.
L8  There is no theorem of L of the form A ABB.
L9  There is no theorem of L of the form A BBA.

L6 - L9 are consequences of L5. We shall prove or disprove theorems about J analogous to L1 - L9 first.

Notice that L is not closed under MP. Let A \sim p p p p p and B \sim (p p p p p) p. AB is an instance of ASU. If L were closed under MP, applying MP to

$$\pi(AB, Bp, A) p$$

twice, Bpp would be obtained in L, contrary to L4.

L is not closed under ASS1 either. Otherwise, App would be derivable, contrary to L4.

That L is not closed under ASS2 can be seen as follows. Let A \sim \pi(p p p p p, p) p; by using A and ASS2, in J we derive B, B \sim \pi(\pi(A, p) p, p p p) p. Let us show that B is not derivable in L.

B is not an instance of ASU.

If B is obtained by LSU from C, then C \sim \pi(\pi(A, p) p, p) p \sim (A(p)) (p p). Let us show that there is no instance of ASS2.

If B is derivable by LPR from \pi(X, F) p, then

$$\pi(\pi(A, p) p, p) p \sim \pi(X, E, E)$$

for some X, E, and F. It is clear that E \sim p.

If F \sim Ap, then \pi(A(p), p) p is derivable in L. But this is neither an axiom nor can it be obtained by LPR or LG. If it is obtained by LSU, then App is derivable, contrary to L4.

If F \sim p, then (A(p)) (p p) is derivable, contrary to L7.

Suppose that B is derivable by LG from \pi(X, Y) p and \pi(Z, E) p. Hence,

$$\pi(\pi(A, p) p, p) p \sim \pi(X, (Y) p E, Z) p$$

If (Y) p E \sim p, then Y is empty, E \sim p and \pi(X, Z) \sim \pi(\pi(A, p) p, p). Now X is not p and Z is not empty (otherwise, pp is derivable). Hence, B is obtained from \pi(A, p) p p and \pi(p, p) p, which is impossible.

If (Y) p E \sim \pi(A, p) p, then \pi(X, Z) \sim \pi(pp, p).

Let Z be empty; then B is obtained from \pi(pp, p, Y) p and Ep, contrary to L4.

Let Z \sim pp; then B is obtained from \pi(Y, p) p and \pi(E, pp) p. Obviously, Y is not empty and E is not pp; hence, E \sim Ap and Y Ep \sim p - a contradiction.

Let Z \sim p then B is obtained from \pi(pp, Y) p and \pi(E, p) p. Since Y cannot be empty, Y p \sim A and E \sim pp, contrary to L6.

This shows that L is not closed under ASS2.

Since JSU = LSU + ASS1 and JG = LG + ASS2, we have J = L + ASS1 + ASS2.
J is a proper extension of L and there is no theorem about J analogous either to L₄ or L₅ or L₇. However, theorems analogous to L₆, L₈, and L₉ still hold true.

No instance of AA is derivable in J

Theorem 6. \((X.p)p\) is derivable in J iff X is nonempty and any member of X is derivable in J.

Proof. Let X be \(\pi(A₁, \ldots, Aₙ)\), \(n > 0\), and let \(A₁, \ldots, Aₙ\) be derivable in J. By ASS₁, \(Aₙpp\) is derivable; if \(n > 1\), by using JG in the form of ASS₂, we derive \((\pi(A₁, \ldots, Aₙ).p)p\), i.e., \((X.p)p\).

Suppose that \((X.p)p\) is derivable. If X is empty, then pp is derivable; however, this is neither an axiom nor can it be obtained by any of the rules. Hence, X is nonempty.

Let \(X \sim \pi(A₁, \ldots, Aₙ)\) and proceed by induction on the weight of the derivation of \((X.p)p\).

Obviously, \((X.p)p\) is neither an instance of ASU nor can it be obtained by JPR. If it is obtained from \((a')\) by JSU, then \((X.p)p \sim (V.p)p\) and \((a')\) is \(V.p\); hence, \(X \sim V.p \sim A₁\), X is nonempty and \(A₁\) is derivable in J.

If \((X.p)p\) is obtained from \((a')\) and \((a'')\) by JG, then \(X.p \sim \pi(U,W,(V.p)C), U\) and W are empty, \(X \sim \pi(A₁, \ldots, Aₙ) \sim \pi(V.p,W_C)\) for some \(A₁, \ldots, Aₙ\), and \((a')\) and \((a'')\) are \(V.p\) and \(W₃.p\) respectively. By induction hypothesis, all members of \(W₃\), say \(W₃ \sim \pi(A₁, \ldots, Aₙ₋₁)\), are derivable in J. Obviously, we can take \(V.p \sim Aₙ\).

This completes the proof of the theorem.

Since J is (as L) closed under uniform substitution, to prove the main theorems of this paper it suffices to prove them under the assumption that there is only one variable in J, say \(p\). Let \(J₁\) be J with just one variable \(p\). In the sequel, if not stated otherwise, "derivable" means "derivable in \(J₁\)."

Theorem 7 (NOID). There is no theorem of \(J₁\) of the form \(\pi((X.p)p^{2k}, X).p\), \(k \in \omega\).

Proof. If there is a theorem of \(J₁\) of this form, then

Hyp 3 there is a formula \((a)\) \(\pi((X.p)p^{2k}, X).p\) of smallest degree derivable in \(J₁\).

Let us consider how \((a)\) could have been obtained. We leave to the reader the verification that \((a)\) cannot be an instance of ASU.

Case I \((a)\) is obtained from \((a')\) by JSU; hence,

\(\pi((X.p)p^{2k}, X) \sim \pi(Y,(Z.p)p)\)

for some \(Y\) and \(Z\).

1.1 \(Y \sim \pi(Y',(X.p)p^{2k})\) and \(X \sim \pi(Y',(Z.p)p)\). Obviously, we have \((a')\)
\[ \pi(\pi((Y', (Z.p)p.p) p^{2k}, Y', Z).p) \]

If both \( Y' \) and \( Z \) are empty, then \((a')\) is \( pqqq^{2k}p \) hence, \( pp \) is derivable by Theorem 6. This is impossible.

If \( Y' \) is empty and \( Z \) nonempty, then \((a')\) is \( \pi((Z.p)pqqq^{2k}, Z).p \), contrary to Hyp 3.

Let \( Y' \) be nonempty and \( Z \) arbitrary. By using ASU and JPR we derive \((b)\)
\[ \pi(\pi(Y', Z).p, (Z.p)p, Y').p \]
If \( k > 0 \), we use JSU to derive \((c)\) \( \pi(\pi(Y', Z).p, \pi((Y', (Z.p)p.p) p^{2k-1}).p \) Hence, by \((c)\), \((a')\), and JTR we derive \( \pi(\pi(Y', Z).p, Y'), Z).p \), contrary to Hyp 3.

I.2 \((X.p)p^{2k} \sim (Z.p)p \) and \( X \sim Y' \). If \( k = 0 \), then \( X \sim Z.p \) and \((a')\) is \( \pi(Z.p, Z).p \) contrary to Hyp 3.

If \( k > 0 \), then \( Z.p \sim (X.p)p^{2k-1} \) and \( Z \sim (X.p)p^{2k-2} \). Hence, we have \((a')\)
\[ \pi((X.p)p^{2k-2}, X).p \]
contrary to Hyp 3.

Case II \((a)\) is obtained by JPR; hence, \( \pi((X.p)p^{2k}, X) \sim \pi(Y, AB, A) \) for some \( Y, A \) and \( B \).

II.1 \( Y \sim \pi(Y', (X.p)p^{2k}) \) and \( X \sim \pi(Y', AB, A) \). Obviously, we have \((a')\)
\[ \pi((\pi(Y', AB, A).p)p^{2k}, Y', B).p \]
Now \( \pi(Bp, AB, A).p \) is an instance of ASU; hence, by JPR we obtain
\[ \pi(\pi(Y', B).p, Y', AB, A).p \]
and then by using JSU we derive \( \pi(\pi(Y', B).p, \pi(Y', AB, A).p)p.p \). If \( k > 0 \), by JSU we get \( \pi(\pi(Y', B).p, \pi(Y', AB, A).p)p^{2k-1}).p \). Hence, using JTR and \((a')\) we obtain \( \pi(\pi(Y', B).p, Y', B).p \), contradicting thus Hyp 3.

II.2 \((X.p)p^{2k} \sim AB \) and \( X \sim \pi(Y, A) \). Hence,
\[ (X.p)p^{2k} \sim (\pi(Y, WA).p)p^{2k} \sim \pi(WA.p, WB).p \]
If \( k > 0 \), then \( WB \) is empty and we have \( B \sim p \), and \( WA.p \sim (\pi(Y, WA).p)p^{2k-1} \); this is impossible.

Let \( k = 0 \); then \( \pi(Y, A) \sim (A, WB) \) and \( Y \sim WB \). Thus, \((a')\) is
\[ \pi(WB.p, WB).p \]
contrary to Hyp 3.

II.3 \((X.p)p^{2k} \sim A \) and \( X \sim \pi(Y, AB) \); this is impossible.

Case III \((a)\) is obtained by JG; hence, \( \pi((X.p)p^{2k}, X) \sim \pi(Y, Z, (U.p)B) \) and both \((a')\) \( \pi(Y, U).p \) and \((a'')\) \( \pi(Z, B).p \) are derivable.

III.1 \( Y \sim \pi(Y', (X.p)p^{2k}) \) and \( X \sim \pi(Y', Z, (U.p)B) \); hence, \((a')\)
From \((a'')\) we obtain \((b)\) \( \pi(U.p, Z, (U.p)B).p \) by using JPR. If necessary, we apply JPR to obtain \((c)\) \( \pi((Y', U).p, Y', Z, (U.p)B).p \) \( k > 0 \), by using JSU we derive \((d)\)
\[ \pi((Y', U).p, \pi(Y', Z, (U.p)B).p)p^{2k-1}).p \]
Hence, by (d), (a’), and JTR we derive \( π(π(Y’, U).p, Y’, U).p \), contrary to Hyp 3.

III.2 \( Z \sim π(Z’, (X.p)p^{2k}) \) and \( X \sim π(Y, Z’, (U.p)B) \); hence, (a’’)

On the other hand, from (a’) we obtain (b) \( π(Y, (U.p)B).B \), by Theorem 2.
Hence, by using JSU we derive (c) \( π(Bp, Y, (U.p)B).p \), and if \( Z’ \) is nonempty, we derive
by using JPR. Now if \( k > 0 \), we can use JSU to obtain (e)

In any case we can use (e), (a’’), and JTR to obtain \( π(π(Z’, B).p, Z’, B).p \), contrary to Hyp 3.

III.3 \( (X.p)p^{2k} \sim (U.p)B \) and \( X \sim π(Y, Z) \). If \( k > 0 \), then \( B \sim p, U.p \sim (X.p)p^{2k-1} \) and \( U \sim (X.p)p^{2k-2} \). Obviously, we have (a’’)
\[ π((π(Y, Z).p)p^{2k-2}, Y).p \]
and (a”) \( π(Z, p).p \). Hence, \( Z \) is nonempty.

III.3.1 Let \( Y \) be empty; then (a’) \( (Z.p)p^{2k-1} \). We derive
\[ (b) \quad π((p, Z.p)p^{2k-1}).p \]
by using (a”) and JSU. Hence, by using (a’), (b), and MP we obtain \( pp \), which is impossible.

III.3.2 Let \( Y \) be nonempty. By using (a”) and JPR we derive
\[ (b) \quad π(Y.p, Y, Z).p \]
hence, by applying JSU to (b) we derive (c) \( π(Y.p, (π(Y, Z).p)p^{2k-1}).p \), and hence
\( π(Y.p, Y).p \) is derivable by using (a’), (c), and JTR, contrary to Hyp 3.

Let \( k = 0 \) and \( B \sim V.p \); then \( X \sim π(U.p, V) \).

III.3.3 \( Y \sim π(Y’, U.p) \) and \( V \sim π(Y’, Z) \). We have
\[ (a’) \quad π(Y’, U.p, U).p \]
and (a”) \( π(π(Y’, Z).p, Z).p \).

If \( Y’ \) is empty, Hyp 3 is violated.

Let \( Y’ \) be nonempty. If \( Z \) is empty, \( (a’’) \) becomes \( (Y’.p)p \) and hence
\( π(U.p, U).p \) is obtained from (a’) and (a”) by JTR, contrary to Hyp 3.

Let \( Z \) be nonempty. By using JPR, from (a’) we obtain
\[ π(π(Z, U).p, U, Y’, Z).p \]
and (a”) \( π(Z, U).p \) by using JTR, contrary to Hyp 3.

III.3.4 \( Z \sim π(Z’, U.p) \) and \( V \sim π(Y, Z’) \). We have
\[ (a’’) \quad π(Y, U).p \]
and (a”) \( π(Z’, U.p, π(Y, Z’).p).p \).

From (a’) and (a”) we obtain \( π(π(Y, Z’).p, Y, Z’).p \) by using JTR, contrary to Hyp 3.

This completes the proof.

**Theorem 8.** There is no theorem of J of the form \( AA \).
Theorem 9. There is no theorem of \( J \) of the form \( AABB \).

Theorem 10. There is no theorem of \( J \) of the form \( ABBA \).

Theorems 8–10 are trivial consequences of NOID.

**No instance of \( AABB \) is derivable in \( J \)**

**Theorem 11 (NOE).** \( \pi(\pi(X,Y),p,Y) \) is derivable in \( J_1 \) if and only if \( \pi(X,Y) \) is derivable in \( J_1 \).

*Proof.* To prove the non-trivial part of the theorem, proceed by induction on the degree of \( \pi(X,Y) \). If \( Y \) is empty, we use Theorem 6. Let us accept the induction hypothesis

Hyp 4. The theorem holds for any \( \pi(X',Y') \) of degree smaller than the degree of \( \pi(X,Y) \).

Suppose that (a) \( \pi(\pi(X,Y),p,Y) \) is derivable in \( J_1 \). By NOID, \( X \) is nonempty. The verification that (a) is not an instance of ASU is left to the reader.

Case I. (a) is obtained by JSU from (a') \( \pi(U,V),p \), where \( \pi(\pi(X,Y),p,Y) \sim \pi(U,(V)p) \).

I.1. \( (V,p)p \sim \pi(X,Y) \) and \( U \sim Y \); obviously, either \( X \) or \( Y \) is empty. But \( X \) is nonempty. If \( Y \) is empty, then by Theorem 6, \( X \sim \pi(A_1,\ldots,A_n) \) for some derivable \( A_1,\ldots,A_n \).

I.2. \( Y \sim \pi((V,p)p,Y') \) and \( U \sim \pi(\pi(X,(V,p)p,Y'),p,Y') \). Obviously, (a) is obtained from (a') \( \pi(\pi(X,(V,p)p,Y'),p,Y') \). Since \( X \) is nonempty, there is a member \( A \) of \( X \). But as an instance of ASU we have \( \pi(\pi(A,V),p,(V,p)p,A) \). By using JPR we derive \( \pi(\pi(X,Y'),p,X,(V,p)p,Y') \). Hence, by using JTR and (a') we obtain \( \pi(\pi(X,Y'),p,Y',Y') \). By Hyp 4, \( X \sim \pi(A_1,\ldots,A_n) \) for some \( A_1,\ldots,A_n \) and \( n \), and \( A_1,\ldots,A_n \) are derivable in \( J_1 \).

Case II. (a) follows by JPR from (a') \( \pi(U,D)p \), where \( \pi(\pi(X,Y),p,Y) \sim \pi(U,CD,C) \).

II.1. \( Y \sim \pi(CD,C,Y') \) and \( U \sim \pi(\pi(X,CD,C,Y'),p,Y') \). But

\[
\pi(\pi(X,D,Y'),p,X,CD,C,Y').p
\]

is easily derivable in \( J_1 \). Hence, by using JTR and (a'), so is

\[
\pi(\pi(X,D,Y'),p,D,Y')p.
\]

Hence, by Hyp 4, \( X \sim \pi(A_1,\ldots,A_n) \) for some \( A_1,\ldots,A_n \) and \( n \), and \( A_1,\ldots,A_n \) are derivable in \( J_1 \).

II.2. \( \pi(X,Y),p \sim CD \) and \( Y \sim \pi(U,C) \); hence, \( \pi(X,C,U)p \sim CD \). It is clear that \( \pi(X,U) \sim W_D \). Now (a') is \( \pi(\pi(X,U),p,U) \) and by Hyp 4, \( X \sim \pi(A_1,\ldots,A_n) \) for some \( A_1,\ldots,A_n \) and \( n \), and thus \( A_1,\ldots,A_n \) are derivable in \( J_1 \).

II.3. \( \pi(X,Y),p \sim C \) and \( Y \sim \pi(CD,Y') \); this is impossible.
Case III (a) follows by JG from (a') \( \pi(U, V).p \) and (a'') \( \pi(W, D).p \), where we have
\[
\pi(\pi(X, Y).p, Y) \sim \pi(U, W).p (V.pD).
\]
III.1 \( \pi(X, Y).p \sim (V.p)D \) and \( Y \sim \pi(U, W) \); hence,
\[
\pi(X, U, W) \sim \pi(V.p, W.D).
\]
III.1.1 \( X \sim \pi(X', V.p), W_D \sim \pi(X', U, W) \), and (a'') is
\[
\pi(\pi(X', U, W).p, W).p
\]
If \( U \) is empty, by Hyp 4 and (a''), \( X' \sim \pi(A_1, \ldots, A_{n-1}) \) for some derivable \( A_1, \ldots, A_{n-1} \) and \( n \). On the other hand, \( (a') \) is \( V.p \) and we may take \( V.p \sim A_n \).

If \( U \) is nonempty, from \( (a') \) we obtain \( \pi((V.p)p, U).p \) and hence
\[
\]
by JPR. Now by using JTR and (a''), we obtain \( \pi(\pi(X, W).p, V, W).p \). Hence, by Hyp 4, \( X \sim \pi(A_1, \ldots, A_4) \) for some derivable \( A_1, \ldots, A_n \).

III.1.2 \( U \sim \pi(V.p, U').p \) and \( W_D \sim \pi(X, U', W) \). Obviously, \( (a') \) and (a'') are
\[
\pi(V.p, U', V).p \text{ and } \pi(W, \pi(X, U', W).p).p,
\]
respectively. Hence, by using JPR and (a''), we easily derive
\[
\]
Now by using (a'') and JTR we get \( \pi(\pi(X, V, W).p, V, W).p \) in \( J_1 \). By Hyp 4 we have that for some derivable \( A_1, \ldots, A_n, X \sim \pi(A_1, \ldots, A_n) \).

III.1.3 \( W \sim \pi(V.p, W').p \) and \( W_D \sim \pi(X, U', W') \). Obviously, \( (a') \) and (a'') are \( \pi(U, V).p \) and \( \pi(V.p, W', \pi(X, U', W').p).p \), respectively. By JTR, we derive \( \pi(\pi(X, U', W')p, U, W').p \). By Hyp 4, \( X \sim \pi(A_1, \ldots, A_n) \) and \( A_1, \ldots, A_n \) for some derivable \( A_1, \ldots, A_n \).

III.2 \( U \sim \pi(\pi(X, Y).p, U') \) and \( Y \sim \pi((V.p)D, U', W). \) Now (a'') is
\[
\pi((\pi(X, V.p)D, U', W).p, U', V).p
\]
By using (a'') \( \pi(W, D).p \) and JPR we derive
\[
\]
Hence, by using JTR and (a''), we obtain \( \pi(\pi(X, U', V).p, U').p \). Hence, \( X \sim \pi(A_1, \ldots, A_n) \), by Hyp 4, for some derivable \( A_1, \ldots, A_n \).

III.3 \( W \sim \pi(\pi(X, Y).p, W') \) and \( Y \sim \pi((V.p)D, U, W'). \) Now, obviously, (a'') is
\[
\pi(\pi(X, V.p)D, U, W').p, W', D).p
\]
From (a') \( \pi(U, V).p \), we obtain \( \pi(U, (V.p)D).D \) by Theorem 2, and
\[
\pi((V.p)D, Dp, U).p
\]
by JSU. Now by repeatedly using JPR, we easily derive
\[
\]
and hence
\[
\]
by using JTR and \((a'')\). By Hyp 4, \(X \sim \pi(A_1, \ldots, A_n)\) for some derivable \(A_1, \ldots, A_n\).

This completes the proof of the theorem.

**COROLLARY** There is no theorem of \(J\) of the form \(AABB\).

**Proof.** Suppose that there are \(A\) and \(B\) such that \(AABB\) is derivable in \(J\). Since \(J\) is closed under uniform substitution, there are \(A_1\) and \(B_1\) such that \(A_1A_1B_1B_1\) is derivable in \(J_1\). By NOE, \(A_1A_1\) is derivable in \(J_1\) and hence in \(J\), contrary to NOID.

In fact, NOE is in \(J\) equivalent to NOID. For, suppose NOE and let \(A\) be a formula such that \(AA\) is derivable in \(J\); then \(AApp\) is derivable, contrary to NOE.

It is known that \(AABB\) is a theorem of \(E_\omega\); hence the name NOE.

A corollary of NOE concerning contraction and the Reicke Law is the following.

**Theorem 12.** \((A(AB))(AB)\) is derivable in \(J\) iff so is \(A\); \(ABAA\) is derivable in \(J\) iff so is \(AB\).

NOE can be generalized to the following theorem.

**Theorem 13.** \((a)\) \(\pi((\pi(X,Y),p)p^{2k},Y).p\) is derivable iff \(X\) is nonempty and every member of \(X\) is derivable.

**Proof.** If \(k = 0\), the theorem is true by NOE.

Let \(k > 0\) and proceed by induction on \(k\). If \(Y\) is empty, we use Theorem 6.

Suppose that \((a)\) is derivable in \(J_1\). By NOID, \(X\) is nonempty. The verification that \((a)\) is not an instance of ASU is left to the reader.

Case I (a) is obtained by JSU from \((a')\) \(\pi(U,V).p\), where \(\pi((\pi(X,Y),p)p^{2k},Y) \sim \pi(U,(V\cdot p)p)\).

1.1 \((V,p)p \sim (\pi(X,Y),p)p^{2k}\) and \(U \sim Y\); obviously, either \(X\) or \(Y\) is empty. But \(X\) is nonempty. If \(Y\) is empty, then by Theorem 6, \(X \sim \pi(A_1, \ldots, A_n)\) for some derivable \(A_1, \ldots, A_n\).

1.2 \(Y \sim \pi((\pi(X,Y),p)p^{2k},V,Y').p\) and \(U \sim \pi((\pi(X,Y),p)p^{2k},V,Y').p\). Obviously, \((a)\) is obtained from \((a')\) \(\pi((\pi(X,(V\cdot p)p),Y').p)p^{2k},V,Y').p\). Since \(X\) is nonempty, there is a member \(A\) of \(X\). But as an instance of ASU we have \(\pi((\pi(A,V),p),(V\cdot p)p)\). By using JPR we derive \(\pi((\pi(X,Y,Y'),p)),(V\cdot p)p)\) and then by using JSU we obtain \(\pi((\pi(X,Y,Y'),p),(V\cdot p)p)\). Hence, by using JTR and \((a')\) we obtain \(\pi((\pi(X,Y,Y'),p),(V\cdot p)p)\). Now we use NOE to conclude that \(X \sim \pi(A_1, \ldots, A_n)\) for some \(A_1, \ldots, A_n\) derivable in \(J_1\).

Case II (a) follows by JPR from \((a')\) \(\pi(U,D).p\), where \(\pi((\pi(X,Y),p)p^{2k},Y) \sim \pi(U,CD,C)\).

II.1 \(Y \sim \pi(CD,C,Y').p\) and \(U \sim \pi((\pi(X,CD,C,Y'),p)p^{2k},Y').p\). But
\[ \pi(\pi(X, D, Y'), p, X, CD, C, Y') \cdot p \]

is easily derivable in \( \mathbf{J}_1 \). By JSU we derive

\[ \pi(\pi(X, D, Y'), p, (\pi(X, CD, C, Y'), p) \cdot p^{2k-1}) \cdot p \]

Hence, by using JTR and \((a')\), we obtain \( \pi(\pi(X, D, Y'), p, D, Y') \cdot p \). Hence, \( X \sim \pi(A_1, \ldots, A_n) \) by NOE, for some \( A_1, \ldots, A_n \) and \( n \), and \( A_1, \ldots, A_n \) are derivable in \( \mathbf{J}_1 \).

II.2 \((\pi(X, Y), p) \cdot p^{2k} \sim CD \) and \( Y \sim \pi(U, C) \); since \( k > 0 \), \( D \sim p \) and \( Y \) is empty. The theorem follows by Theorem 6.

II.3 \((\pi(X, Y), p) \cdot p^{2k} \sim C \) and \( Y \sim \pi(CD, Y') \); this is impossible.

Case III \((a)\) follows by JG from \((a')\) \( \pi(U, V) \cdot p \) and \((a'')\) \( \pi(W, D) \cdot p \), where we have

\[ \pi((\pi(X, Y), p) \cdot p^{2k}, Y) \sim \pi(U, W, (V, p) \cdot D). \]

By induction hypothesis, \( X \sim \pi(A_1, \ldots, A_n) \) for some derivable \( A_1, \ldots, A_n \).

III.2 \( U \sim \pi((\pi(X, Y), p) \cdot p^{2k}, U') \) and \( Y \sim \pi((V, p) \cdot D, U', W) \). Now \((a')\) is

\[ \pi((\pi(X, V, p) \cdot D, U', W') \cdot p \cdot p^{2k}, U', V) \cdot p. \]

By \((a'')\) \( \pi(W, D) \cdot p \) and JPR we derive \( \pi(\pi(X, U', V), p, X, (V, p) \cdot D, U', W) \cdot p, \)

and then we use JSU to obtain \( \pi(\pi(X, U', V), p, (\pi(X, (V, p) \cdot D, U', W') \cdot p \cdot p^{2k-1}) \cdot p. \)

Hence, by using JTR and \((a')\), we obtain \( \pi(\pi(X, U', V), p, U', V) \cdot p \). Hence, \( X \sim \pi(A_1, \ldots, A_n) \) by NOE, for some derivable \( A_1, \ldots, A_n \).

III.3 \( W \sim \pi((\pi(X, Y), p) \cdot p^{2k}, W') \) and \( Y \sim \pi((V, p) \cdot D, U, W') \). Now, obviously, \((a'')\) is

\[ \pi((\pi(X, (V, p) \cdot D, U', W') \cdot p, W', D) \cdot p. \]

From \((a')\) \( \pi(U, V) \cdot p \), we obtain \( \pi(U, (V, p) \cdot D) \cdot p \) by Theorem 2, and

\[ \pi((V, p) \cdot D, Dp, U) \cdot p \]

by JSU. Now by repeatedly using JPR and JSU, we easily derive

\[ \pi(\pi(X, W', D), p, (\pi(X, (V, p) \cdot D, U', W') \cdot p \cdot p^{2k-1}) \cdot p, \]

and hence \( \pi(\pi(X, W', D), p, W', D) \cdot p \) by using JTR and \((a'')\). Now we use NOE to conclude that \( X \sim \pi(A_1, \ldots, A_n) \) for some derivable \( A_1, \ldots, A_n \).

This completes the proof of the theorem.

The difference between \( \mathbf{L} \) and \( \mathbf{J}_1 \) is now clear: by \( L_0 \), there is no theorem of \( \mathbf{L} \) of the form \( \pi((\pi(X, Y), p) \cdot p^{2k}, Y) \cdot p \) by Theorem 13, \( \pi((\pi(X, Y), p) \cdot p^{2k}, Y) \cdot p \) is derivable in \( \mathbf{J}_1 \) if \( X \) is nonempty and every member of \( X \) is derivable in \( \mathbf{J}_1 \).
Two open problems

Let us adjoin to $J$ the axiom-schema $pp$. It is easy to prove that $ASU$, $JSU$, and $JPR$ are redundant. The system $J+ID$ is equivalent to $RW_{\rightarrow}$, defined by MP and the following axiom-schemata:

<table>
<thead>
<tr>
<th>ID</th>
<th>$AA$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASS</td>
<td>$A.ABB$</td>
</tr>
<tr>
<td>TR</td>
<td>$AB.BC.AC$</td>
</tr>
</tbody>
</table>

(the proof is omitted). It is then easy to show that $A\cdot B$ is not true for $J+ID$. From $A.ABB$, by Theorem 2 we obtain $ABB(AB).A.AB$. On the other hand, $A(AB).ABB.AB$ is an instance of $ASU$. Thus there are distinct formulas $C$ and $D$ such that both $CD$ and $DC$ are derivable in $J+ID$. It is therefore natural to raise the following two questions:

Question 1. Is there any proper extension $EX$ of $TW_{\rightarrow}$ such that $A\cdot B$ holds for $EX$?

Question 2. Is there any proper extension $EX$ of $J$ such that $NOID$ holds for $EX$?

REFERENCES